Successions of J-Bessel in spaces with indefinite metric.

OSMIN FERRER[1], LUIS LAZARO[2], JORGE RODRIGUEZ[3]
[1]Departamento de Matemáticas, Facultad de Educación y Ciencias, Universidad de Sucre, Sincelejo, COLOMBIA. https://orcid.org/0000- 0003-1015-6271

[2] Departamento de Ciencias Básicas, Corporación Universitaria del Caribe-CECAR, Sincelejo, Departamento de Matemáticas, Facultad de Educación y Ciencias. Universidad de Sucre, Sincelejo, COLOMBIA. https://orcid.org/0000-0002-6246-6514

[3] Programa de Matemáticas, Facultad de Ciencias Básicas, Universidad del Atlántico, Barranquilla. Universidad del Norte, Departamento de Matemáticas y estadística, Barranquilla, COLOMBIA. https://orcid.org/0000- 0002-1953-5350

Abstract: - A definition of Bessel’s sequences in spaces with an indefinite metric is introduced as a generalization of Bessel’s sequences in Hilbert spaces. Moreover, a complete characterization of Bessel’s sequences in the Hilbert space associated to a space with an indefinite metric is given. The fundamental tools of Bessel’s sequences theory are described in the formalism of spaces with an indefinite metric. It is shown how to construct a Bessel’s sequences in spaces with an indefinite metric starting from a pair of Hilbert spaces, a condition is given to decompose a Bessel’s sequences into in spaces with an indefinite metric so that this decomposition generates a pair of Bessel’s sequences for the Hilbert spaces corresponding to the fundamental decomposition.
In spaces where there was no norm, it seemed impossible to construct Bessel’s sequences. The fact that in [1] frame were constructed for Krein spaces motivated us to construct Bessel’s sequences for spaces of indefinite metric.

Key-Words: - Krein spaces, indefinite metric, $J-$norm, successions de $J-$Bessel, base $J-$orthonormal.

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1 Introduction

In Hilbert spaces there is a large amount of own results and made possible by the Bessel’s sequence, the construction of Riesz bases, frames and Gabor frames support this idea, likewise the p-Schatten operators study [2].

The Bessel's theory of the Hilbert spaces originates from an article by Duffin and Schaeffer [3], such sequence are closely linked with the frame theory which has a great development in [4][5][6], Recently the frame theory is extended to undefined metric spaces in [1] then has a development in [2] [7] [8], the Bessel’s sequences in Hilbert spaces and their relation to frames and Riesz bases makes it a very important tool for future applications.

On the other hand, the theory of indefinite metric spaces which can be seen to be developed in [9][10] has been showing great development in its many applications to physics, and is promising in itself, so it is very novel and promising for future research to extend the Bessel´s sequences theory to indefinite metric spaces, Therefore it is natural that one would want to have for Bessel’s sequences the same tools available for the indefinite metric spaces, it is possible to think of having these tools for the Hilbert spaces generated from an indefinite metric space, however in this paper we develop a theory completely independent of such spaces and then relate it through some operators associated with the indefinite metric spaces.

It is possible to think of having these tools for Hilbert spaces generated from an indefinite metric spaces, however in this work we develop a completely independent theory of said spaces and then we relate it through some operators associated with indefinite metric spaces

2 Preliminary

Definition 2.1:[4] Be $(H,\langle\cdot,\cdot\rangle)$ a Hilbert space, $\{x_n\}_{n\in\mathbb{N}} \subset H$ is a Bessel´s sequence in H, if there is a constant $A > 0$ such that

$$\sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq A\,\|x\|^2, \text{for everything } x \in H.$$

Theorem 2.2:[4][5] Be $(H,\langle\cdot,\cdot\rangle)$ a Hilbert space, $\{x_n\}_{n\in\mathbb{N}} \subset H$, if $\{x_n\}_{n\in\mathbb{N}}$ is a Bessel's sequences in H, then there is $\{e_n\}_{n\in\mathbb{N}} \subset H$ orthonormal base for H and an operator $T : H \to H$ linear bounded such that

$$T(e_n) = x_n, \text{ for everything } n \in \mathbb{N}.$$

Definition 2.3:[9][10] Throughout this paper, K denotes a vector space on the complex plane $\mathbb{C}$. Let be a sesquilinear form $[\cdot,\cdot] : K \times K \to \mathbb{C}$.

The pair $(K,[\cdot,\cdot])$ is called a Krein space if

$K = K^+ \oplus K^-$, where $(K^+,[\cdot,\cdot]),(K^-,-[\cdot,\cdot])$ are Hilbert spaces, and K^+, K^- are orthogonal with respect to $[\cdot,\cdot]$.

Example 2.4:[9][10] Let $K = \mathbb{R}^2$ with undefined internal product $[\cdot,\cdot]:\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by $[(a,b),(c,d)] \coloneqq ac - bd$, (a,b), $(c,d) \in \mathbb{R}^2$. Consider $K^+ \coloneqq \{(x,0): x \in \mathbb{R}\}$ and $K^- \coloneqq \{(0,y): y \in \mathbb{R}\}$ clearly $(K^+,[\cdot,\cdot])$ and $(K^-,-[\cdot,\cdot])$ are Hilbert's spaces. For this reason $(\mathbb{R}^2,[\cdot,\cdot])$ is a Krein space.

Definition 2.5:[9][10] Let K be a Krein space with fundamental decomposition $K = K^+ \oplus K^-$, Two operators $P^+ : K \to K^+$, $P^- : K \to K^-$ are defined as follows $P^+k = k^+$ and $P^-k = k^-$for all $k \in K$, where $k^+\in K^+$, $k^-\in K^-$ and $k = k^+ + k^-$. The operators P^+ and $P^-$are known as fundamental projectors.

Example 2.6:[9][10] For the Krein space $(\mathbb{R}^2,[\cdot,\cdot])$ given in example 2.5 the fundamental projectors P^+ and $P^-$are given by $P^+(x,y) = (x,0)$ and $P^-(x,y) = (0,y)$.

Definition 2.7: [9][10] Let $(K,[\cdot,\cdot])$ be a Krein space, the operator $J : K \to K$ defined by $J = P^+ - P^-$, i.e. for all $k \in K$

$$Jk = P^+k - P^-k = k^+ - k^-.$$

Is called the fundamental symmetry of Krein K space.

Example 2.8: [9][10] For the Krein space $(\mathbb{R}^2,[\cdot,\cdot])$ given in example 2.5 the fundamental symmetry $J = P^+ - P^-$ is given by $J(x,y) = (x,0) - (0,y) = (x,-y)$.

Definition 2.9:[10][11] Let $(K = K^+ \oplus K^-,[\cdot,\cdot])$ a space of Krein and J the fundamental symmetry associated with the given decomposition. The function $[\cdot,\cdot]_J: K \times K \to \mathbb{C}$ is defined by

$$[k_1,k_2]_J = [Jk_1,k_2],\ for\ all\ k_1,k_2 \in K.$$

This function will be called internal $J-$product and it is $[k,k]_J \geq 0$ for all $k \in K$. and the norm induced by the $J-$ internal product $||k||_J = \sqrt{[k,k]_J}$ is called $J-$norm. Furthermore, the space $(K,[\cdot,\cdot]_J)$ turns out to be a Hilbert space.

In this work for a Krein space $(K = K^+ \oplus K^-,[\cdot,\cdot])$ we will notice $||k^+||_+ = \sqrt{[k^+,k^+]} = ||k^+||_J$ and $||k^-||_- = \sqrt{-[k^-,k^-]} = ||k^-||_J$.

Remark 2.10: We can see that equality is fulfilled

$$\begin{aligned} ||k^+||_+{}^2 + ||k^-||_-{}^2 &= [k^+,k^+] - [k^-,k^-] \\ &= ||k||_J{}^2. \end{aligned}$$

Example 2.11:[7][10] For the Krein space $(\mathbb{R}^2,[\cdot,\cdot])$ given in example 2.5 the internal $J-$product is given by

$$\begin{aligned} [(a,b),(c,d)]_J &= [J(a,b),(c,d)] \\ &= [(a,-b),(c,d)] \\ &= ac - (-bd) = ac + bd \end{aligned}$$

And the $J-$norm is

$$||(a,b)||_J = \sqrt{[(a,b),(a,b)]_J} = \sqrt{a^2+b^2}$$

Proposition 2.12:[7][10] Let $(K = K^+ \oplus K^-, [\cdot,\cdot])$ a Krein space with fundamental symmetry J, then:

(1) J is a symmetric operator, i.e. $[Jk_1, k_2] = [k_1, Jk_2]$, for all $k_1,\ k_2 \in K$..

(2) J is an isometric operator, i.e. $[Jk_1, Jk_2] = [k_1, k_2]$, for all $k_1,\ k_2 \in K$.

(3) $J^2 = I$.

(4) J is a $J-$isometric operator, i.e. $[Jk_1, Jk_2]_J = [k_1, k_2]_J$, for all $k_1,\ k_2 \in K$.

Theorem 2.13:[7][10] Be $(K = K^+ \oplus K^-, [\cdot,\cdot])$ a Krein space, for any $J-$norm in K the inequality is fulfilled

$$|[k_1, k_2]| \leq ||k_1||_J\ ||k_2||_J \ \ for\ all\ k_1, k_2 \in K.$$

Theorem 2.14:[9][10] Be$(K = K^+ \oplus K^-, [\cdot,\cdot], J)$ a Krein space and $f : K \to C$ a linear functional boundary. Then there is only one element $y \in K$ such that $f(x) = [x,y]$ for all $x \in K$.

Definition 2.15:[1][6][8] Be $(K_1 = K_1^+ \oplus K_1^-, [\cdot,\cdot], J_1)$ and $(K_2 = K_2^+ \oplus K_2^-, [\cdot,\cdot], J_2)$. A linear operator $T: K_1 \to K_2$ is said to be limited if there is a positive real number c such that for everything $k \in K_1$.

$$||Tk||_{J_2} \leq c\ ||k||_{J_1}, for\ everything\ k \in K.$$

Example 2.16:[9] Let's consider the operator

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

Defined by $T(x,y) = (y,x)$

$$||T(x,y)||_J = ||(y,x)||_J = \sqrt{y^2+x^2}$$

$$\leq c\sqrt{x^2+y^2} = c\ ||k||_J.$$

It is evident that for $c \geq 1$ the inequality is fulfilled, therefore the operator T thus defined is limited.

Definition 2.17:[10] Be $(K_1 = K_1^+ \oplus K_1^-, [\cdot,\cdot]_1, J_1)$ and $(K_2 = K_2^+ \oplus K_2^-, [\cdot,\cdot]_2, J_2)$ Krein spaces. The attachment of the linear operator $T : K_1 \to K_2$ is the only linear operator $T^{[*]}: Dom(T^{[*]}) \subset K_2 \to K_1$ such that for all $x \in K_1, y \in Dom(T^{[*]})$

$$[Tx,y]_2 = [x, T^{[*]}y]_1.$$

Proposition 2.18:[9][10] Be $(K = K^+ \oplus K^-, [\cdot,\cdot], J)$ a Krein space and $T : K \to K$ a linear operator, then $T^{[*]} = JT^{*J}J$.

Proposition 2.19:[3] Be $(K, [\cdot,\cdot], J)$ a Krein space, $T : K \to K$ a linear bounded operator, then $T\,J, JT : K \to K$ are linear bounded operators.

Proof. The operator J is linear, and taking into account that the composition of linear operators is linear and the composition of dimensioned operators is dimensioned, then TJ and JT are linear and dimensioned.

Proposition 2.20:[2][9] Be $(K = K^+ \oplus K^-, [\cdot,\cdot], J)$ a Krein space, $T : K \to K$ a linear operator such that $TK^+ \subset K^+, TK^- \subset K^-$ then $TJ = JT$.

Proof. Be $k \in K$, then there are $k^+, w^+ \in K^+$ and $k^-, w^- \in K^-$ such that $k = k^+ + k^-$, $Tk^+ = w^+$ and $Tk^- = w^-$ then

$$T\,Jk = TJ(k^+ + k^-) = T(k^+ - k^-)$$

$$= Tk^+ - Tk^- = w^+ - w^-$$

$$= J(w^+ + w^-) = J(Tk^+ + Tk^-)$$

$$= JT(k^+ + k^-)$$

$$= JT(k).$$

Remark 2.1: In particular $P^+T = TP^+$ and $P^-\,T = TP^-$.

Definition 2.22:[9][12] Be $(K = K^+ \oplus K^-, [\cdot,\cdot])$ a Krein space with fundamental symmetry J is said to $\{e_n\}_{n\in\mathbb{N}} \subset H$ is orthonormal basis for $(K, [\cdot,\cdot])$ if

$$[e_n, e_m] = \begin{cases} 0 & si\ n \neq m \\ \pm 1 & si\ n = m \end{cases}$$

Example 2.23:[10][12] For the Krein space $(\mathbb{R}^2, [\cdot,\cdot])$ given in example 2.5 an orthogonal base is given by $\{e_n\}_{n\in\mathbb{N}} = \{e_1 = (1,0), e_2 = (0,1)\}$ as it fulfils $[e_1, e_1] = 1$, $[e_1, e_2] = [e_2, e_1] = 0$ and $[e_2, e_2] = -1$.

Definition 2.24:[9][12] Be$(K,[\cdot,\cdot])$ a Krein space with fundamental symmetry J says that $\{e_n\}_{n\in\mathbb{N}} \subset K$ is orthonormal basis for $(K,[\cdot,\cdot]_J)$ if it is fulfilled.

$$[e_n, e_m]_J = \begin{cases} 0 & si\ n \neq m \\ 1 & si\ n = m \end{cases}$$

Example 2.25:[10][12] For the Krein space $(\mathbb{R}^2, [\cdot,\cdot])$ given in example 2.5 a J −ortonormal base is given by $\{e_n\}_{n\in\mathbb{N}} = \{e_1 = (1,0), e_2 = (0,1)\}$ as it meets $[e_1,e_1]_J = 1$, $[e_1,e_2]_J = [e_2,e_1]_J = 0$ and $[e_2,e_2]_J = 1$.

Below, we present some results obtained in this work, which allowed us to introduce the Bessel's sequences in spaces of indefinite metric.

3 successions of J-Bessel in spaces with indefinite metric

Proposition 3.1 Be $(K = K^+ \oplus K^-, [\cdot,\cdot])$ a Krein space with fundamental symmetry J, if $\{e_n = e_n^+ + e_n^-\}_{n\in\mathbb{N}} \subset K$ generate to $(K,[\cdot,\cdot])$, if only if $\{e_n^+\}_{n\in\mathbb{N}} \subset K^+$, $\{e_n^-\}_{n\in\mathbb{N}} \subset K^-$ generate to $(K^+,[\cdot,\cdot])$ *and* $(K^-, -[\cdot,\cdot])$ respectively**.**

Proof. Sea $k^+ \in K^+$ then exists $\propto_n \in \mathbb{C}$ such that

$$k^+ = \sum_{n\in\mathbb{N}} \propto_n e_n$$

then

$$k^+ = P^+k^+ = P^+ \sum_{n\in\mathbb{N}} \propto_n e_n$$

$$= \sum_{n\in\mathbb{N}} \propto_n P^+ e_n = \sum_{n\in\mathbb{N}} \propto_n e_n^+$$

Similarly it is shown that $\{e_n^-\}_{n\in\mathbb{N}}$ generate to $(K^-, -[\cdot,\cdot])$. Now $\{e_n^+\}_{n\in\mathbb{N}} \subset K^+$, $\{e_n^-\}_{n\in\mathbb{N}} \subset K^-$ generate to $(K^+,[\cdot,\cdot])$ *and* $(K^-, -[\cdot,\cdot])$ respectively**.**

then exists $\propto_i, \beta_i \in \mathbb{C}$ and $k = k^+ + k^-$ such that

$$k = k^+ + k^- = \sum_{n\in\mathbb{N}} \propto_n e_n^+ + \sum_{n\in\mathbb{N}} \beta_n e_n^-$$

Proposition 3.2: $\{e_n^+\}_{n\in\mathbb{N}} \subset K^+$, $\{e_n^-\}_{n\in\mathbb{N}} \subset K^-$ generators for K^+ and K^-, We can get $\{e_n^{*+}\}_{n\in\mathbb{N}}$, $\{e_n^{*-}\}_{n\in\mathbb{N}}$, orthonormal bases for K^+ and K^- hence a basis for K. Similarly from $\{e_n = e_n^+ + e_n^-\}_{n\in\mathbb{N}}$ generator for K orthonormal bases are obtained for $(K^+,[\cdot,\cdot])$ and $(K^-, -[\cdot,\cdot])$.

Proposition 3.3: Let $(K = K^+ \oplus K^-, [\cdot,\cdot])$ a Krein space with fundamental symmetry J, $\{k_n^+\}_{n\in\mathbb{N}}, \{k_n^-\}_{n\in\mathbb{N}}$ orthonormal bases for Hilbert spaces $(K^+,[\cdot,\cdot])$ and $(K^-, -[\cdot,\cdot])$ respectively, then $\{\frac{\sqrt{2}}{2}(k_n^+ + k_n^-)\}_{n\in\mathbb{N}}$ is an orthonormal basis for the space of Hilbert $(K,[\cdot,\cdot]_J)$

Proof.

$$[\frac{\sqrt{2}}{2}(k_n^+ + k_n^-), \frac{\sqrt{2}}{2}(k_m^+ + k_m^-)]_J$$

$$= [\frac{\sqrt{2}}{2}(k_n^+ - k_n^-), \frac{\sqrt{2}}{2}(k_m^+ + k_m^-)]$$

$$= [\frac{\sqrt{2}}{2}k_n^+, \frac{\sqrt{2}}{2}k_m^+] - [\frac{\sqrt{2}}{2}k_n^-, \frac{\sqrt{2}}{2}k_m^-]$$

$$= \frac{1}{2}[k_n^+, k_m^+] - \frac{1}{2}[k_n^-, k_m^-] = \delta_{nm}.$$

Then $\{\frac{\sqrt{2}}{2}(k_n^+ + k_n^-)\}_{n\in\mathbb{N}}$ is an orthonormal base for Hilbert's space $(K,[\cdot,\cdot]_J)$.

Proposition 3.4: Be $(K,[\cdot,\cdot])$ a Krein space with fundamental decomposition $K = K^+ \oplus K^-$ and be J the associated fundamental symmetry, $T^+ : K^+ \to K^+$, $T^- : K^- \to K^-$ dimensioned linear operators, then the operator $T : K \to K$ defined by

$$T(k) = T(k^+ + k^-) = T^+k^+ + T^-k^-$$

For each $k \in K$, is linearly dimensioned in Hilbert's space $(K,[\cdot,\cdot]_J)$.

Proof.

Be $k_1, k_2 \in (K,[\cdot,\cdot]_J)$ and $\alpha \in \mathbb{C}$, then

$$T(\alpha k_1 + k_2) = T(\alpha k_1^+ + \alpha k_1^- + k_2^+ + k_2^-)$$

$$= T((\alpha k_1^+ + k_2^+) + (\alpha k_1^- + k_2^-))$$

$$= T^+(\alpha k_1^+ + k_2^+) + T^-(\alpha k_1^- + k_2^-)$$

$$= \alpha T^+k_1^+ + T^+k_2^+ + \alpha T^-k_1^- + T^-k_2^-$$

$$= \alpha T^+k_1^+ + \alpha T^-k_1^- + T^+k_2^+ + T^-k_2^-$$

$$= \alpha T(k_1^+ + k_1^-) + T(k_2^+ + k_2^-)$$

$$= \alpha T(k_1) + T(k_2).$$

Thus T is lineal.

Be $k = k^+ + k^- \in K$ hipotecally T^+ and T^- are limited, so there are real positives c_1, c_2 such that

$$||T^+k^+||_+ \le c_1||k^+||_+ , ||T^-k^-||_- \le c_1||k^-||_-$$

be $\sqrt{Max\{2c_1{}^2, 2c_2{}^2\}} = c.$

Then

$$||Tk||_J^2 = ||T(k^+ + k^-)||_J^2 = ||T^+k^+ + T^-k^-||_J^2$$

$$\le (\,||T^+k^+||_J + ||T^-k^-||_J\,)^2$$

$$\le 2||T^+k^+||_J^2 + 2||T^-k^-||_J^2$$

$$\le 2c_1{}^2||k^+||_J{}^2 + 2c_2{}^2||k^-||_J{}^2$$

$$= 2c_1{}^2||k^+||_+{}^2 + 2c_2{}^2||k^-||_-{}^2$$

$$\le Max\{2c_1{}^2, 2c_2{}^2\}\,(\,||k^+||_+{}^2 + ||k^-||_-{}^2\,)$$

$$\le c^2\,(\,||k||_J^2\,)$$

Whereby $||Tk||_J \le c\,||k||_J$

The following result obtained in this work, motivated us to extend the theory of Bessel’s sequences for Hilbert spaces to indefinite metric spaces.

Theorem 3.5: Be $(K, [\cdot,\cdot])$ a Krein space with fundamental decomposition $K = K^+ \oplus K^-$ and let J be the associated fundamental symmetry if $\{k_n^+\}_{n\in\mathbb{N}} \subset K^+$, $\{k_n^-\}_{n\in\mathbb{N}} \subset K^-$ are Bessel's sequences to $(K^+, [\cdot,\cdot])$ and $(K^-, -[\cdot,\cdot])$ respectively, then $\{\frac{\sqrt{2}}{2}\,(k_n^+ + k_n^-)\}_{n\in\mathbb{N}}$ it's a Bessel´s sequence for Hilbert's space $(K, [\cdot,\cdot]_J)$.

Proof. There are $T^+: K^+ \to K^+$, $T^-: K^- \to K^-$ dimensioned linear operators $\{t_n^+\}_{n\in\mathbb{N}} \subset K^+$, $\{k_n^-\}_{n\in\mathbb{N}} \subset K^-$ orthonormal bases for $(K^+, [\cdot,\cdot])$ and $(K^-, -[\cdot,\cdot])$ respectively, such that $T^+t_n^+ = k_n^+$ and $T^-t_n^- = k_n^-$ for each $n \in \mathbb{N}$.

Let’s define the operator $T : K \to K$ as

$$T(k) = T(k^+ + k^-) = T^+k^+ + T^-k^-$$

For each $k \in K$, the proposition 3.3 guarantees that T is linear and the proposition 3.2 guarantees that

$$\{\tfrac{\sqrt{2}}{2}\,(t_n^+ + t_n^-)\}_{n\in\mathbb{N}} \subset K.$$

is an orthonormal base for Hilbert's space $(K, [\cdot,\cdot]_J)$. In addition

$$T\left(\frac{\sqrt{2}}{2}\,(t_n^+ + t_n^-)\right) = \frac{\sqrt{2}}{2}\;T(\,t_n^+ + t_n^-)$$

$$= \frac{\sqrt{2}}{2}T^+t_n^+ + \frac{\sqrt{2}}{2}T^-t_n^-$$

$$= \frac{\sqrt{2}}{2}k_n^+ + \frac{\sqrt{2}}{2}k_n^-$$

$$= \frac{\sqrt{2}}{2}(k_n^+ + k_n^-).$$

The theorem 2.2 guarantees that $\{\frac{\sqrt{2}}{2}(k_n^+ + k_n^-)\}_{n\in\mathbb{N}}$ it's a Bessel’s sequence for Hilbert's space $(K, [\cdot,\cdot]_J)$.

Definition 3.6: Be $(K = K^+ \oplus K^-, [\cdot,\cdot])$ a Krein space with fundamental symmetry J, $\{k_n\}_{n\in\mathbb{N}} \subset K$, we say that $\{k_n\}_{n\in\mathbb{N}}$ is a J −Bessel’s sequence in K if there is $\{e_n\}_{n\in\mathbb{N}} \subset K$ orthonormal base for $(K, [\cdot,\cdot])$ and a operator for $T : K \to K$ linear dimensioned, such that

$$T(Je_n) = Jk_n, ; for\ all\ n \in \mathbb{N}.$$

Remark 3.7: In the particular case of K being a Hilbert space, $J = I$ and it would have

$$T(e_n) = k_n\,; for\ everything\ \ n \in \mathbb{N}.$$

Which is the result given in theorem 2.2 which is equivalent to definition 2.1.

Example 3.8: Let’s consider the Krein space $(\mathbb{R}^2, [\cdot,\cdot])$ with fundamental symmetry J given in the example 2.5 the orthonormal base $\{e_n\}_{n\in\mathbb{N}} = \{e_1 = (1,0), e_2 = (0,1)\}$, the lineal operator

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

Defined by $T(x, y) = (y, x)$. The example 2.17 guarantees that T is bounded and $\{k_n\}_{n\in\mathbb{N}} = \{\,b_1 = (-1,0),\ b_2 = (0,-1)\,\}$,

In addition

$$TJe_1 = TJ(0,1) = T(0,-1) = (-1,0)$$

$$= J(-1,0) = J(b_1)$$

$$TJe_2 = TJ(1,0) = T(1,-0) = (0,1)$$

$$= J(0,-1) = J(b_2)$$

Then $\{k_n\}_{n\in\mathbb{N}} = \{\, b_1 = (-1,0),\ b_2 = (0,-1)\,\}$ is a J-Bessel's sequence to the Krein space ($\mathbb{R}^2, [\cdot,\cdot]$).

Proposition 3.9: Be $(K = K^+ \oplus K^-, [\cdot,\cdot])$ a Krein space with fundamental symmetry J, if $\{k_n = k_n^+ + k_n^-\}_{n\in\mathbb{N}} \subset K$ is a J-Bessel's sequences for $(K, [\cdot,\cdot])$ and the operator associated with the Bessel sequences leaves K^+ and K^- then $\{k_n^+\}_{n\in\mathbb{N}}$, $\{k_n^-\}_{n\in\mathbb{N}}$ are Bessel's sequences to $(K^+, [\cdot,\cdot])$ and $(K^-, -[\cdot,\cdot])$ respectivelly.

Proof. If $\{k_n = k_n^+ + k_n^-\}_{n\in\mathbb{N}} \subset K$ is a J-Bessel's sequence for $(K, [\cdot,\cdot])$ then there are $\{e_n\}_{n\in\mathbb{N}}$ ortonormal base for $(K, [\cdot,\cdot])$, one linear T operator and $TK^+ \subset K^+, TK^- \subset K^-$ and it complies

$$T(Je_n) = Jk_n, for\ everything\ n \in \mathbb{N}.$$

Then

$$T(e_n^+) = T(Je_n^+) = TJ(P^+(e_n^+ + e_n^-))$$

$$= TJ\big(P^+(e_n)\big) = TP^+(Je_n)$$

$$= P^+T(Je_n) = P^+Jk_n = k_n^+$$

The proposition 3.1 guarantees that $\{e_n^+\}_{n\in\mathbb{N}}$ is an orthonormal base for Hilbert's space $(K^+, [\cdot,\cdot])$.

so $\{k_n^+\}_{n\in\mathbb{N}}$ it's a Bessel´s sequence for Hilbert's space $(K^+, [\cdot,\cdot])$.

Similarly, it is shown that $\{k_n^-\}_{n\in\mathbb{N}}$ it's a Bessel's sequences for Hilbert space $(K^-, -[\cdot,\cdot])$.

The following result shows the consistency of the definition 3.5 of J −Bessel´s sequence given in this paper in that it does not depend on fundamental decomposition

Theorem 3.10: Be $(K, [\cdot,\cdot])$ a Krein space with fundamental decomposition given by $K = K_1^+ \oplus K_1^-$, J_1 the associated fundamental symmetry and $\{k_n\}_{n\in\mathbb{N}}$ a J-Bessel's sequence to $(K, [\cdot,\cdot])$ with regard to J_1. if J_2 is the fundamental symmetry associated with decomposition $K = K_2^+ \oplus K_2^-$, then $\{k_n\}_{n\in\mathbb{N}}$ is also a J-Bessel's sequence for $(K, [\cdot,\cdot])$ with regard to J_2.

Proof. be T the dimensioned linear operator that converts $\{k_n\}_{n\in\mathbb{N}} \subset K$ in a J-Bessel´s sequence to $(K, [\cdot,\cdot])$ with regard to the fundamental simmetry J_1 associated with decomposition $K = K_1^+ \oplus K_1^-$. Then there is $\{e_n\}_{n\in\mathbb{N}}$ Krein space orthonormal base $(K, [\cdot,\cdot])$ and it complies that $T J_1 e_n = J_1 k_n$. Now if J_2 is the fundamental symmetry associated with decomposition $K = K_2^+ \oplus K_2^-$ and $\phi : K \to K$ is the operator defined by

$$\phi := J_2 J_1 T J_1 J_2$$

then ϕ is also linear bounded. Finally it is shown that ϕ makes $\{k_n\}_{n\in\mathbb{N}} \subset K$ a J-Bessel´s sequence with respect to the fundamental symmetry J_2. Indeed, if we consider the orthonormal base $\{e_n\}_{n\in\mathbb{N}}$ it results that

$$\phi(J_2 e_n) = (J_2 J_1 T J_1 J_2)(J_2 e_n)$$

$$= J_2 J_1 T J_1 J_2^2 e_n = J_2 J_1 (T J_1 e_n)$$

$$= J_2 J_1 J_1 k_n = J_2 J_1^2\, k_n$$

$$= J_2 k_n, for\ all\ \ n \in \mathbb{N}.$$

Therefore in definition 3.5 one can simply talk about J-Bessel's sequence for the Krein space $(K, [\cdot,\cdot])$.

The following result guarantees us that if a Krein space has a J-Bessel´s sequence, it shares it with its associated Hilbert space.

Proposition 3.11: Be $(K = K^+ \oplus K^-, [\cdot,\cdot])$ a Krein space with fundamental symmetry J. If $\{k_n\}_{n\in\mathbb{N}}$ is a J-Bessel's sequence to $(K, [\cdot,\cdot])$, then $\{k_n\}_{n\in\mathbb{N}}$ it's a Bessel's sequence for Hilbert's space $(K, [\cdot,\cdot]_J)$.

Proof. Let's suppose that $\{k_n\}_{n\in\mathbb{N}}$ is a J-Bessel's sequence for $(K, [\cdot,\cdot])$, then it exists $\{e_n\}_{n\in\mathbb{N}} \subseteq K$ orthonormal base and an operator $T : K \to K$ linear dimensioned with respect to the $J -$ norm such that $T(Je_n) = Jk_n$ for all $n \in \mathbb{N}$. The proposition 3.2 guarantees that $\{\frac{\sqrt{2}}{2} Je_n\}_{\,n\in\mathbb{N}}$ is J −orthonormal base for $(K, [\cdot,\cdot])$. Let's consider the operator $\phi : K \to K$ defined by $\phi(k) := \sqrt{2}\, JT(k)$, for all $k \in K$.

The proposition 2.20 guarantees that ϕ is linear in size and is such that

$$\begin{aligned}\phi(\frac{\sqrt{2}}{2}Je_n) &= \sqrt{2}\,JT\left(\frac{\sqrt{2}}{2}Je_n\right)\\ &= (\sqrt{2})(\frac{\sqrt{2}}{2})JT(Je_n) = JT(Je_n)\\ &= J(Jk_n) = k_n.\end{aligned}$$

Henceforth $\{k_n\}_{n\in\mathbb{N}}$ is a Bessel´s sequence in $(K,[\cdot,\cdot]_J)$.

Remark 3.12: Note that in particular if $\{Jk_n\}_{n\in\mathbb{N}}$ a Bessel´s sequence to $(K,[\cdot,\cdot])$, then $\{Jk_n\}_{n\in\mathbb{N}}$ it's a Bessel´s sequence for Hilbert's space $(K,[\cdot,\cdot]_J)$.

In the following result we show that fundamental symmetry preserves Bessel's sequence in spaces of indefinite metric

Proposition 3.13: Be $(K = K^+ \oplus K^-,[\cdot,\cdot])$ a Krein space with fundamental symmetry J. If $\{k_n\}_{n\in\mathbb{N}}$ is a Bessel´s sequence for Hilbert space $(K,[\cdot,\cdot]_J)$ then $\{Jk_n\}_{n\in\mathbb{N}}$ it's a Bessel´s sequence for Hilbert's space $(K,[\cdot,\cdot]_J)$.

Proof. Let’s suppose that $\{k_n\}_{n\in\mathbb{N}} \subset K$ its a Bessel´s sequence for Hilbert's space $(K,[\cdot,\cdot]_J)$, then there is $\{e_n\}_{n\in\mathbb{N}} \subset K$ base $J-$orthonormal in K and an operator $T : K \to K$ linear dimensioned with respect to the J-norm such that $T(e_n) = k_n$, $for\ all\ n \in \mathbb{N}$.

Let’s consider the operator $\Gamma : K \to K$ defined by $\Gamma(k) = JT(k)$, the proposition 2.20 guarantees that JT is linear with respect to the J-norm, also

$$\begin{aligned}\Gamma(e_n) = JT(e_n) &= J\big(T(e_n)\big)\\ &= Jk_n\,, for\ all\ n \in \mathbb{N}.\end{aligned}$$

This leads to the conclusion that $\{Jk_n\}_{n\in\mathbb{N}}$ it's a Bessel’s sequence for Hilbert's space $(K,[\cdot,\cdot]_J)$.

In the following result, we show how to build new J-Bessel’s sequence in Krein spaces through a limited operator

Proposition 3.14: If $(K,[\cdot,\cdot])$ is a Krein space with fundamental symmetry J, $\{k_n\}_{n\in\mathbb{N}}$ is a J-Bessel’s sequence in K and $U : K \to K$ is a limited linear operator, then $\{U(Jk_n)\}_{n\in\mathbb{N}}$ is a Bessel’s sequence for $(K,[\cdot,\cdot])$.

Proof. There is an operator $T : K \to K$ linear dimensioning and an orthogonal base $\{e_n\}_{n\in\mathbb{N}}$ for K such that $T(Je_n) = Jk_n$ for all n ∈ N, let’s consider the operator $JUT : K \to K$which is linearly dimensioned, in addition $JUT(Je_n) = JU(Jk_n) = J\big(U(Jk_n)\big)$, for all $n \in \mathbb{N}$. Therefore it is clear that $\{U(Jk_n)\}_{n\in\mathbb{N}}$ is a Bessel’s sequence for $(K,[\cdot,\cdot])$.

Next, we prove that definition 2.1 usually given in Hilbert spaces for Bessel´s sequence is a consequence of definition 3.5 given in this article.

Theorem 3.15: Be $(K,[\cdot,\cdot])$ a Krein space with fundamental symmetry J, if $\{k_n\}_{n\in\mathbb{N}}$ is a J-Bessel’s sequence for $(K,[\cdot,\cdot])$ and the operator associated with the Bessel’s sequence leaves K^+ and K^-, then there is a constant $A > 0$ such that

$$\sum_{n\in\mathbb{N}} |[k,k_n]|^2 \le A\,\|k\|_J^2, for\ all\ k \in K.$$

Proof. As $\{k_n\}_{n\in\mathbb{N}}$ is a J-Bessel’s sequence for $(K,[\cdot,\cdot])$ by proposition 3.8 $\{k_n^+\}_{n\in\mathbb{N}} \subset K^+$,

$\{k_n^-\}_{n\in\mathbb{N}} \subset K^-$are Bessel′s sequence to$(K^+,[\cdot,\cdot])$ and $(K^-,-[\cdot,\cdot])$ respectively, then theorem 3.4 guarantees us that $\{\frac{\sqrt{2}}{2}(k_n^+ + k_n^-)\}_{n\in\mathbb{N}}$ it's a Bessel’s sequence for Hilbert's space $(K,[\cdot,\cdot]_J)$ furthermore proposition 3.12 guarantees that $\{\frac{\sqrt{2}}{2}J(k_n^+ + k_n^-)\}_{n\in\mathbb{N}}$ it's a Bessel’s sequence for Hilbert's space $(K,[\cdot,\cdot]_J)$ Bessel's definition 2.1 of sequence for Hilbert spaces ensures that there is $A_1 > 0$ such that

$$\sum_{n\in\mathbb{N}} |[k,\frac{\sqrt{2}}{2}Jk_n]_J|^2 \le A_1\,\|k\|_J^2$$

For all $k \in K$.

Then

$$\sum_{n\in\mathbb{N}} |[k,k_n]|^2 \le \sqrt{2}\,A_1\,\|k\|_J^2$$

Taking $\sqrt{2}\,A_1 = A$ it has that

$$\sum_{n\in\mathbb{N}} |[k,k_n]|^2 \le A\,\|k\|_J^2$$

For all $k \in K$.

4 Conclusions

In this article, we have introduced the concept of Bessel's sequences in spaces of indefinite metric, starting from the construction of Bessel´s sequences for Hilbert spaces associated with a space of indefinite metric. It is also shown how to construct Bessel's sequences for a Krein space from its fundamental decomposition and that these do not depend on the decomposition of the Krein space. The definition set out in this investigation allows us to construct Bessel's sequences starting from the orthonormal basis. In addition, it is shown that the definition presented in Hilbert spaces is a consequence of the definition given in this work and how fundamental symmetry preserves the Bessel's sequences in Krein spaces and associated Hilbert spaces. Based on the relationship that the Bessel's sequences have with the frames in Hilbert spaces and taking into account that the frame were extended to undefined metric spaces in [1] and that also recent developments in the theory of frames and wavelet [13] [14] show us the solidity of this theory in fields applied to systems and signal processing, this work turns out to be promising to find new applications in more general spaces than that developed in [13] [14].

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