On the Vasyunin Cotangent sums related to Riemann Hypothesis

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Abstract: In this work, we are interested by Vasyunin cotangent-sum $V(p/q)$ encountered in computation of the inner product arising in the Baez-Duarte-Balazard criterion for Riemann hypothesis. By hint of generating functions theory and introduction of double Euclidean algorithm, we give series expansions of $V(p/q)$ and the symmetric sum $S(p,q) = V(p/q) + V(q/p)$. These calculus permit to deduce another reformulation of Vasyunin formula. This studyisacomplementoftherecentworkofM.GoubiconcerningspecialcaseV$(1/q)$.

Key–Words: Vasyunin-cotangent sum, Generating functions, Cauchy product, Double euclidian algorithm.

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1 Introduction

Throughout this work, $p < q$ two positive coprime integers, $\bar{p}$ the inverse of p modulo q ($\bar{p} < q,\ p\bar{p} \equiv 1(q)$) and $\bar{q}$ the inverse of q modulo p. The well-known Vasyunin cotangent sum is defined by

$$V(p/q) = \sum_{k=1}^{q-1} \left\{\frac{pk}{q}\right\} cot\frac{\pi k}{q}, \tag{1}$$

where $\{t\} = t - \lfloor t \rfloor$ is the fractional part of t. These sums admit a natural generalization to Dedekind-Vasyunin sums [1]:

$$V_a\left(\frac{p}{q}\right) = \frac{q^a}{a+1}\sum_{k=1}^{q-1} B_{a+1}\left(\left\{\frac{kp}{q}\right\}\right)\cot\left(\frac{\pi k}{q}\right),$$

where a a positive integer and $B_a(\{t\})$ the normalized Bernoulli polynomials. We observe that $V\left(\frac{p}{q}\right) = V_0\left(\frac{p}{q}\right)$ and $V_a\left(\frac{p}{q}\right)$ is the opposite of the cotangent sum investigated by Bettin-Conrey [2]:

$$C_a\left(\frac{\bar{p}}{q}\right) = -q^a\sum_{k=1}^{q-1}\cot\left(\frac{\pi k\bar{p}}{q}\right)\zeta\left(-a, \frac{k}{q}\right);$$

where $\zeta(s,t)$ is the Hurwitz zeta function [3] given by the series

$$\zeta(s,t) = \sum_{k\geq 0}\frac{1}{(k+t)^s}.$$

To see this relation, we can write

$$C_a\left(\frac{\bar{p}}{q}\right) = -\frac{q^a}{a+1}\sum_{k=1}^{q-1} B_{a+1}\left(\left\{\frac{k}{q}\right\}\right)\cot\left(\frac{\pi k\bar{p}}{q}\right).$$

Letting the function $e_p(t) = \left\{\frac{t}{p}\right\};\ t \geq 0$ and the indicator function χ of the interval $[1,\ +\infty[$ given by

$$\chi(t) := \begin{cases} 1 & \text{if } t \in [1,\infty[, \\ 0 & \text{otherwise.}\end{cases}$$

These functions lie to Hilbert space $\mathcal{H} = L^2(0,\ +\infty; t^{-2}dt)$ (see [4]) with the inner product

$$\langle f, g\rangle = \int_0^{+\infty} f(t)g(t)t^{-2}dt;\ f, g \in \mathcal{H}.$$

The symmetric sum $S(p,q) = V(p/q) + V(q/p)$ arises in the study of the Riemann zeta function by virtue of the Vasyunin formula (see [5])

$$\begin{aligned}\langle e_p, e_q\rangle &= \frac{\log 2\pi - \gamma}{2}\left(\frac{1}{p}+\frac{1}{q}\right) \\ &+ \frac{p-q}{2pq}\log\frac{q}{p} - \frac{\pi}{2pq}S(p,q).\end{aligned} \tag{2}$$

The Baez-Duarte-Balazard criterion states that Riemann hypothesis is true if and only if $\lim_{n\longrightarrow+\infty} d_n = 0$, where d_n is the distance given by the expression

$$d_n^2 = \frac{Gram(\chi, e_1, e_2..., e_n)}{Gram(e_1, e_2..., e_n)}, \tag{3}$$

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where $Gram\left(v_1, v_2..., v_n\right)$ is the Gram determinant of the matrix $\left(\langle v_j, v_k\rangle\right)_{1\leq p,q\leq n}$, and we have

$$Gram\left(v_1, v_2, \cdots v_n\right) = \left|\left\langle v_j, v_k\right\rangle\right|_{1\leq j,k\leq n}.$$

For computing the distance d_n, we need two kinds of inner products. The first is given by the expression (see [4]):

$$\langle\chi, e_p\rangle = \frac{\log p + 1 - \gamma}{p},$$

where $\gamma = 0,5772\cdots$ is the Euler constant. The second is given by the formula (2). In this work we give series expansions of the sums $V\left(p/q\right)$ and $S\left(p,q\right)$ to deduce a new reformulation of Vasyunin formula. The method used is different from the old ones, it is based on the theory of generating functions and the use of arithmetic arguments like double Euclidean algorithm and congruences. The sum $V\left(p/q\right)$ is related to cotangent sum

$$c_0\left(p/q\right) = -\sum_{k=1}^{q-1}\frac{k}{q}cot\frac{\pi kp}{q}, \tag{4}$$

by virtue of the relation (see [6, 7])

$$V\left(p/q\right) = -c_0\left(\bar{p}/q\right). \tag{5}$$

We recall that $c_0\left(p/q\right)$ is the value at $s = 0$ of Estermann zeta function (see [7])

$$E_0\left(0, p/q\right) = \frac{1}{4} + \frac{i}{2}c_0\left(p/q\right), \tag{6}$$

where

$$E_0\left(s, p/q\right) = \sum_{k\geq 1}\frac{d\left(k\right)}{k^s}exp\left(\frac{2\pi ikp}{q}\right); \tag{7}$$

with $d\left(k\right) = \sum_{d|k} d$ is the divisor function. Bettin-Conrey showed that $c_0\left(p/q\right)$ satisfies the following reciprocity formula

$$c_0\left(\frac{p}{q}\right) + \frac{q}{p}c_0\left(\frac{q}{p}\right) - \frac{1}{\pi p} = \frac{i}{2}\psi_0\left(\frac{p}{q}\right), \tag{8}$$

for a chosen function ψ_0 and for further information about this sum, we can refer to [2, 8, 9, 10, 11]. The analogous relation of (8) for the Vasyunin cotangent sum is

$$V\left(\frac{\bar{p}}{q}\right) + \frac{q}{p}V\left(\frac{\bar{q}}{p}\right) + \frac{1}{\pi p} = -\frac{i}{2}\psi_0\left(\frac{p}{q}\right).$$

The first term of this equality is different from $S\left(p,q\right)$; necessary for the Vasyunin formula. It is for this reason that we proceed differently in order to directly find the expression of $S\left(p,q\right)$. M. Th. Rassias investigated the sum $c_0\left(1/q\right)$ and provide that

$$c_0\left(1/q\right) = \frac{q}{\pi}\sum_{\substack{k\geq 1\\ k\neq iq}}\frac{1-2\left\{k/b\right\}}{k}, \tag{9}$$

which can be expressed in the equivalent form

$$c_0\left(1/q\right) = \frac{q}{\pi}\sum_{\substack{k\geq 1\\ k\neq iq}}\left[\frac{q}{k}\left(1+2\left\lfloor\frac{k}{q}\right\rfloor\right)-2\right]. \tag{10}$$

Recently the second author [12] of this paper investigated the sum $V\left(1/q\right)$ and provide another series expansion:

$$V\left(1/q\right) = \frac{q}{\pi}\sum_{k\geq 0}\frac{(q-1)(q-2)c_k}{\left(k+1\right)\left(k+2\right)\left(k+q\right)\left(k+q+1\right)},$$

where c_k is the integer sequence defined the recursive formulae

$$c_k - 2c_{k-1} + c_{k-2} = 0,\ 2\leq q\leq k-1,\ k = q+1,$$

$$c_q - 2c_{q-1} + c_{q-2} = 1$$

and for $k\geq q+2$;

$$c_k - 2c_{k-1} + c_{k-2} - c_{k-q} + 2c_{k-q-1} - c_{k-q-2} = 0,$$

with initial terms $c_0 = 1$ and $c_1 = 2$. Explicit formula of c_k is given in the work [13] by the relation

$$c_k = \left(k+1-\frac{q}{2}\left\lfloor\frac{k}{q}\right\rfloor\right)\left(\left\lfloor\frac{k}{q}\right\rfloor+1\right). \tag{11}$$

If k is a multiple of q, we have $\left\lfloor\frac{k}{q}\right\rfloor = \frac{k}{q}$ and then

$$c_k = \frac{1}{2q}\left(k+2\right)\left(k+q\right).$$

Furthermore we obtain

$$V\left(1/q\right) = \frac{1}{2\pi}\sum_{q|k}\frac{(q-1)(q-2)}{\left(k+1\right)\left(k+q+1\right)}$$
$$+\frac{q}{\pi}\sum_{k\neq iq}\frac{(q-1)(q-2)c_k}{\left(k+1\right)\left(k+2\right)\left(k+q\right)\left(k+q+1\right)}.$$

2 Series expansion of $V\left(p/q\right)$

Let us denoting $R_{q,p}(t)$ the rational function

$$R_{q,p}(t) = \frac{P_{q,p}(t)}{\left(1-t^q\right)\left(1-t^p\right)^2},$$

where $P_{q,p}(t)$ is another rational function which takes the following form

$$P_{q,p}(t) = (q-1)\,t^{qp+p-1} - qt^{qp-1} + t^{p-1} - t^{p+q-1} + \frac{qt^{2p+q-1} - (q-1)t^{p+q-1}}{t^{pq}}.$$

It is obvious that $P_{q,1}(t)$ and $t^{pq}P_{q,p}(t)$ are polynomials. Otherwise, we quote from the work [14] the following expression

$$V\,(p/q) = \frac{1}{\pi q}\sum_{r=1}^{q-1}\sum_{k\geq 0}\frac{r(1-2r\bar{p}/q)}{(k+1-r\bar{p}/q)(k+r\bar{p}/q)}. \quad (12)$$

According to identity (12) and the function $R_{q,p}(t)$; an integral representation of Vasyunin cotangent sum is given by following proposition

Proposition 1

$$V\,(p/q) = \frac{1}{\pi}\int_0^1 R_{q,\bar{p}}(t)dt. \quad (13)$$

Proof. From the identity (12) we deduce that

$$\begin{aligned} V\,(p/q) &= \frac{1}{\pi}\sum_{r=0}^{q-1}\sum_{k\geq 0} r\left(\frac{1}{qk+r\bar{p}} - \frac{1}{q\,(k+1)-r\bar{p}}\right) \\ &= \frac{1}{\pi}\sum_{r=0}^{q-1}\int_0^1 \sum_{k\geq 0} r\left(t^{qk+r\bar{p}-1} - t^{q(k+1)-r\bar{p}-1}\right)dt \\ &= \frac{1}{\pi}\int_0^1 \frac{\sum_{r=0}^{q-1} rt^{r\bar{p}-1} - t^q\sum_{r=0}^{q-1} rt^{-r\bar{p}-1}}{1-t^q}dt \\ &= \frac{1}{\pi}\int_0^1 \frac{t^{-1}\sum_{r=0}^{q-1} r\,(t^{\bar{p}})^r - t^{q-1}\sum_{r=0}^{q-1} r\,(t^{-\bar{p}})^r}{1-t^q}dt \end{aligned}$$

We quote from the work [14, p. 170] the expression

$$\sum_{r=0}^{q-1} rt^{r-1} = \frac{(q-1)\,t^q - qt^{q-1} + 1}{(1-t)^2},$$

which becomes

$$\sum_{r=0}^{q-1} rt^{r} = \frac{(q-1)\,t^{q+1} - qt^{q} + t}{(1-t)^2}.$$

Applying this identity for $t^{\bar{p}}$ and $t^{-\bar{p}}$, we will have

$$\sum_{r=0}^{q-1} r\,(t^{\bar{p}})^r = \frac{(q-1)\,t^{(q+1)\bar{p}} - qt^{q\bar{p}} + t^{\bar{p}}}{(1-t^{\bar{p}})^2}$$

and

$$\sum_{r=0}^{q-1} r\,(t^{-\bar{p}})^r = \frac{(q-1)\,t^{-(q-1)\bar{p}} - qt^{-(q-2)\bar{p}} + t^{\bar{p}}}{(1-t^{\bar{p}})^2}.$$

Letting

$$\sigma = t^{-1}\sum_{r=0}^{q-1} r\,(t^{\bar{p}})^r - t^{q-1}\sum_{r=0}^{q-1} r\,(t^{-\bar{p}})^r,$$

then

$$\begin{aligned} (1-t^{\bar{p}})^2\,\sigma &= (q-1)\,t^{q\bar{p}+\bar{p}-1} - qt^{q\bar{p}-1} + t^{\bar{p}-1} \\ &- (q-1)\,t^{\bar{p}-q\bar{p}+q-1} + qt^{2\bar{p}-q\bar{p}+q-1} \\ &- t^{\bar{p}+q-1} \end{aligned}$$

and the desired result follows.

Remark 2 *For $p \equiv 1\,(q)$, we have $\bar{p} = 1$ and $P_{q,1}(t) = (q-2)\,t^q - qt^{q-1} + qt - q + 2$. Furthermore the identity $(5,5)$ [14, Proposition 5.2] is immediate.*

In order to evaluate the series expansion of the Vasyunin cotangent sum V, we must investigate the generating function $f(t) = 1/\,(1-t^q)\,(1-t^{\bar{p}})^2$ and compute the sequence u_k such that $f(t) = \sum_{k\geq 0} u_k t^k$, we attract attention that $u_k = c_k$ if $p \equiv 1(q)$.

2.1 Double Euclidean algorithm and series expansion of f

The second author, introduced the notion of the double Euclidean algorithm over the pair (p,q) which extends the classical Euclidean algorithm.

Definition 3 *Let k be a positive integer, the double Euclidean algorithm of k over the pair (q,p) is defined by*

$$k = a_k q + b_k p + r_k; \quad (14)$$

with the conditions that $0 \leq b_k p + r_k < q$ and $0 \leq r_k < p$.

The Euclidean algorithm of k over q is written under the form $k = \lfloor k/q \rfloor\, q + r_k$. Similarly, we have

$$k = \lfloor k/q \rfloor\, q + \lfloor (k - \lfloor k/q \rfloor\, q)\,/p \rfloor\, p + r_k.$$

The proof is left as an exercises for the reader. for $p = 1$; we reproduce the Euclidean algorithm.

Example 1 *The double Euclidean algorithm of* 31 *over the pair* $(7,4)$ *is given by* $33 = 4\times 7 + 1\times 4 + 1$. *Then the remainder of this division is* 1.

This kind of division is important for computing the explicit formula of the sequence u_k generated by the function $f(t)$.

Proposition 4 *For every positive integer k we have $k = a_k q + b_k \bar{p} + r_k$ and then*

$$u_k = \begin{cases} \left(\left\lfloor \frac{a_k}{\bar{p}} \right\rfloor + 1\right)\left(b_k + 1 + \frac{q}{2}\left\lfloor \frac{a_k}{\bar{p}} \right\rfloor\right) \text{ if } r_k = 0, \\ 0 \text{ otherwise.} \end{cases}$$

proof. Letting $f_1(t) = \frac{1}{1-t^q}$ and $f_2(t) = \frac{1}{1-t^{\bar{p}}}$. Then $f(t) = f_1(t) f_2^2(t)$. But $f_1(t) = \sum_{q|k} t^k$ and $f_2(t) = \sum_{\bar{p}|k} t^k$. It follows from the Cauchy product of generating functions (see [15]), that $f_2^2(t) = \sum_{\bar{p}|k}\left(\frac{k}{\bar{p}} + 1\right) t^k$. According to expressions of f_1 and f_2^2, we deduce that

$$f_1(t) = \sum_{k \geq 0} \left\lfloor \frac{1}{k - a_k q + 1} \right\rfloor t^k$$

and

$$f_2^2(t) = \sum_{k \geq 0} \left(\frac{k}{\bar{p}} + 1\right) \left\lfloor \frac{1}{k - \left\lfloor \frac{k}{\bar{p}} \right\rfloor \bar{p} + 1} \right\rfloor t^k.$$

Then

$$u_k = \sum_{i=0}^{k} \left(\frac{k-i}{\bar{p}} + 1\right) \left\lfloor \frac{1}{i - \left\lfloor \frac{i}{q} \right\rfloor q + 1} \right\rfloor \times \left\lfloor \frac{1}{k - i - \left\lfloor \frac{k-i}{\bar{p}} \right\rfloor \bar{p} + 1} \right\rfloor.$$

After discussion, the above sum reduced to the sum over all $0 \leq i \leq k$ such that $q|i$ and $\bar{p}|k-i$. Then $u_k \neq 0$ if and only if $r_k = 0$. In this case we have

$$u_k = \sum_{i=0}^{k} \left(\frac{a_k q + b_k \bar{p} - i}{\bar{p}} + 1\right) \left\lfloor \frac{1}{i - \left\lfloor \frac{i}{q} \right\rfloor q + 1} \right\rfloor \times \left\lfloor \frac{1}{a_k q + b_k \bar{p} - i - \left\lfloor \frac{a_k q + b_k \bar{p} - i}{\bar{p}} \right\rfloor \bar{p} + 1} \right\rfloor.$$

Then

$$u_k = \sum_{\substack{i=0 \\ q|i}}^{k} \left(\frac{a_k q + b_k \bar{p} - i}{\bar{p}} + 1\right) \times \left\lfloor \frac{1}{a_k q + b_k \bar{p} - i - \left\lfloor \frac{a_k q + b_k \bar{p} - i}{\bar{p}} \right\rfloor \bar{p} + 1} \right\rfloor.$$

Letting $i = qj$, thus

$$u_k = \sum_{j \leq a_k} \left(\frac{(a_k - j)q + b_k \bar{p}}{\bar{p}} + 1\right) \times \left\lfloor \frac{1}{(a_k - j)q + b_k \bar{p} - \left\lfloor \frac{(a_k - j)q + b_k \bar{p}}{\bar{p}} \right\rfloor \bar{p} + 1} \right\rfloor$$

Thereafter (by taking $i = a_k - j$)

$$u_k = \sum_{i \leq a_k} \left(\frac{iq + b_k \bar{p}}{\bar{p}} + 1\right) \times \left\lfloor \frac{1}{iq + b_k \bar{p} - \left\lfloor \frac{iq + b_k \bar{p}}{\bar{p}} \right\rfloor \bar{p} + 1} \right\rfloor.$$

But the terms corresponding to i not multiples of $\bar{p}$ are zero, then

$$u_k = \sum_{\substack{i \leq a_k \\ \bar{p}|i}} \left(\frac{iq + b_k \bar{p}}{\bar{p}} + 1\right) \times \left\lfloor \frac{1}{iq + \left(b_k - \left\lfloor \frac{iq + b_k \bar{p}}{\bar{p}} \right\rfloor\right) \bar{p} + 1} \right\rfloor$$

and for $k = a_k q + b_k \bar{p}$ we have

$$u_k = \sum_{i=0}^{\left\lfloor \frac{a_k}{\bar{p}} \right\rfloor} (iq + b_k + 1).$$

Furthermore

$$u_k = (b_k + 1)\left(\left\lfloor \frac{a_k}{\bar{p}} \right\rfloor + 1\right) + \frac{q}{2}\left(\left\lfloor \frac{a_k}{\bar{p}} \right\rfloor + 1\right)\left\lfloor \frac{a_k}{\bar{p}} \right\rfloor,$$

and the desired result follows.

It is well-known that (see [3])

$$\left\lfloor \frac{1}{n} \right\rfloor = \sum_{d|n} \mu(d), \tag{15}$$

where μ is the möbius function defined as follows: $\mu(1) = 1$; if $n = p_1^{m_1} \cdots p_k^{m_k}$. Then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } a_1 = a_2 = \cdots = a_k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Letting $N_i = iq + \left(b_k - \left\lfloor \frac{iq + b_k \bar{p}}{\bar{p}} \right\rfloor\right) \bar{p} + 1$ and regarding the proof of Proposition 4. The following corollary is immediate

Corollary 5 *According to Möbius function we have*

$$u_k = \sum_{\substack{i \leq a_k \\ \bar{p}|i}} \sum_{d|N_i} \left(\frac{iq + b_k \bar{p}}{\bar{p}} + 1\right) \mu(d). \tag{16}$$

The extended binomial coefficient to a complex number α is given by

$$\binom{\alpha}{k} = \frac{(\alpha)_k}{k!}, \tag{17}$$

where $(\alpha)_k = (\alpha)(\alpha - 1)(\alpha - 2) \ldots (\alpha - k + 1)$ is a falling number. A second expression of u_k is given by the following proposition

Proposition 6

$$u_k = \sum_{j,q|k+j(q-\bar{p})} (j+1) \tag{18}$$

Proof. It is well known that

$$(1+t)^{\alpha} = \sum_{k\geq 0} \binom{\alpha}{k} t^k, \ \alpha \in \mathcal{C}, \ |t| < 1.$$

According to this identity we deduce that

$$(1-t^q)^{-1} = \sum_{k\geq 0} t^{qk}$$

and

$$(1-t^{\bar{p}})^{-2} = \sum_{k\geq 0} \binom{-2}{k} (-1)^k t^{\bar{p}k} = \sum_{k\geq 0} (k+1)\, t^{\bar{p}k}.$$

The Cauchy product of $(1-t^q)^{-1}$ and $(1-t^{\bar{p}})^{-2}$ conducts to

$$f(t) = \sum_{i\geq 0} \sum_{j=0}^{i} (j+1)\, t^{j\bar{p}+(i-j)q},$$

putting $k = j\bar{p} + (i-j)\, q$ and the desired follows.

According to identity (18) and the proposition 4, the following corollary is immediate

Corollary 7 *For $r_k = 0$ we have*

$$\sum_{j,q|k-j\bar{p}} (j+1) = \left(\left\lfloor \frac{a_k}{\bar{p}} \right\rfloor + 1 \right) \left(b_k + 1 + \frac{q}{2} \left\lfloor \frac{a_k}{\bar{p}} \right\rfloor \right). \tag{19}$$

Otherwise if $r_k \neq 0$, then

$$k - j\bar{p} \equiv 0(q) \tag{20}$$

has no solutions in the set of integers.

Letting $\alpha(q,p) = (2pq-q-p)(q-2pq+2p)$ and $\beta(q,p) = pq^2 - q^2$. According to Proposition 4, a series expansion of $V(p/q)$ is given by the following theorem

Theorem 8 *For every positive integer k, letting $k = a_k q + b_k \bar{p} + r_k$, the double euclidian division of k over $(q, \bar{p})$, then we have*

$$V(p/q) = \frac{q}{\pi} \sum_{k\equiv 0(q,\bar{p})} \left(\left\lfloor \frac{a_k}{\bar{p}} \right\rfloor + 1 \right) \times \left(b_k + 1 + \frac{q}{2} \left\lfloor \frac{a_k}{\bar{p}} \right\rfloor \right) M_{q,\bar{p}}(k), \tag{21}$$

with

$$N_{q,\bar{p}}(k) M_{q,\bar{p}}(k) = \alpha(q,\bar{p})(k+\bar{p})(k+\bar{p}+q) - \beta(q,\bar{p})(k+q-\bar{p}q+2\bar{p})(k+\bar{p}q).$$

and

$$N_{q,\bar{p}}(k) = (k+\bar{p}-\bar{p}q+q)(k+q-\bar{p}q+2\bar{p})(k+\bar{p}q) \times (k+\bar{p})(k+\bar{p}+q)(k+\bar{p}q+\bar{p})$$

Proof. From the integral representation and the expression of u_k given in Theorem 1, we deduce that

$$\begin{aligned} V(p/q) &= \frac{1}{\pi} \int_0^1 P_{q,\bar{p}}(t) \left(\sum_{r_k=0} u_k t^k \right) dt \\ &= \frac{1}{\pi} \sum_{r_k=0} u_k \int_0^1 P_{q,\bar{p}}(t) t^k dt. \end{aligned}$$

Using the expression of $P_{q,\bar{p}}(t)$ we deduce that

$$\int_0^1 P_{q,\bar{p}}(t) t^k dt = (q-1) \left(\frac{1}{k+\bar{p}q+\bar{p}} - \frac{1}{k+q-\bar{p}q+\bar{p}} \right) + q \left(\frac{1}{k+q-\bar{p}q+2\bar{p}} - \frac{1}{k+\bar{p}q} \right) + \frac{1}{k+\bar{p}} - \frac{1}{k+q+\bar{p}}.$$

If we consider the well-known identity:

$$\frac{1}{b} - \frac{1}{b+c} = \frac{c}{b(b+c)}$$

and carry out the calculations, we can express the above integral in the equivalent form

$$\int_0^1 P_{q,\bar{p}}(t) t^k dt = A + B,$$

with

$$A = \frac{\bar{p}q(q-2\bar{p}q+2\bar{p})(2\bar{p}q-q-2\bar{p})}{\ell_k(\bar{p},q)},$$

$$\ell_k(\bar{p},q) = (k+\bar{p}q+\bar{p})(k+\bar{p}-\bar{p}q+q) \times (k+q-\bar{p}q+2\bar{p})(k+\bar{p}q)$$

and

$$B = \frac{\bar{p}q(\bar{p}q^2 - q^2)}{(k+\bar{p})(k+\bar{p}+q)(k+\bar{p}q+\bar{p})(k+\bar{p}-\bar{p}q+q)}$$

furthermore

$$\int_0^1 P_{q,\bar{p}}(t) t^k dt = M_{q,\bar{p}}(k)$$

and the desired result follows.

3 Series expansion of $S(p,q)$

Using directly the integral and series representations of $V(p/q)$ and $V(q/p)$ for computing similar formulae for $S(p,q)$ is far from the good estimation. For example, from the integral representations of $V(p/q)$ and $V(q/p)$, we obtain

$$S(p,q)=\frac{1}{\pi}\int_0^1\left(Q_{q,\bar{p}}(t)+Q_{p,\bar{q}}(t)\right)dt.$$

Thank's to symmetric function G introduced by Goubi and al.[14]:

$$G(p,q)=\sum_{k\geq 1}\frac{pq}{k(k+1)}\left\{\frac{k}{p}\right\}\left\{\frac{k}{q}\right\}, \tag{22}$$

and hint of the relation (see [1])

$$\begin{aligned}\pi S(p,q) &= G(p,p)+G(q,q)-2G(p,q)\\ &+ (q-p)\log\frac{p}{q},\end{aligned}\tag{23}$$

we give another integral and series expansion of $S(p,q)$. For simplifying calculus, we consider the following axillary function $\theta_{p,q}$ given by

$$\begin{aligned}\theta_{p,q}(r)=&\left(p\left\{\frac{r}{p}\right\}-q\left\{\frac{r}{q}\right\}\right)\\ \times&\left(p\left\{\frac{r}{p}\right\}-q\left\{\frac{r}{q}\right\}+q-p\right).\end{aligned}\tag{24}$$

We consider respectively $\dot{r}$ and $\ddot{r}$ the rest of Euclidean algorithm of r over p and q, then

$$\theta_{p,q}(r)=(\dot{r}-\ddot{r})(\dot{r}-\ddot{r}+q-p).$$

In the work [16] the authors provide that the series expansion of $\frac{\log p}{p}$ is

$$\frac{\log p}{p}=\sum_{k\geq 1}\frac{1}{k(k+1)}\left\{\frac{k}{p}\right\}. \tag{25}$$

Substituting the expression (24) and (25) in the identity (23), we deduce that

$$\pi S(p,q)=\sum_{k\geq 1}\frac{\theta_{p,q}(k)}{k(k+1)}.$$

We have already proved the following theorem

Theorem 9 *The series expansion of $S(p,q)$ is given by the following equivalent identities*

$$S(p,q)=\frac{1}{\pi}\sum_{k\geq 1}\frac{\left(\dot{k}-\ddot{k}\right)\left(\dot{k}-\ddot{k}+q-p\right)}{k(k+1)}, \tag{26}$$

and

$$S(p,q)=\frac{1}{\pi}\sum_{k\equiv 0(pq)}\sum_{r=p}^{pq-1}\frac{(\dot{r}-\ddot{r})(\dot{r}-\ddot{r}+q-p)}{(k+r)(k+r+1)}. \tag{27}$$

Consequently, the integral representation of $S(p,q)$ is given by the following corollary

Corollary 10

$$S(p,q)=\frac{1}{\pi}\int_0^1\frac{\sum_{r=p}^{pq-1}(\dot{r}-\ddot{r})(\dot{r}-\ddot{r}+q-p)(1-t)t^{r-1}}{1-t^{pq}}dt.$$

proof Since we have

$$\frac{1}{(k+r)(k+r+1)}=\int_0^1\left(t^{k+r-1}-t^{k+r}\right)dt,$$

Letting $\sum=\sum_{k\equiv 0(pq)}\frac{1}{(k+r)(k+r+1)}$ then

$$\begin{aligned}\sum &= \int_0^1\left(t^{r-1}-t^r\right)\sum_{i\geq 0}t^{ipq}dt\\ &= \int_0^1\frac{(t^{r-1}-t^r)}{1-t^{pq}}dt.\end{aligned}$$

According to identity (27) Theorem 9, the result (28) holds true.

Under the expression of $S(p,q)$, another reformulation of Vasyunin formula is given by the following expressions

$$\begin{aligned}\langle e_p,e_q\rangle &= \frac{\log 2\pi-\gamma}{2}\left(\frac{1}{p}+\frac{1}{q}\right)+\frac{p-q}{2pq}\log\frac{q}{p}\\ &- \frac{1}{2pq}\sum_{k\equiv 0(pq)}\sum_{r=p}^{pq-1}\frac{(\dot{r}-\ddot{r})(\dot{r}-\ddot{r}+q-p)}{(k+r)(k+r+1)},\end{aligned}$$

and

$$\begin{aligned}\langle e_p,e_q\rangle &= \frac{\log 2\pi-\gamma}{2}\left(\frac{1}{p}+\frac{1}{q}\right)+\frac{p-q}{2pq}\log\frac{q}{p}\\ &- \frac{1}{2pq}\sum_{k\geq 1}\frac{\left(\dot{k}-\ddot{k}\right)\left(\dot{k}-\ddot{k}+q-p\right)}{k(k+1)}.\end{aligned}$$

4 Conclusion

This work deals with the Vasyunin cotangent sum $V_0\left(\frac{p}{q}\right)$ related to Riemann hypothesis via the Baez-Duarte-Balazard criterion. Since $V\left(\frac{p}{q}\right)$ is opposite to $c_0\left(\frac{\overline{p}}{q}\right)$, we can translate the results found on $c_0\left(\frac{p}{q}\right)$

to $V\left(\frac{p}{q}\right)$. Better still, we used the same techniques that M. Goubi gave for the study of $V_0\left(\frac{1}{q}\right)$ in order to find series expansion of $V_0\left(\frac{p}{q}\right)$ and the symmetric sum $S(p,q)$. These results led to another reformulation of Vasyunin formula. This study is the fruit of the use of generating functions and the introduction of Double Euclidean algorithm. As a future directions for this work; we propose to evaluate the series expansion of $V_a\left(\frac{p}{q}\right)$, $S_a\left(p,q\right) = V_a\left(\frac{p}{q}\right) + V_a\left(\frac{q}{p}\right)$ and study their arithmetic properties. The results obtained can be used to understand the arithmetic behavior of the zeta function and the nature of its zeros.

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