Contra-continuous functions defined through Λ_I-closed sets

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Abstract: - We introduce some variants of contra-continuity in terms of Λ_I-closed sets, namely contra-Λ_I-continuous, contra quasi-Λ_I-continuous and contra Λ_I-irresolute functions. The relationships between these functions are investigated and their respective characterizations are established. Moreover, we study their behavior of several topological notions under the direct and inverse images of these functions.

Key-Words: - Ideal; local function, Λ_I-set, Λ_I-closed set, contra Λ_I-continuous function.

Received: July 8, 2020. Revised: December 6, 2020. Accepted: December 17, 2020. Published: December 29, 2020.

1 Introduction

In 1986, Maki [8] introduced and studied the notions of Λ-sets in topological spaces. Later, in 1997, Arenas et al. [[1]] defined and studied the notions of λ-closed and λ-open sets, using Λ-sets and closed sets. Particularly, these authors used λ-closed sets to characterize the axiom $T_{1/2}$. On the other hand, in 1933, Kuratowski [7] introduced a generalization of the closure, called the *local function*, by the ideal theory on topological spaces. In 1992, Jankovic and Hamlett [6], introduced the concept of I-open set via the local function, which is independent of the notion of open set and is a generalization of the notion of pre-open set given by Mashhour et al. [9]. Replacing the class of open sets by the class of I-open sets, in 2011, Noiri and Keskin [[10]] introduced and studied modifications of Λ-sets and λ-closed sets in the context of topological spaces equipped with an ideal, which they called Λ_I-sets and Λ_I-closed sets, and so characterized two separation properties called spaces I-T_1 and spaces I-$T_{1/2}$.

In 1996, Dontchev [2] introduced the notion of contra-continuous function in topological spaces and established interesting results which related contra-continuity with compact spaces, S-closed spaces and strongly-S-closed spaces. The main purpose of this work is to introduce new variants of contra-continuous functions and characterize its by using Λ_I-closed sets, in contrast with the variants of continuity studied by Sanabria et al. [[11]]. Also, we study and investigate the preservation of several separation properties, connectedness and compactness, through direct and inverse images of such functions.

2 Preliminaries

Troughout this paper, (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assummed, unless explicitely stated. In the same form: $P(X)$, Int(A) and Cl(A) denote the power set of X, the interior of A and the closure of A, respectively. An ideal I on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies the following two properties:

1. $A \in I$ and $B \subset A$ implies $B \in I$;
2. $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X,τ) with an ideal I on X is called an ideal topological space and is denoted by (X,τ,I). Given (X,τ,I), the application $(.)^\star : P(X) \to P(X)$ defined as $A^\star(I,\tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I. Briefly, we will write $A^\star$ for $A^\star(I,\tau)$. In general, $X^\star$ is a proper subset of X. The equality $X = X^\star$ is equivalent to the equality $\tau \cap I = \emptyset$, see [5]. Following [[10]], we call the ideal topological spaces which satisfy this hypothesis Hayashi-Samuels spaces (briefly H.S.S.). Observe that $\mathrm{Cl}^\star(A) = A \cup A^\star(I,\tau)$ defines a Kuratowski closure for a topology $\tau^\star(I)$ (also denoted $\tau^\star$ when there is no chance for confusion), finer than τ. The elements of $\tau^\star$ are called $\tau^\star$-open and the complement of a $\tau^\star$-open is called $\tau^\star$-closed. Observe that a subset A of an ideal topological space (X,τ,I) is $\tau^\star$-closed if and only if $A^\star \subset A$, see [5]. The subset A of (X,τ,I) is said to be I-open [6] if $A \subset \mathrm{Int}(A^\star)$. Note that X is not a I-open set, in general. A is an I-closed set if its complement is an I-open set. The collection of all I-open sets of an ideal topological space (X,τ,I) is denoted by $\mathrm{IO}(X,\tau)$. Following to [10], for a subset A of (X,τ,I) we define $\Lambda_I(A)$ as $\Lambda_I(A) = \cap\{U : A \subset U, U \in \mathrm{IO}(X,\tau)\}$. Also, a subset A is said to be a Λ_I-set if $A = \Lambda_I(A)$, while A is said to be a Λ_I-closed set if $A = U \cap F$, where U is a Λ_I-set and F is an $\tau^\star$-closed set. The complement of a Λ_I-closed set is called Λ_I-open set. In [10] the following implications are shown:

$$I\text{-open} \Longrightarrow \Lambda_I\text{-set} \Longrightarrow \Lambda_I\text{-closed}$$

Lemma 2.1. *[11, Lemma 2] If (X,τ,I) is a H.S.S., the every $\tau^\star$-open set is Λ_I-open.*

Lemma 2.2. *[11, Lemma 3] Let $\{B_\alpha : \alpha \in \Delta\}$ be a collection of subsets of the ideal topological space (X,τ,I). If B_α is Λ_I-open for each $\alpha \in \Delta$, then $\bigcup\{B_\alpha : \alpha \in \Delta\}$ is Λ_I-open.*

Next, we present the definitions and characterizations of Λ_I-continuous, quasi-Λ_I-continuous and Λ_I-irresolute functions given in [11].

Definition 2.3. *Let (X,τ,I) and (Y,σ,J) be two ideal topological spaces. A function $f : (X,\tau,I) \longrightarrow (Y,\sigma,J)$ is said to be:*

1. *Λ_I-continuous, if $f^{-1}(V)$ is a Λ_I-open set in (X,τ,I) for each open set V in (Y,σ,J).*
2. *Quasi-Λ_I-continuous, if $f^{-1}(V)$ is a Λ_I-open set in (X,τ,I) for each $\sigma^\star$-open set V in (Y,σ,J).*
3. *Λ_I-irresolute, if $f^{-1}(V)$ is a Λ_I-open set in (X,τ,I) for each Λ_J-open set V in (Y,σ,J).*

Theorem 2.4. *Let (X,τ,I) and (Y,σ,J) be two ideal topological spaces and $f : (X,\tau,I) \to (Y,\sigma,J)$ be a function. The following statements are equivalent:*

1. *f is Λ_I-continuous (resp. quasi-Λ_I-continuous, Λ_I-irresolute).*
2. *$f^{-1}(B)$ is a Λ_I-closed set in (X,τ,I) for each closed (resp. $\sigma^\star$-closed, Λ_J-closed) set B in (Y,σ,J).*
3. *For each $x \in X$ and each open (resp. $\sigma^\star$-open, Λ_J-open) set V in (Y,σ) such that $f(x) \in V$, there exists a Λ_I-open set U in (X,τ,I) with $x \in U$ such that $f(U) \subset V$.*

Proof. See Theorems 4, 5 and 6 of [11]. □

3 Contra Λ_I-continuous functions

The concept of contra-continuous function in topological spaces was given by Dontchev in [4]. A function $f : X \to Y$ is said to be contra-continuous, if the preimage of each open set in Y is a closed set in X. In this section we use open sets, $\tau^\star$-open sets, Λ_I-open and Λ_I-closed sets to introduce and characterize new variants of contra-continuous function, called contra Λ_I-continuous, contra quasi-Λ_I-continuous and contra Λ_I-irresolute functions.

Definition 3.1. *Let (X,τ,I) and (Y,σ,J) be two ideal topological spaces and $f : (X,\tau,I) \to (Y,\sigma,J)$ be a function. Then, f is said to be:*

1. *Contra Λ_I-continuous, if $f^{-1}(V)$ is a Λ_I-closed subset of X for each open subset V of Y.*
2. *Contra quasi-Λ_I-continuous, if $f^{-1}(V)$ is a Λ_I-closed subset of X for each $\sigma^\star$-open set V of Y.*
3. *Contra Λ_I-irresolute, if $f^{-1}(V)$ is a Λ_I-closed set of X for each Λ_J-open set V of Y.*

Theorem 3.2. *Let (X,τ,I) and (Y,σ,J) be two ideal topological spaces and $f : (X,\tau,I) \to (Y,\sigma,J)$ be a function, where (Y,σ,J) is a H.S.S. If f is contra Λ_I-irresolute, then f is contra quasi-Λ_I-continuous.*

Proof. Let V be a $\sigma^\star$-open subset of Y, then by Lemma 2.1, we have V is a Λ_J-open set of Y and since f is contra Λ_I-irresolute, it follows that $f^{-1}(V)$ is a Λ_I-closed set of X. Therefore, f is contra quasi-Λ_I-continuous. □

In the following example, we show a function that is contra quasi-Λ_I-continuous but is not contra Λ_I-irresolute.

Example 3.3. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset,\{a,c\},X\}$, $I = \{\emptyset,\{c\}\}$, $\sigma = \{\emptyset,X,\{c\},\{b,c\}\}$, $J = \{\emptyset,\{a\}\}$. Then, the collection of all Λ_I-closed sets of X is $\{\emptyset,X,\{b\},\{c\},\{b,c\}\}$, the collection of all $\sigma^\star$-open sets of X is $\{\emptyset,X,\{c\},\{b,c\}\}$, the collection of all Λ_J-open sets of X is $\{\emptyset,X,\{c\},\{b,c\},\{a,c\}\}$ and, we have the identity function $f : (X,\tau,I) \to (X,\sigma,J)$ is contra quasi-Λ_I-continuous, but is not contra Λ_I-irresolute, because $f^{-1}(\{a,c\}) = \{a,c\}$ is not a Λ_I-closed set of X.*

The following example shows that in Theorem 3.2, the condition that (Y, σ, J) is a H.S.S., cannot be omitted.

Example 3.4. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{b, c\}\}$, $I = \{\emptyset, \{b\}\}$. *Note that* (X, τ, I) *is not a H.S.S. because* $\tau \cap I = \{\emptyset, \{b\}\}$. *In addition, the collection of all* Λ_I*-open sets of* X *is* $\{\emptyset, X\}$, *the collection of all* Λ_I*-closed sets of* X *is* $\{\emptyset, X\}$, *the collection of all* $\tau^\star$*-open sets of* X *is* $\{\emptyset, X, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$ *and, we have the identity function* $f : (X, \tau, I) \to (X, \tau, I)$ *is contra* Λ_I*-irresolute, but is not contra quasi-*Λ_I*-continuous, because* $f^{-1}(\{b\})$, $f^{-1}(\{c\})$, $f^{-1}(\{a, c\})$ *and* $f^{-1}(\{b, c\})$ *are not* Λ_I*-closed sets of* X.

Theorem 3.5. *Let* (X, τ, I) *and* (Y, σ, J) *be two ideal topological spaces and* $f : (X, \tau, I) \to (Y, \sigma, J)$ *be a function. If* f *is contra quasi-*Λ_I*-continuous function, then* f *is contra* Λ_I*-continuous.*

Proof. Let V be an open set of Y. Then V is a $\sigma^\star$-open set of Y and, as f is contra quasi-Λ_I-continuous, it follows that $f^{-1}(V)$ is a Λ_I-closed set of X. Hence, f is contra Λ_I-continuous. □

Now we show an example of a contra Λ_I-continuous function that is not contra quasi-Λ_I-continuous.

Example 3.6. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{c\}, \{b, c\}\}$, $I = \{\emptyset, \{c\}\}$ *and* $J = \{\emptyset, \{c\}, \{b, c\}, \{b\}\}$. *Then, the collection of all* $\sigma^\star$*-open sets of* X *is* $\{\emptyset, X, \{c\}, \{b, c\}, \{a\}, \{a, b\}, \{a, c\}\}$, *the collection of all* Λ_I*-closed sets of* X *is* $\{\emptyset, X, \{c\}, \{b, c\}, \{a, c\}, \{b\}\}$ *and, we have the identity function* $f : (X, \tau, I) \to (X, \sigma, J)$ *is contra* Λ_I*-continuous, but is not contra quasi-*Λ_I*-continuous, because* $f^{-1}(\{a\})$ *and* $f^{-1}(\{a, c\})$ *are not* Λ_I*-closed sets of* X.

Corollary 3.7. *Let* (X, τ, I) *and* (Y, σ, J) *be two ideal topological spaces and* $f : (X, \tau, I) \to (Y, \sigma, J)$ *be a function, where* (Y, σ, J) *is a H.S.S. If* f *is contra* Λ_I*-irresolute, then* f *is contra* Λ_I*-continuous.*

Proof. The proof follows from Theorems 3.2 and 3.5. □

According to the previous results, given an H.S.S., we obtain the following diagram, where none of the implications is reversible:

$$\begin{array}{ccc} \text{Contra } \Lambda_I\text{-irresolute} & \Longrightarrow & \text{Contra quasi } \Lambda_I\text{-continuous} \\ & & \Downarrow \\ & & \text{Contra } \Lambda_I\text{-continuous} \end{array}$$

Theorem 3.8. *Let* (X, τ, I), (Y, σ, J) *and* (Z, θ, K) *be three ideal topological spaces,* $f : (X, \tau, I) \to (Y, \sigma, J)$ *and* $g : (Y, \sigma, J) \to (Z, \theta, K)$ *be two functions. The following statements hold:*

1. $g \circ f$ *is contra* Λ_I*-irresolute, if* f *is* Λ_I*-irresolute and* g *is contra* Λ_J*-irresolute.*
2. $g \circ f$ *is contra* Λ_I*-irresolute, if* f *is contra* Λ_I*-irresolute and* g *is* Λ_J*-irresolute.*
3. $g \circ f$ *is* Λ_I*-irresolute, if* f *is contra* Λ_I*-irresolute and* g *is contra* Λ_J*-irresolute.*
4. $g \circ f$ *is contra* Λ_I*-continuous, if* f *is contra* Λ_I*-continuous and* g *es continuous.*
5. $g \circ f$ *is contra* Λ_I*-continuous, if* f *is contra* Λ_I*-irresolute and* g *es* Λ_I*continuous.*
6. $g \circ f$ *is contra* Λ_I*-continuous, if* f *is* Λ_I*-irresolute and* g *is contra* Λ_J*-continuous.*
7. $g \circ f$ *is contra* Λ_I*-continuous, if* f *is* Λ_I*-continuous and* g *is contra continuous.*
8. $g \circ f$ *is* Λ_I*-continuous, if* f *is contra* Λ_I*-continuous and* g *is contra continuous.*
9. $g \circ f$ *is* Λ_I*-continuous, if* f *is contra* Λ_I*-irresolute and* g *is contra* Λ_J*-continuous.*
10. $g \circ f$ *is contra quasi-*Λ_I*-continuous, if* f *is* Λ_I*-irresolute and* g *is contra quasi-*Λ_J*-continuous.*
11. $g \circ f$ *is contra quasi-*Λ_I*-continuous, if* f *is contra* Λ_I*-irresolute and* g *is quasi* Λ_J*-continuous.*
12. $g \circ f$ *is quasi* Λ_I*-continuous, if* f *is contra* Λ_I*-irresolute and* g *is contra quasi-*Λ_J*-continuous.*

Proof. (1) Let V be a Λ_K-open set of Z. Since g is contra Λ_J-irresolute, then $g^{-1}(V)$ is a Λ_J-closed set of Y and as f is Λ_I-irresolute, then by Theorem 2.4, we have $f^{-1}(g^{-1}(V))$ is a Λ_I-closed set of X. But $(g \circ f)^{-1}(V) = (f^{-1} \circ g^{-1})(V) = (f^{-1}(g^{-1}(V))$ and hence, $(g \circ f)^{-1}(V)$ is a Λ_I-closed set of X. This shows that gof is contra Λ_I-irresolute.

The proofs of (2)-(12) are analogous to the case (1). □

In the next three theorems, we characterize contra Λ_I-continuous, contra quasi-Λ_I-continuous and contra Λ_I-irresolute functions, respectively.

Theorem 3.9. *Let* $f : (X, \tau, I) \to (Y, \sigma)$ *be a function. The following statements are equivalent:*

1. f *is contra* Λ_I*-continuous.*
2. $f^{-1}(F)$ *is a* Λ_I*-open set of* X *for each closed set* F *of* Y.

3. *For each $x \in X$ and each closed set F of Y such that $f(x) \in F$, there exists a Λ_I-open set U of X with $x \in U$ and $f(U) \subset F$.*

Proof. (1) $\Rightarrow$ (2) Let F be any closed subset of Y, then $V = Y - F$ is an open subset of Y and since f is contra Λ_I-continuous, $f^{-1}(V)$ is a Λ_I-closed subset of X, but $f^{-1}(V) = f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F) = X - f^{-1}(F)$ and hence, $f^{-1}(F)$ is a Λ_I-open subset of X.

(2) $\Rightarrow$ (3) Let V be any open subset of Y, then $F = Y - V$ is a closed subset of Y and by hypothesis, we have $f^{-1}(F)$ is a Λ_I-open subset of X, but $f^{-1}(F) = f^{-1}(Y-V) = f^{-1}(Y)-f^{-1}(V) = X-f^{-1}(V)$ and so, $f^{-1}(V)$ is a Λ_I-closed subset of X. This shows that f is contra Λ_I-continuous.

(1) $\Rightarrow$ (3) Let $x \in X$ and F be a closed subset of Y such that $f(x) \in F$, then $x \in f^{-1}(F)$ and since f is a contra Λ_I-continuous function, $f^{-1}(F)$ is a Λ_I-open subset of X. If $U = f^{-1}(F)$, then U is a Λ_I-open subset of X such that $x \in U$ and $f(U) = f(f^{-1}(F)) \subset F$.

(3) $\Rightarrow$ (1) Let F be any closed subset of Y and $x \in f^{-1}(F)$, then $f(x) \in F$ and by (3), there exists a Λ_I-open subset U_x of X such that $x \in U_x$ and $f(U_x) \subset F$. Thus, $x \in U_x \subset f^{-1}(f(U)) \subset f^{-1}(F)$ and hence, $f^{-1}(F) = \bigcup\{U_x : x \in f^{-1}(F)\}$. By Lemma 2.2, we obtain that $f^{-1}(F)$ is a Λ_I-open subset of X. □

Theorem 3.10. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function, the following statements are equivalent:*

1. *f is contra quasi-Λ_I-continuous.*
2. *$f^{-1}(F)$ is a Λ_I-open set of X for each $\sigma^\star$-closed set F of Y.*
3. *For each $x \in X$ and for each $\sigma^\star$-closed set F of Y such that $f(x) \in F$, there exists a Λ_I-open set U of X with $x \in U$ and $f(U) \subset F$..*

Proof. It is proven in a similar way to the Theorema 3.9. □

Theorem 3.11. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function, the following statements are equivalent:*

1. *f is contra Λ_I-irresolute.*
2. *$f^{-1}(F)$ is a Λ_I-open set of X for each Λ_J-closed set F of Y.*
3. *For each $x \in X$ and each Λ_J-closed set F of Y such that $f(x) \in F$, there exists a Λ_I-open set U of X with $x \in U$ and $f(U) \subset F$.*

Proof. It is proven in a similar way to the Theorema 3.9. □

4 Preservation of notions under direct or inverse images

In this section we study the behavior of some topological notions under direct or inverse images of the new variants of contra-continuity introduced in the Section 3. Before continuing our study, we must remember the following definitions introduced in [11]. An ideal topological space (X, τ, I) is said to be Λ_I-connected (resp. $\tau^\star$-connected) if X cannot be written as a disjoint union of two nonempty Λ_I-open (resp. $\tau^\star$-open) sets.

Theorem 4.1. *If $f : (X, \tau, I) \to (Y, \sigma)$ is a contra Λ_I-continuous and surjective function and (X, τ, I) is a Λ_I-connected space having more than one element, then (Y, σ) is not a discrete space.*

Proof. Suppose that (Y, σ) is a discrete space and A be any nonempty proper subset of Y. So, A is an open and closed subset of Y and as f is a contra Λ_I-continuous function, it follows that $f^{-1}(A)$ is a Λ_I-open and Λ_I-closed set of X. Since (X, τ, I) is a Λ_I-connected space, by [11, Theorem 11], $\emptyset$ and X are the only subsets of X which are both Λ_I-open and Λ_I-closed . Thus, $f^{-1}(A) = \emptyset$ or $f^{-1}(A) = X$. If $f^{-1}(A) = \emptyset$, then this contradicts the fact that $A \neq \emptyset$ and f is surjective. If $f^{-1}(A) = X$, then f is not a function and, hence, (Y, σ) is not a discrete space. □

Theorem 4.2. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is a surjective function, then the following properties hold:*

1. *If f is contra Λ_I-irresolute and (X, τ, I) is a Λ_I-connected space, then (Y, σ, J) is a Λ_J-connected space.*
2. *If f is a contra quasi-Λ_I-continuous function and (X, τ, I) is a Λ_I-connected space, then (Y, σ, J) is a $\sigma^\star$-connected space.*
3. *If f is a contra Λ_I-continuous function and (X, τ, I) is a Λ_I-connected space, then (Y, σ) is connected.*

Proof. (1) Assume that (X, τ, I) is a Λ_I-connected space and $f : (X, \tau, I) \to (Y, \sigma)$ is surjective contra Λ_I-irresolute function. Suppose that (Y, σ) is not Λ_J-connected. Then, there exist nonempty Λ_J-open subsets A and B of Y such that $A \cap B = \emptyset$ and $Y = A \cup B$. Thus, $B = Y - A$ and $A = Y - B$ are nonempty Λ_J-closed subsets of Y, and as f is a contra Λ_I-irresolute function, we have $f^{-1}(A)$ and $f^{-1}(B)$ are Λ_I-open subsets of X such that $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ and $f^{-1}(A) \cup f^{-1}(B) = X$. This contradicts the fact that (X, τ, I) is a Λ_I-connected space. Therefore, (Y, σ, J) is Λ_J-connected.

The proofs of (2) and (3) are similar to case (1). □

Theorem 4.3. *An ideal topological space (X, τ, I) is Λ_I-connected, if each contra Λ_I-continuous function $f : (X, \tau, I) \to (Y, \sigma)$, where (Y, σ) is a T_0-space, is a constant function.*

Proof. Suppose that (X, τ, I) is not a Λ_I-connected space and each contra Λ_I-continuous function $f : (X, \tau, I) \to (Y, \sigma)$, where (Y, σ) is a T_0-space, is a constant function. Since (X, τ, I) is not Λ_I-connected, by [11, Theorem 11], there exists a nonempty proper subset A of X which is both Λ_I-open and Λ_I-closed. Let $Y = \{a, b\}$, $\sigma = \{Y, \emptyset, \{a\}, \{b\}\}$ be a topology on Y and $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function such that $f(A) = \{a\}$ and $f(X - A) = \{b\}$. Then f is a non-constant contra Λ_I-continuous function such that (Y, σ) is a T_0-space, which is a contradiction. Therefore, (X, τ, I) is a Λ_I-connected space. □

Theorem 4.4. *If $f : (X, \tau, I) \to (Y, \sigma)$ is a contra Λ_I-continuous function and (Y, σ) is a regular space, then f is Λ_I-continuous.*

Proof. Let $x \in X$ and V be an open set of Y such that $f(x) \in V$. Since (Y, σ) is a regular space, there exists an open set W of Y such that $f(x) \in W \subset Cl(W) \subset V$. Now, since f is a contra Λ_I-continuous function, then by Theorem 2.4, there exists a Λ_I-open set U of X such that $x \in U$ and $f(U) \subset Cl(W) \subset V$. By Theorem 3.9, we conclude that f is a Λ_I-continuous function. □

Definition 4.5. *An ideal topological space (X, τ, I) is said to be Λ_I-normal, if for each pair of disjoint closed subsets A and B of X, there exist Λ_I-open subsets U and V of X such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

Remark 4.6. *Let (X, τ, I) be a H.S.S. If (X, τ) is normal, then (X, τ, I) is Λ_I-normal.*

Recall that a topological space (X, τ) is said to be ultra normal [12], if for each pair of nonempty disjoint closed subsets A and B of X, there exist two clopen subsets G and H of X such that $A \subset G$, $B \subset H$ and $U \cap V = \emptyset$.

Theorem 4.7. *If $f : (X, \tau, I) \to (Y, \sigma)$ is an injective, closed and contra Λ_I-continuous function and (Y, σ) is an ultra normal space, then (X, τ, I) is a Λ_I-normal space.*

Proof. Let A and B be two disjoint closed subsets of X. Since f is closed and injective, then $f(A)$ and $f(B)$ are disjoint closed subsets of Y and as (Y, σ) is an ultra normal space, there exist two clopen subsets G and H of Y such that $f(A) \subset G$, $f(B) \subset H$ and $G \cap H = \emptyset$. Now, since f es contra Λ_I-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are Λ_I-closed subsets of X and also, $A \subset f^{-1}(f(A)) \subset f^{-1}(G)$, $B \subset f^{-1}(f(B)) \subset f^{-1}(H)$ and $f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\emptyset) = \emptyset$. In consequence, (X, τ, I) is Λ_I-normal. □

Definition 4.8. *A space (X, τ, I) is said to be Λ_I-T_2, if for each pair of distinct points $x, y \in X$, there exist Λ_I-open subsets U and V of X such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.*

Remark 4.9. *Let (X, τ, I) be a H.S.S. If (X, τ) is T_2, then (X, τ, I) is Λ_I-T_2.*

Recall that a topological space (X, τ) is said to be Urysohn [13], if for each pair of distinct points $x, y \in X$, there exist two open subsets U and V of X such that $x \in U$, $y \in V$ and $Cl(U) \cap Cl(V) = \emptyset$. The following result shows that, the inverse image of an Urysohn space under an injective and contra Λ_I-continuous function, is a Λ_I-T_2-space.

Theorem 4.10. *If $f : (X, \tau, I) \to (Y, \sigma)$ is an injective and contra Λ_I-continuous function and (Y, σ) is an Urysohn space, then (X, τ, I) is a Λ_I-T_2-space.*

Proof. Consider x and y be two points of X with $x \neq y$. By the injectivity of f, we have $f(x) \neq f(y)$ and as (Y, σ) is an Urysohn space, there exist two open subsets U and V of Y such that $f(x) \in U$, $f(y) \in V$ and $Cl(U) \cap Cl(V) = \emptyset$. By Theorem 3.9, there exist two Λ_I-open subsets A and B of X such that $x \in A$, $y \in B$, $f(A) \subset Cl(U)$ and $f(B) \subset Cl(V)$. Thus, $f(A) \cap f(B) \subset Cl(U) \cap Cl(V) = \emptyset$, which implies that $f(A \cap B) = f(A) \cap f(B) = \emptyset$ and hence, $A \cap B = \emptyset$. This shows that (X, τ, I) is a Λ_I-T_2-space. □

Recall that a topological space (X, τ) is locally indiscrete [3], if each open subset of X is closed. In the following definition some modifications of a locally indiscrete space are introduced in order to investigate related properties with the functions defined in the Section 3.

Definition 4.11. *We say that an ideal topological space (X, τ, I) is:*

1. *Locally $\tau^\star$-indiscrete, if each $\tau^\star$-open subset of X is closed in X.*
2. *Locally Λ_I-indiscrete, if each Λ_I-open subset of X is closed in X.*
3. *Λ_I-space, if each Λ_I-open subset of X is open in X.*

Proposition 4.12. *Let (X, τ, I) be an ideal topological space. The following statements hold:*

1. *If (X, τ, I) is an H.S.S. locally Λ_I-indiscrete, then (X, τ, I) is locally $\tau^\star$-indiscrete.*

2. *If (X, τ, I) is a locally $\tau^\star$-indiscrete space, then (X, τ) is locally indiscrete.*

3. *(X, τ, I) is locally $\tau^\star$-indiscrete space if and only if each $\tau^\star$-closed subset of X is open in X.*

4. *(X, τ, I) is locally Λ_I-indiscrete space if and only if each Λ_I-closed subset of X is open in X.*

5. *(X, τ, I) is Λ_I-space if and only if each Λ_I-closed subset of X is closed in X.*

Theorem 4.13. *Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and $f : (X, \tau, I) \to (Y, \sigma, J)$ be a contra Λ_I-continuous function. The following statements hold:*

1. *If (X, τ, I) is locally Λ_I-indiscrete, then f is a continuous function.*

2. *If (X, τ, I) is a Λ_I-space, then f is a contra-continuous function.*

Proof. (1) Let B be a closed subset of Y. Since f is a contra Λ_I-continuous function, $f^{-1}(B)$ is a Λ_I-open subset of X and as (X, τ, I) is locally Λ_I-indiscrete, then $f^{-1}(B)$ is a closed subset of X. Therefore, f is a continuous function.

The proof of (2) is similar to case (1). □

The following result shows that, the direct image of a Λ_I-space under a surjective, closed and contra Λ_I-irresolute (resp. contra quasi-Λ_I-continuous, contra Λ_I-continuous) function is a locally Λ_J-indiscrete (resp. locally $\sigma^\star$-indiscrete, locally indiscrete) space.

Theorem 4.14. *Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and $f : (X, \tau, I) \to (Y, \sigma, J)$ be a surjective and closed function. The following statements hold:*

1. *If f is contra Λ_I-irresolute and (X, τ, I) is a Λ_I-space, then (Y, σ, J) is a locally Λ_J-indiscrete.*

2. *If f is contra quasi-Λ_I-continuous and (X, τ, I) is a Λ_I-space, then (Y, σ, J) is a locally $\sigma^\star$-indiscrete.*

3. *If f is contra Λ_I-continuous and (X, τ, I) is a Λ_I-space, then (Y, σ) is a locally indiscrete.*

Proof. Straightforward. □

Recall that a topological space (X, τ) is said to be strongly S-closed [2], if each closed cover of X has a finite subcover. Now we introduce a modification of a strongly S-closed space using Λ_I-closed sets.

Definition 4.15. *We say that an ideal topological space (X, τ, I) is strongly S-Λ_I-closed, if each cover of X by Λ_I-closed sets has a finite subcover.*

Remark 4.16. *Let (X, τ, I) be a H.S.S. If (X, τ, I) is strongly S-Λ_I-closed, then (X, τ) is strongly S-closed.*

The notions of Λ_I-compact space and $\tau^\star$-compact space were introduced in [11]. We say that an ideal topological space (X, τ, I) is Λ_I-compact (resp. $\tau^\star$-compact), if each cover of X by Λ_I-open (resp. $\tau^\star$-open) sets has a finite subcover. The following result shows that, the direct image of a strongly S-Λ_I-closed space under a surjective and contra Λ_I-irresolute (resp. contra quasi-Λ_I-continuous, contra Λ_I-continuous) function, is a Λ_J-compact (resp. $\sigma^\star$-compact, compact) space.

Theorem 4.17. *Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and $f : (X, \tau, I) \to (Y, \sigma, J)$ be a surjective function. The following statements hold:*

1. *If f is contra Λ_I-irresolute and (X, τ, I) is strongly S-Λ_I-closed, then (Y, σ, J) is Λ_J-compact.*

2. *If f is contra quasi-Λ_I-continuous and (X, τ, I) is strongly S-Λ_I-closed, then (Y, σ, J) is $\sigma^\star$-compact.*

3. *If f is contra Λ_I-continuous and (X, τ, I) is strongly S-Λ_I-closed, then (Y, σ) is compact.*

Proof. (1) Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of Y by Λ_J-open sets. Since f is contra Λ_I-irresolute, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a cover of X by Λ_I-closed and as (X, τ, I) is strongly S-Λ_I-closed, there exists a finite subcollection $\{f^{-1}(V_{\alpha_i}) : i = 1, \dots, n\}$ of $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ such that $X = \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$. Thus, $Y = f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right) = \bigcup_{i=1}^{n} f(f^{-1}(V_{\alpha_i})) \subset \bigcup_{i=1}^{n} V_{\alpha_i}$ and hence, (Y, σ, J) is Λ_J-compact.

The proofs of (2) and (3) are similar to case (1). □

5 Conclusion

The notion of continuous function is one of the most important in the study of general topology and the lines of research derived from it by the use of ideals on topological spaces. This has made it possible to establish some interesting advances in topics related to computing and image design, particularly in digital topology. The antagonistic concept to the continuity of a function, is the contra-continuity of a function, which appeared due to the need to analyze the behavior of certain spaces defined in terms of coverings. In this research work, new classes of contra-continuous functions were studied, which were defined using the notions of open set, $\tau^\star$-open set, Λ_I-open set and Λ_I-closed set. The relationships between these classes of functions were studied and some compositions involving them were analyzed; as well as characterizations of such functions were established. Similarly, the behavior of modifications of connected spaces, normal spaces, T_2-spaces, λ-spaces, locally indiscrete spaces and strongly S-closed spaces, under direct or inverse images of the new classes of functions defined in this work. It should be noted that in some results obtained that involved the use of notions described by means of I -open sets, the additional condition had to be requested that the topological space (X, τ, I) used was Hayashi Samuels to guarantee that X would turn out to be an I-open set and, therefore, each $\tau^\star$-open set would be Λ_I-open.

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