Existence and Global Behavior of Decaying Solutions of a Nonlinear Elliptic Equation

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Abstract: - This paper is devoted to study the following radial equation

$$(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + \alpha|u|^{q-1}u + \beta r(|u|^{q-1}u)' = 0, \quad r > 0.$$

where $p > 2$, $q > 1$, $N \geq 1$, $\alpha > 0$ and $\beta > 0$.
Our purpose is to give existence results of decaying solutions of the above equation and their asymptotic behavior near infinity. The study depends strongly of the sign of $N\beta - \alpha$ and the comparison between $\frac{\alpha}{q\beta}$, $\frac{p}{q+1-p}$ and $\frac{N-p}{p-1}$. More precisely, we prove that if $N\beta - \alpha > 0$, there is a positive solution u which has one of the following behaviors near infinity:

(i) $u(r) \underset{+\infty}{\sim} Lr^{-\frac{\alpha}{q\beta}}$, where $L > 0$.

$$(ii)\; u(r) \underset{+\infty}{\sim} \left((\frac{p-1}{q\beta})(q+1-p)(\frac{N-p}{p-1} - \frac{\alpha}{q\beta}) \left(\frac{\alpha}{q\beta}\right)^{p-1} \right)^{\frac{1}{q+1-p}} r^{-\frac{\alpha}{q\beta}} (\ln r)^{\frac{1}{q+1-p}}.$$

$$(iii)\; u(r) \underset{+\infty}{\sim} \left(\frac{(p-1)\left(\frac{p}{q+1-p}\right)^{p-1}\left(\frac{N-p}{p-1} - \frac{p}{q+1-p}\right)}{\alpha - q\beta\frac{p}{q+1-p}} \right)^{\frac{1}{q+1-p}} r^{\frac{-p}{q+1-p}}.$$

Key-Words: - Porous medium equation; fast diffusion equation; radial self-similar solutions; shooting method; decaying solutions; energy function.

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1 Introduction and Main Results

The aim of this paper is to investigate the structure of positive radial solutions to

$$\Delta_p u + \alpha|u|^{q-1}u + \beta x.\nabla(|u|^{q-1}u) = 0, \quad x \in \mathbb{R}^N, \tag{1}$$

where $p > 2$, $q > 1$, $N \geq 1$, $\alpha > 0$ and $\beta > 0$.
As usual $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the $p-laplacien$ operator.
The idea of this work comes from the study of radial self-similar solutions to the following parabolic equation

$$v_t = \Delta_p v^m \qquad in \quad \mathbb{R}^N \times (0, +\infty), \tag{2}$$

where $p > 2$ and $0 < m < 1$.
When $p = 2$, this equation becomes the porous medium equation, it appears in many physical models and it has been treated extensively in the literature, see [25] and [26]. When $m > 1$, it is the slow diffusion equation, when $0 < m < 1$ it is the fast diffusion equation. When $m = 1$, equation (2) reduce to the heat equation. See for example works of [1], [2], [3], [4], [7], [8], [9], [14], [15] and [21].
The radial self-similar solution to the parabolic equation (2) are the form

$$v(x,t) = t^{-\alpha}W(t^{-\beta}|x|). \tag{3}$$

Where

$$\alpha = \frac{\beta p - 1}{1 - m(p-1)} \tag{4}$$

and W is a radial positive solution which satisfies

$$\Delta_p W^m(r) + \alpha W(r) + \beta r W'(r) = 0, \qquad r > 0. \tag{5}$$

For simplicity in the notation, we set $u = W^m$ and $q = \dfrac{1}{m}$ in (5), we obtain

$$\Delta_p u + \alpha u^q + \beta r(u^q)' = 0, \qquad r > 0. \tag{6}$$

The question of the existence of a self-similar of equation (2) arises. We will prove that (2) admits a radial positive self-similar solution v if $\dfrac{\alpha}{\beta} < N$ and $\dfrac{\alpha}{q\beta} < \dfrac{p}{q+1-p}$.
To obtain this result, we carry out a careful analysis of radial solutions of equation (1). Many authors have studied equation (1). If $p = 2$, $\alpha = 1$ and $\beta = 0$, the first study is due to Emden-Fowler, see for example [10], [11] and [12]). He proved the existence results and give a classification of entire radial solutions. In the case $p = 2$, $\alpha > 0$ and $\beta > 0$, equation (1) was studied by [18], [19], and [20]. When $p > 2$, $\alpha = 1$ and $\beta = 0$, the first results are due to Ni and Serrin [23]. Guedda and Veron [16] studied the existence of entire solutions in radial case. The non radial case was investigated by Bidaut- Veron and Pohozaev [5]. When $p > 1$, $\alpha > 0$ and $\beta = 1$, equation (1) was studied by [22]. In the present work, we are interested in radial solutions of equation (1), we will study the following initial value problem.
Problem (P): Find a function u defined on $[0, +\infty[$ such that $|u'|^{p-2}u'$ is in $C^1([0, +\infty[)$ and

$$(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + \alpha|u|^{q-1}u + \beta r(|u|^{q-1}u)' = 0, \tag{7}$$

$$u(0) = a > 0, \quad u'(0) = 0. \tag{8}$$

By reducing the problem (P) to a fixed point for a suitable integral operator see (for example [6]), we prove that for each $a > 0$, the problem (P) has a unique global solution $u(., a, \alpha, \beta)$.
We focus our study to the case $N\beta - \alpha \geq 0$. If $N\beta - \alpha = 0$ and $q \geq p-1$, we find explicit solution of problem (P)

$$u(r,a) = \begin{cases} a e^{\frac{-(p-1)}{p}\beta^{\frac{1}{p-1}} r^{\frac{p}{p-1}}} & \text{if } q = p-1 \\ \left(a^{\frac{p-1-q}{p-1}} + \frac{q+1-p}{p}\beta^{\frac{1}{p-1}} r^{\frac{p}{p-1}}\right)^{\frac{-(p-1)}{q+1-p}} & \text{if } q > p-1. \end{cases}$$

If $N\beta - \alpha = 0$ and $q < p-1$, the solution $u(r,a)$ has compact support.
If $N\beta - \alpha > 0$, we prove that $u(r,a)$ is a decaying solution for each $a > 0$, i.e it is strictly positive and strictly decreasing on $(0, +\infty)$.
We are interested also to give asymptotic behavior of decaying solutions of problem (P). For this purpose, let us represent equation (7) as an equivalent form.
For any real c, we set

$$v_c(t) = r^c u(r) \ \ where \ \ r > 0 \ \ and \ \ t = \ln(r). \tag{9}$$

Then, v_c satisfies

$$w_c'(t) + A_c w_c(t) + \alpha e^{K_c t}|v_c|^{q-1}v_c(t) + q\beta e^{K_c t}|v_c|^{q-1}h_c(t) = 0 \tag{10}$$

where

$$w_c(t) = |h_c|^{p-2}h_c(t), \tag{11}$$

$$h_c(t) = v_c'(t) - c v_c(t) = r^{c+1}u'(r), \tag{12}$$

$$A_c = N - p - c(p-1) \text{ and } K_c = c(p-1-q) + p. \tag{13}$$

We remark that three critical values of the parameter c will be involved, $\dfrac{\alpha}{q\beta}$, $\dfrac{N-p}{p-1}$ and $\dfrac{p}{q+1-p}$.
These values play an important role in the study of asymptotic behavior of positive solution of problem (P). The main results are the following.

Theorem 1.1. *Let $a > 0$. Then problem (P) has a unique global solution $u(.,a,\alpha,\beta)$. Moreover,*

$$(|u'|^{p-2}u')'(0) = \frac{-\alpha a^q}{N}.$$

Theorem 1.2. *Assume $\frac{\alpha}{\beta} < N$. Let u be a solution of problem (P). Then, u is a decaying solution and has one of the following asymptotic behaviors.*

(i) *If* $\frac{\alpha}{q\beta} < \frac{p}{q+1-p}$,

$$\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}} u(r) = L_1 > 0$$

and

$$\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}+1} u'(r) = \frac{-\alpha}{q\beta} L_1 < 0.$$

(ii) *If* $\frac{\alpha}{q\beta} = \frac{p}{q+1-p}$,

$$\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}} u(r)(\ln(r))^{\frac{-1}{q+1-p}} =$$

$$\left((\frac{p-1}{q\beta})(q+1-p)\left(\frac{N-p}{p-1} - \frac{\alpha}{q\beta}\right)\left(\frac{\alpha}{q\beta}\right)^{p-1}\right)^{\frac{1}{q+1-p}}$$

(iii) *If* $\frac{\alpha}{q\beta} > \frac{p}{q+1-p}$,

$$\lim_{r\to+\infty} r^{\frac{p}{q+1-p}} u(r) =$$

$$\left(\frac{(p-1)\left(\frac{p}{q+1-p}\right)^{p-1}\left(\frac{N-p}{p-1} - \frac{p}{q+1-p}\right)}{\alpha - q\beta\frac{p}{q+1-p}}\right)^{\frac{1}{q+1-p}}$$

and

$$\lim_{r\to+\infty} r^{\frac{p}{q+1-p}+1} u'(r) =$$

$$\frac{-p}{q+1-p}\left(\frac{(p-1)\left(\frac{p}{q+1-p}\right)^{p-1}\left(\frac{N-p}{p-1} - \frac{p}{q+1-p}\right)}{\alpha - q\beta\frac{p}{q+1-p}}\right)^{\frac{1}{q+1-p}}$$

Now, we consider the problem

$$(Q)\begin{cases} v_t = \Delta_p v^m & in \quad \mathbb{R}^N \times (0,+\infty) \\ v(0,1) = b \end{cases}$$

where $p > 2,\ N \geq 1,\ 0 < m < \frac{1}{p-1}$ and $b > 0$.

Theorem 1.3. *Assume $0 < \frac{\alpha}{\beta} < N$ and $\frac{\alpha}{q\beta} < \frac{p}{q+1-p}$. Then for every $b > 0$, problem (Q) admits a radial strictly positive self-similar solution $U_b(x,t) = t^{-\alpha} u^{\frac{1}{m}}(t^{-\beta}|x|)$, where $\alpha = \frac{\beta p - 1}{1-m(p-1)}$ and u is solution of problem (P). Moreover, there exists $L(b) > 0$ such that*

$$\lim_{t\to 0^+} U_b(x,t) = L(b)|x|^{\frac{-\alpha}{\beta}} \quad \textit{for each } x \neq 0.$$

The paper is organized as follows. Section 2 is devoted to existence and uniqueness of global solutions of problem (P), more precisely we give the proof of Theorem 1.1. In section 3, we present fundamental properties of solution u of problem (P) and we study also the monotonicity and behavior of $r^c u(r)$ where c is a positive constant that we compare with the values $\frac{\alpha}{q\beta}$, $\frac{N-p}{p-1}$ and $\frac{p}{q+1-p}$. In section 4 we prove existence of decaying solutions of problem (P) and we describe their asymptotic behavior as $r \to +\infty$ in the three cases, $\frac{\alpha}{q\beta} < \frac{p}{q+1-p}$, $\frac{\alpha}{q\beta} = \frac{p}{q+1-p}$ and $\frac{\alpha}{q\beta} > \frac{p}{q+1-p}$. The obtained results prove the Theorem 1.2. Finally, in section 5 we give the proof of Theorem 1.3 by applying the obtained results in the previous sections related to the parabolic equation (2).

2 Existence of Global Solutions

In this section, we establish the existence of global solutions of problem (P).

Theorem 2.1. *Let $a > 0$. Then problem (P) has a unique global solution $u(.,a,\alpha,\beta)$. Moreover,*

$$(|u'|^{p-2}u')'(0) = \frac{-\alpha a^q}{N}. \tag{14}$$

Proof. The proof will be done in three steps.
Step 1: Existence of a local solution.
Multiply equation (7) by r^{N-1}, we obtain

$$(r^{N-1}|u'|^{p-2}u' + \beta r^N|u|^{q-1}u)' = (\beta N - \alpha)r^{N-1}|u|^{q-1}u. \tag{15}$$

Integrating (15) twice from 0 to r and taking into account (8), we see that problem (P) is equivalent to the equation

$$u(r) = a - \int_0^r G(F[u](s))\, ds, \tag{16}$$

where

$$G(s) = |s|^{(2-p)/(p-1)}s, \qquad s \in \mathbb{R} \tag{17}$$

and the nonlinear mapping F is given by the formula

$$F[u](s) = \beta s|u|^{q-1}u(s) + (\alpha - \beta N)s^{1-N}\int_0^s \sigma^{N-1}|u|^{q-1}u(\sigma)\,d\sigma. \tag{18}$$

Now we consider for $a > M > 0$, the complete metric space

$$E_{a,M,R} = \{\varphi \in C([0,R]) : ||\varphi - a||_0 \leq M\}. \tag{19}$$

Next we define the mapping Ψ on $E_{a,M,R}$ by

$$\Psi[\varphi](r) = a - \int_0^r G(F[\varphi](s))\,ds. \tag{20}$$

Claim 1. Ψ maps $E_{a,M,R}$ into itself for some small M and $R > 0$.
Obviously $\Psi[\varphi] \in C([0,R])$. From the definition of the space $E_{a,M,R}$, $\varphi(r) \in [a-M, a+M]$, for any $r \in [0,R]$. Simple calculations show that for small M, $F[\varphi]$ has a constant sign in $[0,R]$ for every $\varphi \in E_{a,M,R}$. More precisely,

$$F[\varphi](s) \geq Ks \qquad \text{for all } \ s \in [0,R], \tag{21}$$

where $K = \dfrac{\alpha}{2N}a^q$.

Taking into account that the function $r \to \dfrac{G(r)}{r}$ is decreasing on $(0,+\infty)$, we have

$$|\Psi[\varphi](r) - a| \leq \int_0^r \frac{G(F[\varphi](s))}{F[\varphi](s)}|F[\varphi](s)|\,ds$$
$$\int_0^r \frac{G(Ks)}{Ks}|F[\varphi](s)|\,ds$$

for $r \in [0,R]$. On the other hand,

$$|F[\varphi](s)| \leq Cs, \text{ where } C = \left[\beta + \left|\frac{\alpha}{N} - \beta\right|\right](a+M)^q.$$

We thus get

$$|\Psi[\varphi](r) - a| \leq \frac{p-1}{p}CK^{\frac{2-p}{p-1}}r^{\frac{p}{p-1}}$$

for every $r \in [0,R]$. Choose R small enough such that

$$|\Psi[\varphi](r) - a| \leq M, \quad \varphi \in E_{a,M,R}.$$

And thereby $\Psi[\varphi] \in E_{a,M,R}$. The claim is thus proved.

Claim 2. Ψ is a contraction in some interval $[0, r_a]$.
According to Claim 1, if r_a is a small enough, the space E_{a,M,r_a} applies into itself. For such r_a and any $\varphi, \psi \in E_{a,M,r_a}$ we have

$$|\Psi[\varphi](r) - \Psi[\psi](r)| \leq \int_0^r |G(F[\varphi](s)) - G(F[\psi](s))|\,ds \tag{22}$$

where $F[\varphi]$ is given by (18). Next, let

$$\Phi(s) = min(F[\varphi](s), F[\psi](s)).$$

As a consequence of estimate (21), we have

$$\Phi(s) \geq Ks \qquad \text{for } \ 0 \leq s \leq r < r_a$$

and then

$$\begin{aligned} |G(F[\varphi](s)) - G(F[\psi](s))| &\leq \frac{G(\Phi(s))}{\Phi(s)}|F[\varphi](s) - F[\psi](s)| \\ &\leq \frac{G(Ks)}{Ks}|F[\varphi](s) - F[\psi](s)|. \end{aligned} \tag{23}$$

Moreover,

$$|F[\varphi](s) - F[\psi](s)| \leq C'||\varphi - \psi||_0\, s, \tag{24}$$

where

$$C' = q\left[\beta + |\frac{\alpha}{N} - \beta|\right](a+M)^{q-1}.$$

Combining (22), (23) and (24), we have

$$|\Psi[\varphi](s) - \Psi[\psi](s)| \leq \frac{p-1}{p}C'K^{\frac{2-p}{p-1}}r^{\frac{p}{p-1}}||\varphi - \psi||_0 \tag{25}$$

for any $r \in [0, r_a]$. Choosing r_a small enough, Ψ, is a contraction. This proves the claim.
The Banach Fixed Point Theorem then implies the existence of a unique fixed point of Ψ in E_{a,M,r_A}, which is a solution of (16) and, consequently, of problem (P). As usual, this solution can be extended to a maximal interval $[0, r_{max}[$, $0 < r_{max} \leq +\infty$.
Step 2: Existence of a global solution.
We define the following energy function

$$E(r) = \frac{p-1}{p}|u'|^p + \frac{\alpha}{q+1}|u|^{q+1}(r). \tag{26}$$

According to equation (7), we get

$$E'(r) = -ru'^2\left[\frac{N-1}{r^2}|u'|^{p-2} + q\beta|u|^{q-1}(r)\right]. \tag{27}$$

Since $N \geq 1$ and $\beta > 0$ then E is decreasing, hence it is bounded. Consequently, u and u' are also bounded and the local solution constructed above can be extended to $\mathbb{R}^+$.

Step 3: $(|u'|^{p-2}u')'(0) = \dfrac{-\alpha a^q}{N}$.

Integrating (15) between 0 and r, we get

$$\frac{|u'|^{p-2}u'}{r} = -\beta|u|^{q-1}u(r) + (\beta N-\alpha)r^{-N}\int_0^r s^{N-1}|u|^{q-1}u(s)\,ds.$$

Hence using L'Hopital's rule and letting $r \to 0$, we obtain the desired result. The proof of Theorem is complete. □

3 Fundamental Properties

Proposition 3.1. *Assume $N > 1$. Let u be a solution of problem (P). Then,*

$$\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} u'(r) = 0. \tag{28}$$

Proof. We show that $\lim_{r\to+\infty} E(r) = 0$. Since $E'(r) \leq 0$ and $E(r) \geq 0$ for all $r > 0$, there exists a constant $l \geq 0$ such that $\lim_{r\to+\infty} E(r) = l \geq 0$. Suppose $l > 0$. Then, there exists $r_1 > 0$, such that

$$E(r) \geq \frac{l}{2} \qquad for \quad r \geq r_1. \tag{29}$$

Now consider the function

$$D(r) = E(r) + \frac{N-1}{2r}|u'|^{p-2}u'(r)u(r) + \frac{q\beta(N-1)}{2(q+1)}|u|^{q+1}(r). \tag{30}$$

Then

$$D'(r) = -q\beta r|u|^{q-1}(r)u'^2 - \frac{N-1}{2r}\left[|u'|^p + \frac{N}{r}|u'|^{p-2}u'u + \alpha|u|^{q+1}(r)\right]. \tag{31}$$

Since $\beta > 0$, we have

$$D'(r) \leq -\frac{N-1}{2r}\left[|u'|^p + \alpha|u|^{q+1}(r) + \frac{N}{r}|u'|^{p-2}u'u\right].$$

Recalling that u and u' are bounded (because E is bounded), we have

$$\lim_{r\to+\infty} \frac{|u'|^{p-2}u'u(r)}{r} = 0.$$

Moreover, by (26) and (29) we have

$$|u'|^p + \alpha|u|^{q+1}(r) \geq E(r) \geq \frac{l}{2} \quad for \quad r \geq r_1.$$

Consequently, there exist two constants $c > 0$ and $r_2 \geq r_1$ such that

$$D'(r) \leq -\frac{c}{r} \quad for \quad r \geq r_2.$$

Integrating this last inequality between r_2 and r, we get

$$D(r) \leq D(r_2) - c\ln\left(\frac{r}{r_2}\right) \quad for \quad r \geq r_2.$$

In particular, we obtain $\lim_{r\to+\infty} D(r) = -\infty$. Since

$$E(r) + \frac{N-1}{2r}|u'|^{p-2}u'(r)u(r) \leq D(r),$$

we get $\lim_{r\to+\infty} E(r) = -\infty$. This is impossible, hence the conclusion. □

Proposition 3.2. *Let u be a solution of problem (P) and let $S_u := \{r > 0 : u(r) > 0\}$. Then $u'(r) < 0$ for any $r \in S_u$.*

Proof. We argue by contradiction. Let $r_0 > 0$ be the first zero of u'. Since by (14) $u'(r) < 0$ for $r \sim 0$, we have by continuity and the definition of r_0, there exists a left neighborhood $]r_0-\varepsilon, r_0[$ (for some $\varepsilon > 0$) where u' is strictly increasing and strictly negative, that is $(|u'|^{p-2}u')'(r) > 0$ for any $r \in]r_0-\varepsilon, r_0[$, hence by letting $r \to r_0$ we get $(|u'|^{p-2}u')'(r_0) \geq 0$. But by equation (7), we have $(|u'|^{p-2}u')'(r_0) = -\alpha|u|^{q-1}u(r_0) < 0$ since $u(r_0) > 0$, $u'(r_0) = 0$ and $\alpha > 0$. This is a contradiction. The proof is complete. □

Proposition 3.3. *Let u be a strictly positive solution of problem (P), then u and u' have the same behavior (28).*

Proof. If $N > 1$, then by Proposition 3.1, $\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} u'(r) = 0$.
If $N = 1$. Let

$$\phi(r) = |u'|^{p-2}u'(r) + \beta r|u|^{q-1}u(r). \tag{32}$$

Then by equation (7),

$$\phi'(r) = (\beta-\alpha)|u|^{q-1}u(r). \tag{33}$$

Since u is strictly positive then it is strictly decreasing. Therefore $\lim_{r\to+\infty} u(r) \in [0,+\infty[$. Suppose that, $\lim_{r\to+\infty} u(r) = L > 0$. Since the energy function E given by (26) converges, then necessarily, $\lim_{r\to+\infty} u'(r) = 0$. Therefore $\lim_{r\to+\infty} \phi(r) = +\infty$. Using L'Hopital's rule, we have

$$\lim_{r\to+\infty} \phi'(r) = \lim_{r\to+\infty} \frac{\phi(r)}{r}.$$

That is
$$(\beta-\alpha)L^q=\beta\,L^q.$$
Therefore, $-\alpha L^q=0$. But This contradicts the fact that $L>0$. Hence, $\lim\limits_{r\to+\infty}u(r)=0$. ☐

Now for any $c>0$, define the function
$$E_c(r)=cu(r)+ru'(r),\quad r>0. \tag{34}$$
It is clear that
$$(r^c u(r))'=r^{c-1}E_c(r),\quad r>0. \tag{35}$$
Hence, using (7), we have for any $r>0$ such that $u'(r)\neq 0$,
$$\begin{aligned}(p-1)|u'|^{p-2}E_c'(r)&=(p-1)(c-\frac{N-p}{p-1})|u'|^{p-2}u'-\\&\alpha r|u|^{q-1}u-q\beta r^2|u|^{q-1}u'(r)\\&=(p-1)(c-\frac{N-p}{p-1})|u'|^{p-2}u'(r)-\\&q\beta r|u|^{q-1}E_{\frac{\alpha}{q\beta}}(r).\end{aligned} \tag{36}$$
Consequently, if $E_c(r_0)=0$ for some $r_0>0$, equation (7) gives
$$(p-1)|u'|^{p-2}(r_0)E_c'(r_0)=r_0|u|^{q-1}u(r_0)\Big[(q\beta c-\alpha)+(p-1)c^{p-1}\left(\frac{N-p}{p-1}-c\right)\frac{|u|^{p-q-1}(r_0)}{r_0^p}\Big]. \tag{37}$$
From which the sign of $E_c(r)$ for large r can be obtained.

Lemma 3.4. *Let u be a strictly positive solution of problem (P). Then $E_c(r)\neq 0$ for large r in the following cases.*
(i) $c=\frac{\alpha}{q\beta}\neq\frac{N-p}{p-1}$.
(ii) $c\neq\frac{\alpha}{q\beta}$ and $q\le p-1$.
(iii) $c\neq\frac{\alpha}{q\beta}$, $q>p-1$ and $\lim\limits_{r\to+\infty}r^{\frac{p}{q+1-p}}u(r)=+\infty$.
(iv) $c\neq\frac{N-p}{p-1}$, $q>p-1$ and $\lim\limits_{r\to+\infty}r^{\frac{p}{q+1-p}}u(r)=0$.

Proof. Assume that there exists a large r_0 such that $E_c(r_0)=0$. Using the fact that $u>0$, $\lim\limits_{r\to+\infty}u(r)=0$, then according to (37) and our hypotheses, we get $E_c'(r_0)\neq 0$ and thereby $E_c(r)\neq 0$ for large r. ☐

Lemma 3.5. *Assume $0<c<\frac{\alpha}{q\beta}$. Let u be a strictly positive solution of problem (P). If $q\le p-1$ or $q>p-1$ and $\lim\limits_{r\to+\infty}r^{\frac{p}{q+1-p}}u(r)=+\infty$, then $E_c(r)<0$ for large r and $\lim\limits_{r\to+\infty}r^c u(r)=0$*

Proof. We know by Lemma 3.4, that $E_c(r)\neq 0$ for large r. Suppose that $E_c(r)>0$ for large r, hence
$$r|u'(r)|<c\,u(r)\qquad for\ large\ r. \tag{38}$$
Using this last inequality and the fact that $u>0$, we obtain according to (7)
$$(|u'|^{p-2}u')'(r)<u^q\Big[(q\beta c-\alpha)+(N-1)c^{p-1}\frac{u^{p-1-q}}{r^p}\Big]. \tag{39}$$
If $q\le p-1$ or $q>p-1$ and $\lim\limits_{r\to+\infty}r^{\frac{p}{q+1-p}}u(r)=+\infty$, we have $\lim\limits_{r\to+\infty}\frac{u^{p-1-q}}{r^p}=0$. Then, $(|u'|^{p-2}u')'(r)\underset{+\infty}{\sim}(q\beta c-\alpha)u^q(r)<0$. Since $u'(r)<0$, then $\lim\limits_{r\to+\infty}|u'|^{p-2}u'(r)\in[-\infty,0[$, but this contradicts the fact that $\lim\limits_{r\to+\infty}u'(r)=0$. Then, $E_c(r)<0$ for large r and $\lim\limits_{r\to+\infty}r^c u(r)\in[0,+\infty[$. Suppose that $\lim\limits_{r\to+\infty}r^c u(r)>0$, then $\lim\limits_{r\to+\infty}r^{c+\varepsilon}u(r)=+\infty$ for $0<c+\varepsilon<\frac{\alpha}{q\beta}$. This is impossible, and therefore $\lim\limits_{r\to+\infty}r^c u(r)=0$. The proof of lemma is complete. ☐

Lemma 3.6. *Assume $\frac{N-p}{p-1}\ge\frac{\alpha}{q\beta}$. Let u be a strictly positive solution of problem (P). Then $E_{\frac{\alpha}{q\beta}}(r)>0$ for any $r>0$.*

Proof. We distinguish two cases.
Case 1. $\frac{N-p}{p-1}>\frac{\alpha}{q\beta}$.
We have $E_{\frac{\alpha}{q\beta}}(0)=\frac{\alpha}{q\beta}u(0)>0$. Let $r_0>0$ be the first zero of $E_{\frac{\alpha}{q\beta}}(r)$. Therefore $E_{\frac{\alpha}{q\beta}}(r)>0$ in $[0,r_0[$, $E_{\frac{\alpha}{q\beta}}(r_0)=0$ and $E'_{\frac{\alpha}{q\beta}}(r_0)\le 0$. But using the fact that $u(r_0)>0$ and $\frac{N-p}{p-1}>\frac{\alpha}{q\beta}$, we have by (37), $E'_{\frac{\alpha}{q\beta}}(r_0)>0$, which is a contradiction.
Case 2. $\frac{N-p}{p-1}=\frac{\alpha}{q\beta}$.
we have by (36),
$$(p-1)|u'|^{p-2}E'_{\frac{\alpha}{q\beta}}(r)=-q\beta r|u|^{q-1}E_{\frac{\alpha}{q\beta}}(r). \tag{40}$$
Let $r_0>0$. We introduce the following function
$$f(r)=\frac{q\beta}{p-1}\int_{r_0}^{r}s|u'|^{2-p}(s)|u|^{q-1}(s)\,ds. \tag{41}$$
By (40), we obtain
$$E'_{\frac{\alpha}{q\beta}}(r)+f'(r)E_{\frac{\alpha}{q\beta}}(r)=0. \tag{42}$$

Hence,

$$\left(e^{f(r)}E_{\frac{\alpha}{q\beta}}(r)\right)' = 0. \tag{43}$$

Integrating this last equality from r_0 to r, we obtain

$$E_{\frac{\alpha}{q\beta}}(r) = E_{\frac{\alpha}{q\beta}}(r_0)e^{-f(r)} \qquad \forall r > r_0. \tag{44}$$

Since $E_{\frac{\alpha}{q\beta}}(r_0) > 0$ for any $r_0 > 0$ close to 0, then $E_{\frac{\alpha}{q\beta}}(r) > 0$ for any $r > 0$.
This completes the proof of lemma. □

Lemma 3.7. *Assume* $0 < \dfrac{N-p}{p-1} < \dfrac{\alpha}{q\beta}$ *and* $E_{\frac{\alpha}{q\beta}}(r) > 0$ *for large* r. *Let* u *be a strictly positive solution of problem* (P). *Then* $E_{\frac{N-p}{p-1}}(r) > 0$ *for any* $r > 0$.

Proof. We have $E_{\frac{N-p}{p-1}}(0) > 0$. Suppose that there exists $r_0 > 0$ the first zero of $E_{\frac{N-p}{p-1}}$. Then, by 37, $E'_{\frac{N-p}{p-1}}(r_0) < 0$. Therefore, $E_{\frac{N-p}{p-1}}(r) < 0\ \forall r > r_0$. On the other hand, since $E_{\frac{\alpha}{q\beta}}(r) > 0$ for large r, then by (36), we have $E'_{\frac{N-p}{p-1}}(r) < 0$ for large r. Hence, $\lim_{r\to+\infty} E_{\frac{N-p}{p-1}}(r) \in [-\infty, 0[$, which implies that $\lim_{r\to+\infty} ru'(r) \in [-\infty, 0[$, but this contradicts the fact that $\lim_{r\to+\infty} u(r) = 0$. Consequently, $E_{\frac{N-p}{p-1}}(r) > 0$ for any $r > 0$. □

Proposition 3.8. *Assume* $\dfrac{\alpha}{\beta} < N$, $\dfrac{\alpha}{q\beta} = \dfrac{p}{q+1-p}$ *and* $\lim_{r\to+\infty} r^{\frac{p}{q+1-p}}u(r) = +\infty$. *Let* u *be a strictly positive solution of problem* (P). *Then*

$$\lim_{r\to+\infty} \frac{ru'(r)}{u(r)} = \frac{-\alpha}{q\beta}. \tag{45}$$

Proof. Since $\dfrac{\alpha}{\beta} < N$ and $\dfrac{\alpha}{q\beta} = \dfrac{p}{q+1-p}$, then $\dfrac{N-p}{p-1} > \dfrac{N}{q} > \dfrac{\alpha}{q\beta}$, therefore using the fact that $E_{\frac{\alpha}{q\beta}}(r) > 0$ for any $r > 0$ by lemma 3.6, we obtain

$$\frac{-\alpha}{q\beta}u(r) < ru'(r) < 0 \qquad for\ any\ r > 0. \tag{46}$$

Let $c > 0$ and

$$g(r) = \frac{E_c(r)}{u(r)} = c + \frac{ru'(r)}{u(r)}, \quad r > 0. \tag{47}$$

then

$$c - \frac{\alpha}{q\beta} < g(r) < c \qquad for\ any\ r > 0. \tag{48}$$

Consequently g is bounded for large r. We prove that g converges. Assume by contradiction that it oscillates, that is there exist two sequences $\{\eta_i\}$ and $\{\xi_i\}$ going to $+\infty$ as $i \to +\infty$ such that g has a local minimum in η_i and a local maximum in ξ_i satisfying $\eta_i < \xi_i < \eta_{i+1}$ and

$$\begin{aligned}&\liminf_{r\to+\infty} g(r) = \lim_{i\to+\infty} g(\eta_i) = \gamma_1 < \\ &\limsup_{r\to+\infty} g(r) = \lim_{i\to+\infty} g(\xi_i) = \gamma_2.\end{aligned} \tag{49}$$

Therefore, by (48), we have

$$c - \frac{\alpha}{q\beta} \le \gamma_1 < \gamma_2 \le c. \tag{50}$$

Since $g'(\xi_i) = 0$, then

$$\frac{E'_c(\xi_i)}{u'(\xi_i)} = \frac{E_c(\xi_i)}{u(\xi_i)} = g(\xi_i). \tag{51}$$

Therefore

$$\lim_{i\to+\infty} \frac{E'_c(\xi_i)}{u'(\xi_i)} = \gamma_2. \tag{52}$$

On the other hand, we have by (36) and the fact that $u'(r) < 0$,

$$\begin{aligned}\frac{E'_c(r)}{u'(r)} = &\left(c - \frac{N-p}{p-1}\right) + \\ &\frac{q\beta}{p-1}\frac{ru^q(r)}{|u'|^{p-1}(r)}\left[\frac{\alpha}{q\beta} + \frac{ru'(r)}{u(r)}\right].\end{aligned} \tag{53}$$

As $E_{\frac{\alpha}{q\beta}}(r) > 0\ \forall r > 0$, then

$$\frac{|u'(r)|^{p-1}}{ru^q(r)} < \left(\frac{\alpha}{q\beta}\right)^{p-1} r^{-p}u^{p-1-q}. \tag{54}$$

Since $\lim_{r\to+\infty} r^{\frac{p}{q+1-p}}u(r) = +\infty$, then

$$\lim_{r\to+\infty} \frac{ru^q(r)}{|u'(r)|^{p-1}} = +\infty. \tag{55}$$

Moreover, we have

$$\begin{aligned}\lim_{i\to+\infty}\left(\frac{\alpha}{q\beta} + \frac{\xi_i u'(\xi_i)}{u(\xi_i)}\right) &= \frac{\alpha}{q\beta} + \lim_{i\to+\infty} g(\xi_i) - c \\ &= \frac{\alpha}{q\beta} + \gamma_2 - c > 0.\end{aligned} \tag{56}$$

Then, by (53)

$$\lim_{i\to+\infty} \frac{E'_c(\xi_i)}{u'(\xi_i)} = +\infty. \tag{57}$$

But this contradicts (52). Then $g(r)$ converges as $r \to +\infty$, and consequently $\frac{ru'(r)}{u(r)}$ converges also. Let $\lim_{r\to+\infty} \frac{ru'(r)}{u(r)} = -d \leq 0$, then by (46), $0 \leq d \leq \frac{\alpha}{q\beta}$. Suppose that $d < \frac{\alpha}{q\beta}$, then

$$\lim_{r\to+\infty} \left(\frac{\alpha}{q\beta} + \frac{ru'(r)}{u(r)}\right) = \frac{\alpha}{q\beta} - d > 0. \qquad (58)$$

Therefore, by (53) and (55),

$$\lim_{r\to+\infty} \frac{E_c'(r)}{u'(r)} = +\infty. \qquad (59)$$

Using Hospital's rule, we get

$$\lim_{r\to+\infty} \frac{E_c'(r)}{u'(r)} = \lim_{r\to+\infty} \frac{E_c(r)}{u(r)} = \lim_{r\to+\infty} \left(c + \frac{ru'(r)}{u(r)}\right) = c - d. \qquad (60)$$

This contradicts (59). Consequently $\lim_{r\to+\infty} \frac{ru'(r)}{u(r)} = \frac{-\alpha}{q\beta}$. The proof is complete. □

Proposition 3.9. *Assume* $\frac{\alpha}{\beta} < N$, $\frac{\alpha}{q\beta} = \frac{p}{q+1-p}$ *and* $\lim_{r\to+\infty} r^{\frac{p}{q+1-p}} u(r) = +\infty$. *Let u be a strictly positive solution of problem* (P). *Then,*

- *if* $0 < c < \frac{\alpha}{q\beta}$, $\lim_{r\to+\infty} r^c u(r) = \lim_{r\to+\infty} r^{c+1} u'(r) = 0$.
- *if* $c > \frac{\alpha}{q\beta}$, $\lim_{r\to+\infty} r^c u(r) = +\infty$ *and* $\lim_{r\to+\infty} r^{c+1} u'(r) = -\infty$.

Proof. First, we show that $E_c'(r) \neq 0$ for large r. If $E_c'(r) = 0$ for some large r, then

$$(p-1)|u'|^{p-2}E_c''(r) = ru^{q-1}|u'| \Bigg[q\beta(\frac{\alpha}{q\beta} - c) - q\beta(q-1)\frac{E_{\frac{\alpha}{q\beta}} E_{\frac{1}{q-1}}}{r|u'|u} - q\beta(p-1)(c - \frac{N-p}{p-1})\frac{E_{\frac{\alpha}{q\beta}}}{r|u'|} + (p-1)(N-1)\left(c - \frac{N-p}{p-1}\right)\frac{|u'|^{p-1}}{ru^q}\frac{u}{r|u'|}\Bigg] \qquad (61)$$

We know by Proposition 3.8 that $\lim_{r\to+\infty} \frac{ru'(r)}{u(r)} = \frac{-\alpha}{q\beta}$, then

$$\lim_{r\to+\infty} \frac{E_{\frac{1}{q-1}}(r)}{u(r)} = \frac{1}{q-1} - \frac{\alpha}{q\beta} \qquad (62)$$

and

$$\lim_{r\to+\infty} \frac{E_{\frac{\alpha}{q\beta}}(r)}{r|u'|} = 0. \qquad (63)$$

On the other hand, since $E_{\frac{\alpha}{q\beta}}(r) > 0$, $\forall r > 0$ (by Lemma 3.6) and $\lim_{r\to+\infty} r^{\frac{p}{q+1-p}} = 0$, we obtain

$$\lim_{r\to+\infty} \frac{|u'(r)|^{p-1}}{ru^q(r)} = 0. \qquad (64)$$

Therefore, using the fact that $\lim_{r\to+\infty} \frac{u(r)}{r|u'|} = \frac{q\beta}{\alpha}$, we get

$$\lim_{r\to+\infty} \frac{|u'|^{p-1}(r)}{ru^q(r)} \frac{u(r)}{r|u'(r)|} = 0. \qquad (65)$$

Using (62), (63) and (65), we get from (61), $E_c''(r) \neq 0$ if $c \neq \frac{\alpha}{q\beta}$. Consequently, if $c \neq \frac{\alpha}{q\beta}$, we have $E_c'(r) \neq 0$ for large r. We distinguish two cases.

Case 1. $0 < c < \frac{\alpha}{q\beta}$.

We have by Lemma 3.5, $E_c(r) < 0$ for large r and $\lim_{r\to+\infty} r^c u(r) = 0$. If $E_c'(r) < 0$ for large r, then $\lim_{r\to+\infty} E_c(r) \in [-\infty; 0[$, this is impossible since $\lim_{r\to+\infty} u(r) = 0$ and $\lim_{r\to+\infty} ru'(r) = 0$. Therefore, $E_c'(r) > 0$ for large r. On the other hand, we have

$$(r^{c+1}u')' = r^c E_c'(r), \qquad (66)$$

Then the function $r^{c+1}u'$ is negative and increasing for large r and therefore, using L'Hopital's rule, we obtain $\lim_{r\to+\infty} r^{c+1}u'(r) = \lim_{r\to+\infty} r^c u(r) = 0$.

Case 2. $c > \frac{\alpha}{q\beta}$.

We have $E_c(r) > 0$, $\forall r > 0$ (by Lemma 3.6). If $E_c'(r) > 0$ for large r, $\lim_{r\to+\infty} E_c(r) \in]0; +\infty]$, this is also impossible. Therefore, $E_c'(r) < 0$ for large r. Hence, $\lim_{r\to+\infty} r^c u(r) \in]0, +\infty]$ and $\lim_{r\to+\infty} r^{c+1}u'(r) \in [-\infty, 0[$. Suppose that $-\infty < \lim_{r\to+\infty} r^{c+1}u'(r) < 0$, then by L'Hopital's rule, $0 < \lim_{r\to+\infty} r^c u(r) < +\infty$.

Using logarithmic change (9), we have v_c and h_c converge, $A_c > 0$ and $K_c < 0$ and by letting $t \to +\infty$ in equation (10), we obtain $\lim_{t\to+\infty} w_c'(t) > 0$. But this contradicts the fact that w converges. Therefore $\lim_{r\to+\infty} r^{c+1}u'(r) = -\infty$ and $\lim_{r\to+\infty} r^c u(r) = +\infty$. The proof is complete. □

Proposition 3.10. *Let u be a solution of problem* (P). *If there exists* $c > 0$ *such that* $r^c u(r)$ *is* *monotone for large* r *and* $\lim_{r\to+\infty} r^c u(r) = d$. *Then* $\lim_{r\to+\infty} r^{c+1}u'(r) = -c\,d$.

Proof. According to logarithmic change (9) and 34), we have

$$v_c'(t) = r^c\,E_c(r). \tag{67}$$

Then, the function $v_c(t)$ is monotone for large t and $\lim_{t\to+\infty} v_c(t) = d$. Therefore for large t_0, the integral $\int_{t_0}^{t} |v_c'(s)|\,ds$ converges as $t \to +\infty$. Therefore, $\lim_{t\to+\infty} v_c'(t) = 0$. Hence, by (12) $\lim_{t\to+\infty} h_c(t) = -c\,d$, that is $\lim_{r\to+\infty} r^{c+1}u'(r) = -c\,d$. □

4 Asymptotic Behavior at infinity

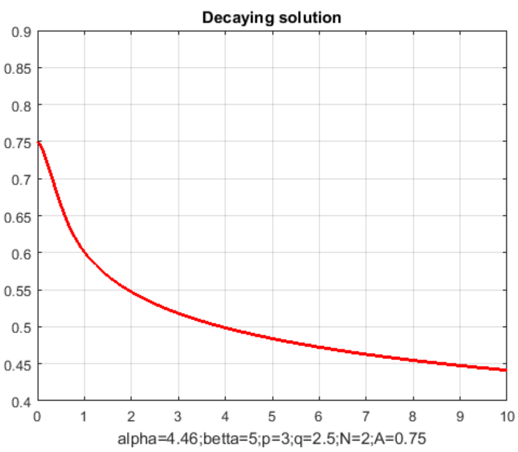
In this section we study the asymptotic behavior near infinity of positive solutions of problem (P).

Theorem 4.1. *Assume* $\frac{\alpha}{\beta} < N$. *Then any solution of problem* (P) *is a decaying solution.*

Proof. We have $u(0) > 0$. Assume by contradiction that there exists $r_0 > 0$ such that $u(r_0) = 0$ (where r_0 is the first zero of u). Then, $u'(r_0) \le 0$. On the other hand, integrating (15) between 0 and r_0, we obtain

$$r_0^{N-1}|u'|^{p-2}u'(r_0) = (\beta N - \alpha)\int_0^{r_0} s^{N-1}u^q(s)\,ds. \tag{68}$$

The right-hand side of the previous equality is strictly positive, but this contradicts the fact that $u'(r_0) \le 0$. Therefore u is strictly positive and therefore it is strictly decreasing by Proposition 3.2. Hence u is a decaying solution. The theorem is proved. □



Theorem 4.2. *Assume* $\frac{\alpha}{\beta} < N$ *and* $\frac{\alpha}{q\beta} < \frac{p}{q+1-p}$. *Let* u *be a solution of problem* (P). *Then*

$$\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}}u(r) = L_1 > 0. \tag{69}$$

and

$$\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}+1}u'(r) = \frac{-\alpha}{q\beta}L_1 < 0. \tag{70}$$

Proof. Recall by Theorem 4.1 that u is strictly positive and then it strictly decreasing. Set

$$\begin{aligned} I(r) &= r^{\frac{\alpha}{\beta}}\left[\frac{\beta}{\alpha}u^q(r) + \frac{1}{\alpha r}|u'|^{p-2}u'(r)\right] \\ &= \frac{\beta}{\alpha}r^{\frac{\alpha}{\beta}}u^q\left[1 - \frac{1}{\beta}\frac{|u'|^{p-1}(r)}{ru^q(r)}\right] \end{aligned} \tag{71}$$

A simple calculation gives

$$I'(r) = \frac{-1}{\alpha}(N - \frac{\alpha}{\beta})r^{\frac{\alpha}{\beta}-2}|u'|^{p-2}u'(r). \tag{72}$$

Since $N > \frac{\alpha}{\beta}$ and $u'(r) < 0$, then $I'(r) > 0\ \forall r > 0$. Moreover, using (14), the fact that $u(0) = a > 0$, we get $\lim_{r\to 0} I(r) = 0$. Therefore, $I(r) > 0\ \forall r > 0$, hence $\lim_{r\to+\infty} I(r) \in]0, +\infty]$ and then there exists $c > 0$ such that $I(r) \ge c$ for large r. As $u'(r) < 0$, then

$$r^{\frac{\alpha}{\beta}}u^q(r) \ge \frac{\alpha}{\beta}c \quad for\ large\ r. \tag{73}$$

On the other hand, we know that by Lemma 3.4 and Lemma 3.6 that $E_{\frac{\alpha}{q\beta}}(r) \ne 0$ for large r. Then, from (73), necessarily $\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}}u(r) \in]0; +\infty]$. Suppose that $\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}}u(r) = +\infty$, then necessarily $E_{\frac{\alpha}{q\beta}}(r) > 0$ for large r and therefore

$$0 < \frac{|u'|^{p-1}(r)}{ru^q(r)} < \left(\frac{\alpha}{q\beta}\right)^{p-1}\frac{1}{r^p u^{q+1-p}(r)}. \tag{74}$$

As, $\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}}u(r) = +\infty$, then $\lim_{r\to+\infty} r^p u^{q+1-p}(r) = +\infty$, which implies according to (74) that $\lim_{r\to+\infty}\frac{|u'|^{p-1}(r)}{ru^q(r)} = 0$. This leads from (71) that $I(r) \underset{+\infty}{\sim} \frac{\beta}{\alpha}r^{\frac{\alpha}{\beta}}u^q(r)$ and therefore $\lim_{r\to+\infty} I(r) = +\infty$. Let

$$0 < \sigma < min\left(\frac{\alpha}{q\beta}; \frac{1}{p-1}\left[\frac{\alpha}{q\beta}(p-1-q) + p\right]\right). \tag{75}$$

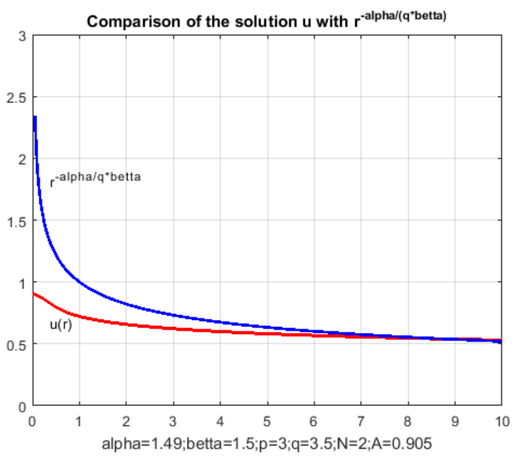
By Lemma 3.5, we obtain, $\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}-\sigma}u(r) = 0$. Then

$$u(r) \le r^{\sigma-\frac{\alpha}{q\beta}} \quad for\ large\ r.$$

Using this last inequality and the fact that $E_{\frac{\alpha}{q\beta}}(r) > 0$ for large r, we get

$$0 < I'(r) < \frac{1}{\alpha}(N-\frac{\alpha}{\beta})\left(\frac{\alpha}{q\beta}\right)^{p-1} r^{\sigma(p-1)+\frac{\alpha}{q\beta}(q+1-p)-p-1}.$$

Since $\sigma(p-1)+\frac{\alpha}{q\beta}(q+1-p)-p < 0$, then $\lim_{r\to+\infty} I(r)$ is finite, which contradicts the fact that $\lim_{r\to+\infty} I(r) = +\infty$. Consequently, $\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}}u(r) = L_1 > 0$. Moreover, since $E_{\frac{\alpha}{q\beta}}(r) \neq 0$ for large r, then $r^{\frac{\alpha}{q\beta}}u(r)$ is monotone for large r. Therefore, by Proposition 3.10, $\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}+1}u'(r) = \frac{-\alpha}{q\beta}L_1 < 0$. This completes the proof. □



Theorem 4.3. *Assume* $\frac{\alpha}{\beta} < N$ *and* $\frac{\alpha}{q\beta} = \frac{p}{q+1-p}$. *Let u be a solution of problem (P). Then*

$$\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}}u(r)(\ln(r))^{\frac{-1}{q+1-p}} = \left((\frac{p-1}{q\beta})(q+1-p)\left(\frac{N-p}{p-1}-\frac{\alpha}{q\beta}\right)\left(\frac{\alpha}{q\beta}\right)^{p-1}\right)^{\frac{1}{q+1-p}} \tag{76}$$

Proof. First we show that $\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}}u(r) = +\infty$. Since u is strictly positive, we introduce this following function

$$\varphi(r) = r^{N-1}|u'|^{p-2}u'(r) + \beta r^N u^q(r). \tag{77}$$

then by (15), we get

$$\varphi'(r) = (\beta N-\alpha)r^{N-1}u^q(r). \tag{78}$$

Since $N\beta > \alpha$ and $u(r) > 0$, then $\varphi'(r) > 0$ and as $\varphi(0) = 0$, we have $\varphi(r) > 0\ \forall r > 0$. That is, for any $r > 0$,

$$|u'|^{p-2}u'(r) > -\beta\, r\, u^q(r) \tag{79}$$

As $u'(r) < 0$, then for any $r > 0$

$$u'(r)u^{\frac{-q}{p-1}} > -\beta^{\frac{1}{p-1}} r^{\frac{1}{p-1}}. \tag{80}$$

Integrating (80) twice from r_0 to r and taking into account $q > p-1$, we obtain

$$u^{\frac{p-1-q}{p-1}}(r) - u^{\frac{p-1-q}{p-1}}(r_0) < \frac{q+1-p}{p}\beta^{\frac{1}{p-1}}\left(r^{\frac{p}{p-1}} - r_0^{\frac{p}{p-1}}\right). \tag{81}$$

Then there exists $C > 0$ such that

$$r^{\frac{p}{q+1-p}}u(r) > C \quad for\ large\ r. \tag{82}$$

As $\frac{\alpha}{\beta} < N$ and $\frac{\alpha}{q\beta} = \frac{p}{q+1-p}$, then $\frac{N-p}{p-1} > \frac{\alpha}{q\beta}$. Hence, by Lemma 3.6, $E_{\frac{\alpha}{q\beta}}(r) > 0$ for any $r > 0$. Consequently $\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}}u(r) \in]0,+\infty]$. Suppose that $\lim_{r\to+\infty} r^{\frac{\alpha}{q\beta}}u(r) = l > 0$. Using equation (7), we get

$$r^{N-1}|u'|^{p-2}u'(r) + \beta r^N u^q(r) = (\beta N-\alpha)\int_0^r s^{N-1}u^q(s)\,ds. \tag{83}$$

Then

$$r^{\frac{\alpha}{\beta}-1}|u'|^{p-2}u'(r) + \beta r^{\frac{\alpha}{\beta}}u^q(r) = (\beta N-\alpha)r^{\frac{\alpha}{\beta}-N}\int_0^r s^{N-1}u^q(s)\,ds. \tag{84}$$

Since

$$\int_0^r s^{N-1}u^q(s)\,ds > \frac{1}{N}r^N u^q = \frac{1}{N}r^{\frac{\alpha}{\beta}}u^q r^{N-\frac{\alpha}{\beta}} \underset{r\to+\infty}{\longrightarrow} +\infty$$

Then, using L'Hopital's rule, we obtain

$$\lim_{r\to+\infty}\frac{\int_0^r s^{N-1}u^q(s)\,ds}{r^{N-\frac{\alpha}{\beta}}} = \lim_{r\to+\infty}\frac{r^{N-1}u^q(r)}{\left(N-\frac{\alpha}{\beta}\right)r^{N-\frac{\alpha}{\beta}-1}} = \frac{\beta l^q}{\beta N-\alpha}. \tag{85}$$

Therefore, from (84) we have

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}-1}|u'|^{p-2}u'(r) = 0. \qquad (86)$$

that is,

$$\lim_{r\to+\infty} r^{\frac{\frac{\alpha}{\beta}-p}{p-1}+1}u'(r) = 0. \qquad (87)$$

Using L'Hopital's rule and the fact that $\frac{\frac{\alpha}{\beta}-p}{p-1} = \frac{p}{q+1-p}$, we obtain $\lim_{r\to+\infty} r^{\frac{p}{q+1-p}}u(r) = 0$. But this contradicts the fact that $\lim_{r\to+\infty} r^{\frac{p}{q+1-p}}u(r) = l > 0$. Consequently, $\lim_{r\to+\infty} r^{\frac{p}{q+1-p}}u(r) = +\infty$.
Using the fact that u is strictly positive and decreasing, we obtain by (36),

$$\frac{E'_{\frac{\alpha}{q\beta}}(r)}{u'} = \left(\frac{\alpha}{q\beta} - \frac{N-p}{p-1}\right) + \frac{q\beta}{p-1}\frac{r\,u^q}{|u'|^{p-1}}\frac{E_{\frac{\alpha}{q\beta}}(r)}{u} = \left(\frac{\alpha}{q\beta} - \frac{N-p}{p-1}\right) + \frac{q\beta}{p-1}\left(\frac{u}{r|u'|}\right)^{p-1} r^p\,u^{q+1-p}\frac{E_{\frac{\alpha}{q\beta}}(r)}{u}. \qquad (88)$$

We introduce the following variable change

$$V(r) = r^{\frac{\alpha}{q\beta}}u(r), \quad r > 0. \qquad (89)$$

It's easy to see by (35)that

$$r\,V^{q-p}V'(r) = r^p\,u^{q+1-p}\frac{E_{\frac{\alpha}{q\beta}}(r)}{u}. \qquad (90)$$

Then, by (88)

$$\frac{E'_{\frac{\alpha}{q\beta}}(r)}{u'} = \left(\frac{\alpha}{q\beta} - \frac{N-p}{p-1}\right) + \frac{q\beta}{p-1}\left(\frac{u}{r|u'|}\right)^{p-1} r\,V^{q-p}V'(r). \qquad (91)$$

Using L'Hopital's rule and proposition 3.8, we get

$$\lim_{r\to+\infty}\frac{E'_{\frac{\alpha}{q\beta}}(r)}{u'} = \lim_{r\to+\infty}\frac{E_{\frac{\alpha}{q\beta}}(r)}{u} = \frac{\alpha}{q\beta} + \frac{ru'}{u} = 0. \qquad (92)$$

Therefore by (91)

$$\lim_{r\to+\infty} rV^{q-p}V'(r) = \frac{p-1}{q\beta}\left(\frac{N-p}{p-1} - \frac{\alpha}{q\beta}\right)\left(\frac{\alpha}{q\beta}\right)^{p-1}. \qquad (93)$$

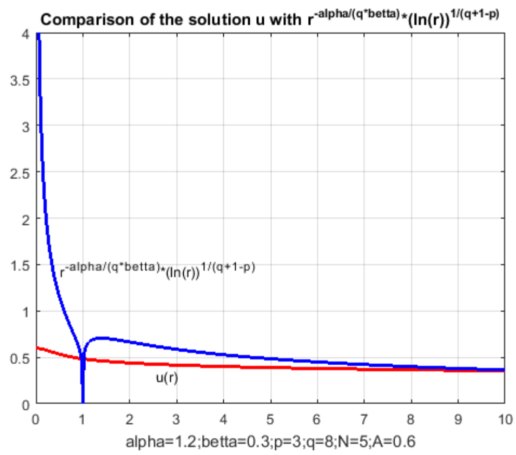
That is to say

$$\lim_{r\to+\infty}\frac{\left(\frac{V^{q+1-p}(r)}{q+1-p}\right)'}{(\ln(r))'} = \frac{p-1}{q\beta}\left(\frac{N-p}{p-1} - \frac{\alpha}{q\beta}\right)\left(\frac{\alpha}{q\beta}\right)^{p-1}. \qquad (94)$$

Using L'Hopital's rule (because $\lim_{r\to+\infty} V(r) = +\infty$), we get

$$\lim_{r\to+\infty}\frac{V^{q+1-p}(r)}{\ln(r)} = \frac{p-1}{q\beta}(q+1-p)\left(\frac{N-p}{p-1} - \frac{\alpha}{q\beta}\right)\left(\frac{\alpha}{q\beta}\right)^{p-1}. \qquad (95)$$

The result follows and the proof is complete. □



Theorem 4.4. *Assume* $\frac{\alpha}{\beta} < N$ *and* $\frac{\alpha}{q\beta} > \frac{p}{q+1-p}$. *Let* u *be a solution of problem* (P). *Then*

$$\lim_{r\to+\infty} r^{\frac{p}{q+1-p}}u(r) = L_2 \qquad (96)$$

and

$$\lim_{r\to+\infty} r^{\frac{p}{q+1-p}+1}u'(r) = \frac{-p}{q+1-p}L_2, \qquad (97)$$

where

$$L_2 = \left(\frac{(p-1)\left(\frac{p}{q+1-p}\right)^{p-1}\left(\frac{N-p}{p-1} - \frac{p}{q+1-p}\right)}{\alpha - q\beta\frac{p}{q+1-p}}\right)^{\frac{1}{q+1-p}}. \qquad (98)$$

Proof. As $\frac{\alpha}{\beta} < N$ and $\frac{\alpha}{q\beta} > \frac{p}{q+1-p}$, then $\frac{N-p}{p-1} > \frac{p}{q+1-p}$. First, we show that $E_{\frac{p}{q+1-p}}(r) > 0\, \forall r > 0$. Let $r_0 > 0$ the first zero of $E_{\frac{p}{q+1-p}}(r)$. Then we have $E_{\frac{p}{q+1-p}}(r) > 0$ $\forall r \in [0, r_0)$, $E_{\frac{p}{q+1-p}}(r_0) = 0$ and $E'_{\frac{p}{q+1-p}}(r_0) \leq 0$. Therefore using (37)

$$\left(q\beta\frac{p}{q+1-p} - \alpha\right) + (p-1)\left(\frac{p}{q+1-p}\right)^{p-1} \left(\frac{N-p}{p-1} - \frac{p}{q+1-p}\right)\frac{u^{p-q-1}(r_0)}{r_0^p} \leq 0. \tag{99}$$

Hence

$$r_0^{\frac{p}{q+1-p}}\, u(r_0) \geq L_2, \tag{100}$$

Where L_2 is given by (98). On the other hand, since $u(r) > 0$, then integrating (15) on $(0, r_0)$, we obtain

$$r_0^{N-1}|u'|^{p-2}u'(r_0) + \beta r_0^N u^q(r_0) = (\beta N - \alpha)\int_0^{r_0} s^{N-1}u^q(s)\, ds. \tag{101}$$

Therefore,

$$\beta\left(r_0^{\frac{p}{q+1-p}}u(r_0)\right)^q = r_0^{\frac{pq}{q+1-p}-1}|u'|^{p-1}(r_0) + (\beta N-\alpha)r_0^{\frac{pq}{q+1-p}-N}\int_0^{r_0} s^{N-1}u^q(s)\, ds. \tag{102}$$

As $E_{\frac{p}{q+1-p}}(r) > 0$ $\forall r \in [0, r_0)$ and $E_{\frac{p}{q+1-p}}(r_0) = 0$, then $\left(r^{\frac{p}{q+1-p}}u(r)\right)' > 0$ $\forall r \in (0, r_0)$ and $|u'(r_0)| = \frac{p}{q+1-p}\frac{u(r_0)}{r_0}$. Then,

$$\beta\left(r_0^{\frac{p}{q+1-p}}u(r_0)\right)^q \leq \left(\frac{p}{q+1-p}\right)^{p-1} r^{\frac{pq}{q+1-p}-p}u^{p-1}(r_0) + (\beta N-\alpha)r_0^{\frac{pq}{q+1-p}-N} r^{\frac{pq}{q+1-p}}u^q(r_0)\int_0^{r_0} s^{N-1-\frac{pq}{q+1-p}}\, ds. \tag{103}$$

Taking into account $N > \frac{pq}{q+1-p}$, we obtain

$$\beta\left(r_0^{\frac{p}{q+1-p}}u(r_0)\right)^q \leq \left(\frac{p}{q+1-p}\right)^{p-1} r^{\frac{p(p-1)}{q+1-p}}u^{p-1}(r_0) + \frac{\beta N-\alpha}{N-\frac{pq}{q+1-p}}\left(r_0^{\frac{p}{q+1-p}}u(r_0)\right)^q. \tag{104}$$

Therefore

$$r_0^{\frac{p}{q+1-p}}u(r_0) \leq \left[\left(\frac{p}{q+1-p}\right)^{p-1}\left(\frac{1}{\beta - \frac{\beta N-\alpha}{N-\frac{pq}{q+1-p}}}\right)\right]^{\frac{1}{q+1-p}} = L_2. \tag{105}$$

Hence by (100) and (105),

$$r_0^{\frac{p}{q+1-p}}u(r_0) = L_2. \tag{106}$$

As $E_{\frac{p}{q+1-p}}(r_0) = 0$, then

$$r_0^{\frac{p}{q+1-p}+1}u'(r_0) = \frac{-p}{q+1-p}L_2. \tag{107}$$

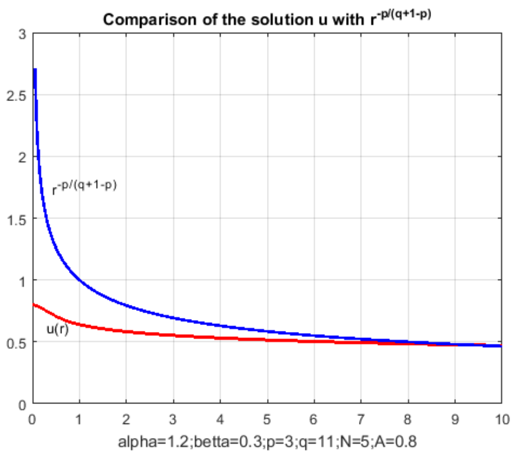
Recalling the logarithmic change (9), $v(t) = r^{\frac{p}{q+1-p}}u(r)$, we obtain by (10), (11) and (12) and (13), the system

$$\begin{cases} v'(t) = \frac{p}{q+1-p}v(t) + |w(t)|^{\frac{2-p}{p-1}}w(t) \\ w'(t) = -\left(N-p-\frac{p}{q+1-p}(p-1)\right)w(t) - \\ \qquad \alpha v^q(t) - q\beta v^{q-1}(t)|w(t)|^{\frac{2-p}{p-1}}w(t). \end{cases} \tag{108}$$

This system has a non trivial equilibrium point $\left(L_2, -\left(\frac{p}{q+1-p}L_2\right)^{p-1}\right)$ and admits a unique solution, but $v(t_0) = L_2$ and $w(t_0) = -\left(\frac{p}{q+1-p}L_2\right)^{p-1}$ (because $h(t_0) = \frac{-p}{q+1-p}L_2$), where $t_0 = \ln(r_0)$, then necessarily $v(t) = L_2$ and $w(t) = -\left(\frac{p}{q+1-p}L_2\right)^{p-1}$, therefore $v'(t) = 0$ and by (67), $E_{\frac{p}{q+1-p}}(r) = 0$, $\forall r > 0$. This is a contradiction. We deduce that $E_{\frac{p}{q+1-p}}(r) > 0\, \forall r > 0$ and $\lim_{r\to+\infty} r^{\frac{p}{q+1-p}}u(r) \in]0, +\infty]$. That is,

$$|u'(r)| < \frac{p}{q+1-p}\frac{u(r)}{r}, \quad \forall r > 0.$$

In the same way, by integrating (15) on $(0, r)$ we obtain $r^{\frac{p}{q+1-p}} u(r) \leq L_2 \quad \forall r > 0$. Then, $\lim_{r \to +\infty} r^{\frac{p}{q+1-p}} u(r) = d \in]0, L_2]$. We show that $d = L_2$. According to Proposition 3.10, since $v'(t) > 0$ and $\lim_{t \to +\infty} v(t) = d > 0$, then $\lim_{t \to +\infty} h(t) = \frac{-p}{q+1-p} d < 0$. Therefore, $\lim_{t \to +\infty} w(t) = -\left(\frac{p}{q+1-p} L_2\right)^{p-1}$ and necessarily $\lim_{t \to +\infty} w'(t) = 0$. Hence, letting $t \to +\infty$ in the second equation of system (108), we get $d = L_2$ and the proof is complete. □



5 Application to the parabolic problem

In this section, we prove the existence of radial strictly positive self-similar solution of the following parabolic problem

$$(Q) \begin{cases} v_t = \Delta_p v^m & in \quad \mathbb{R}^N \times (0, +\infty) \\ v(0, 1) = b \end{cases}$$

where $p > 2$, $N \geq 1$, $0 < m < \frac{1}{p-1}$ and $b > 0$.

Theorem 5.1. *Assume $0 < \frac{\alpha}{\beta} < N$ and $\frac{\alpha}{q\beta} < \frac{p}{q+1-p}$. Then, for every $b > 0$, problem (Q) admits a radial strictly positive self-similar solution $U_b(x, t) = t^{-\alpha} u^{\frac{1}{m}}(t^{-\beta}|x|)$, where $\alpha = \frac{\beta p - 1}{1 - m(p-1)}$ and u is solution of problem (P). Moreover, there exists $L(b) > 0$ such that*

$$\lim_{t \to 0^+} U_b(x, t) = L(b)|x|^{\frac{-\alpha}{\beta}} \quad \textit{for each } x \neq 0. \tag{109}$$

Proof. The Existence and uniqueness of U_b follow from Theorem 2.1 with $b = a^q$ and $m = \frac{1}{q}$. The positivity follows easily from Theorem 4.1.
Put $y = t^{-\beta}|x|$, then

$$|x|^{\frac{\alpha}{\beta}} U_b(x, t) = y^{\frac{\alpha}{\beta}} u^q(y).$$

According to Theorem 4.2, we have $\lim_{y \to +\infty} y^{\frac{\alpha}{\beta}} u^q(y) = L_1^q > 0$. Therefore, there exists $L(b) = L_1^q > 0$, such that

$$\lim_{t \to 0^+} |x|^{\frac{\alpha}{\beta}} U_b(x, t) = \lim_{y \to +\infty} y^{\frac{\alpha}{\beta}} u^q(y) = L(b).$$

The proof is complete. □

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