Continuity for multifunctions in ideal topological spaces

CHAWALIT BOONPOK
Mathematics and Applied Mathematics Research Unit
Department of Mathematics, Faculty of Science, Mahasarakham University
Maha Sarakham, 44150
THAILAND

PRAPART PUE-ON
Mathematics and Applied Mathematics Research Unit
Department of Mathematics, Faculty of Science, Mahasarakham University
Maha Sarakham, 44150
THAILAND

Abstract: This article presents the concept of $\imath^\star$-continuous multifunctions in ideal topological spaces. Especially, several characterizations of $\imath^\star$-continuous multifunctions are investigated. Moreover, the relationships between $\imath^\star$-continuity and the other types of continuity for multifunctions are established.

Key–Words: $\star$-open set, $\imath^\star$-continuous multifunction, almost $\imath^\star$-continuous multifunction, weakly $\imath^\star$-continuous multifunction

Received: April 8, 2020. Revised: December 5, 2020. Accepted: December 19, 2020. Published: December 29, 2020.

1 Introduction

The branch of mathematics called topology is concerned with all questions directly or indirectly related to continuity. Continuity is an important concept for the study and investigation in the theory of classical point set topology. Generalization of this concept by using weaker forms of open sets. The notion of weak continuity due to Levine [13] is one of the most important weak forms of continuity in topological spaces. Rose [17] introduced the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. The present authors introduced and studied weakly quasi-continuous multifunctions [14], weakly precontinuous multifunctions [15] and weakly β-continuous multifunctions [16]. These multifunctions have similar properties. The analogy in their definitions and results suggests the need of formulating a unified theory.

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [12] and Vaidyanathswamy [19]. The topology τ of a space is enlarged to a topology $\tau^\star$ using an ideal $\mathscr{I}$ whose members are disjoint with the members of τ. Every topological space is an ideal topological space and all the results of ideal topological spaces are generalizations of the results established in topological spaces. Some early applications of ideal topological spaces can be found in various branches of mathematics, like a generalization of Cantor-Bendixson theorem by Freud [7], or in measure theory by Scheinberg [18]. In 1990, Janković and Hamlett [11] introduced the notion of $\mathscr{I}$-open sets in ideal topologial spaces. Abd El-Monsef et al. [1] further investigated $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açikgöz et al. [2] introduced and investigated the notions of weakly-$\mathscr{I}$-continuous and weak*-$\mathscr{I}$-continuous functions in ideal topological spaces. Donthev [6] introduced the notion of pre-$\mathscr{I}$-open sets and obtained a decomposition of $\mathscr{I}$-continuity. Hatir and Noiri [10] introduced the notions of semi-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets and β-$\mathscr{I}$-open sets via idealization and using these sets obtained new decompositions of continuity. In 2005, Hatir and Noiri [9] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuity. Hatir et al. [8] introduced and investigated the notions of strong β-$\mathscr{I}$-open sets and strongly β-$\mathscr{I}$-continuous functions.

The purpose of the present article is to introduce the notions of $\imath^\star$-continuous multifunctions, almost $\imath^\star$-continuous multifunctions and weakly $\imath^\star$-continuous multifunctions. In particular, some characterizations of $\imath^\star$-continuous multifunctions, almost $\imath^\star$-continuous multifunctions and weakly $\imath^\star$-continuous multifunctions are investigated. Furthermore, the relationships between $\imath^\star$-continuity, almost $\imath^\star$-continuity and weak $\imath^\star$-continuity are discussed.

2 Preliminaries

Throughout the present paper, spaces (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. In a topological space (X,τ), the closure and the interior of any subset A of X will denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathscr{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A\in\mathscr{I}$ and $B\subseteq A$ implies $B\in\mathscr{I}$; (2) $A\in\mathscr{I}$ and $B\in\mathscr{I}$ implies $A\cup B\in\mathscr{I}$. A topological space (X,τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X,\tau,\mathscr{I})$. For an ideal topological space $(X,\tau,\mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows: $A^\star(\mathscr{I})=\{x\in X : U\cap A\notin\mathscr{I}$ for every open neighbourhood U of $x\}$. In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [12], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\mathrm{Cl}^\star(A)=A^\star\cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$. For every ideal topological space $(X,\tau,\mathscr{I})$, there exists a topology $\tau^\star(\mathscr{I})$ finer than τ, generated by $\mathscr{B}(\tau,\mathscr{I})=\{U-I_0 \mid U\in\tau$ and $I_0\in\mathscr{I}\}$, but in general $\mathscr{B}(\tau,\mathscr{I})$ is not always a topology [11]. A subset A is said to be $\star$-closed [11] if $A^\star\subseteq A$. The complement of a $\star$-closed set is said to be $\star$-open. The interior of a subset A in $(X,\tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$.

By a multifunction $F:X\to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x)\neq\emptyset$ for all $x\in X$. For a multifunction $F:X\to Y$, following [3], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B)=\{x\in X \mid F(x)\subseteq B\}$$

and $F^-(B)=\{x\in X \mid F(x)\cap B\neq\emptyset\}$. In particular, $F^-(y)=\{x\in X \mid y\in F(x)\}$ for each point $y\in Y$. For each $A\subseteq X$, $F(A)=\cup_{x\in A}F(x)$. Then F is said to be surjection if $F(X)=Y$, or equivalent, if for each $y\in Y$ there exists $x\in X$ such that $y\in F(x)$ and F is called injection if $x\neq y$ implies $F(x)\cap F(y)=\emptyset$.

Let $\mathscr{P}(X)$ be the collection of all nonempty subsets of X. For any $\star$-open set V of an ideal topological space $(X,\tau,\mathscr{I})$, we denote

$$V^+=\{B\in\mathscr{P}(X) \mid B\subseteq V\}$$

and $V^-=\{B\in\mathscr{P}(X) \mid B\cap V\neq\emptyset\}$.

Definition 1. [5] *A subset A of an ideal topological space $(X,\tau,\mathscr{I})$ is said to be:*

(i) R-$\mathscr{I}^\star$-open if $A=\mathrm{Int}^\star(\mathrm{Cl}^\star(A))$;

(ii) R-$\mathscr{I}^\star$-closed if its complement is R-$\mathscr{I}^\star$-open;

(iii) $\mathscr{I}^\star$-preopen if $A\subseteq\mathrm{Int}^\star(\mathrm{Cl}^\star(A))$;

(iv) $\mathscr{I}^\star$-preclosed if its complement is $\mathscr{I}^\star$-preopen.

Definition 2. [4] *A subset A of an ideal topological space $(X,\tau,\mathscr{I})$ is said to be:*

(i) semi-$\mathscr{I}^\star$-open if $A\subseteq\mathrm{Cl}^\star(\mathrm{Int}^\star(A))$;

(ii) semi-$\mathscr{I}^\star$-closed if its complement is semi-$\mathscr{I}^\star$-open;

(iii) semi-$\mathscr{I}^\star$-preopen if $A\subseteq\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(A)))$;

(iv) semi-$\mathscr{I}^\star$-preclosed if its complement is semi-$\mathscr{I}^\star$- preopen.

3 On characterizations of $\imath^\star$-continuous multifunctions

In this section, we introduce the concept of $\imath^\star$-continuous multifunctions and investigate some characterizations of $\imath^\star$-continuous multifunctions.

Definition 3. *A multifunction $F:(X,\tau,\mathscr{I})\to(Y,\sigma,\mathscr{J})$ is said to be $\imath^\star$-continuous if for each $x\in X$ and each $\star$-open sets V_1,V_2 of Y such that $F(x)\in V_1^+\cap V_2^-$, there exists a $\star$-open set U of X containing x such that $F(U)\subseteq V_1$ and $F(z)\cap V_2\neq\emptyset$ for every $z\in U$.*

Theorem 4. *For a multifunction $F:(X,\tau,\mathscr{I})\to(Y,\sigma,\mathscr{J})$, the following properties are equivalent:*

(1) F is $\imath^\star$-continuous;

(2) $x\in\mathrm{Int}^\star[F^+(V_1)\cap F^-(V_2)]$ for every $\star$-open set V_1,V_2 of Y such that $F(x)\subseteq V_1$ and

$$F(x)\cap V_2\neq\emptyset;$$

(3) $F^+(V_1)\cap F^-(V_2)$ is $\star$-open in X for any $\star$-open set V_1,V_2 of Y;

(4) $F^-(K_1)\cup F^+(K_2)$ is $\star$-closed in X for any $\star$-closed sets K_1,K_2 of Y;

(5)

$$\mathit{Cl}^\star[F^-(B_1)\cup F^+(B_2)] \subseteq F^-(\mathit{Cl}^\star(B_1))\cup F^+(\mathit{Cl}^\star(B_2))$$

for any subsets B_1, B_2 of Y;

(6)

$$F^-(\mathit{Int}^\star(B_1))\cap F^+(\mathit{Int}^\star(B_2)) \subseteq \mathit{Int}^\star[F^-(B_1)\cap F^+(B_2)]$$

for any subsets B_1, B_2 of Y.

Proof. (1) $\Rightarrow$ (2): Let V_1, V_2 be any $\star$-open sets of Y such that $F(x)\subseteq V_1$ and $F(x)\cap V_2\neq\emptyset$. Then, there exists a $\star$-open set U of X containing x such that $F(U)\subseteq V_1$ and $F(z)\cap V_2\neq\emptyset$ for every $z\in U$. Thus, $U\subseteq F^+(V_1)\cap F^-(V_2)$ and hence

$$x\in \mathrm{Int}^\star[F^+(V_1)\cap F^-(V_2)].$$

(2) $\Rightarrow$ (3): Let V_1, V_2 be any $\star$-open sets of Y and $x\in F^+(V_1)\cap F^-(V_2)$. Then $F(x)\subseteq V_1$ and $F(x)\cap V_2\neq\emptyset$. By (2), we have

$$x\in \mathrm{Int}^\star[F^+(V_1)\cap F^-(V_2)]$$

and hence

$$F^+(V_1)\cap F^-(V_2)\subseteq \mathrm{Int}^\star[F^+(V_1)\cap F^-(V_2)].$$

This shows that $F^+(V_1)\cap F^-(V_2)$ is $\star$-open in X.

(3) $\Rightarrow$ (4): This follows from the fact that $F^-(Y-B)=X-F^+(B)$ and

$$F^+(Y-B)=X-F^-(B)$$

for every subset B of Y.

(4) $\Rightarrow$ (5): Let B_1, B_2 be any subsets of Y. Then $\mathrm{Cl}^\star(B_1)$ and $\mathrm{Cl}^\star(B_2)$ are $\star$-closed in Y. By (4), we have

$$\begin{aligned}&\mathrm{Cl}^\star[F^-(B_1)\cup F^+(B_2)]\\ &\subseteq \mathrm{Cl}^\star[F^-(\mathrm{Cl}^\star(B_1))\cup F^+(\mathrm{Cl}^\star(B_2))]\\ &= F^-(\mathrm{Cl}^\star(B_1))\cup F^+(\mathrm{Cl}^\star(B_2)).\end{aligned}$$

(5) $\Rightarrow$ (6): Let B_1, B_2 be any subsets of Y. By (5), we have

$$\begin{aligned}&X-\mathrm{Int}^\star[F^-(B_1)\cap F^+(B_2)]\\ &=\mathrm{Cl}^\star[X-[F^-(B_1)\cap F^+(B_2)]]\\ &=\mathrm{Cl}^\star[(X-F^-(B_1))\cup(X-F^+(B_2))]\\ &=\mathrm{Cl}^\star[F^+(Y-B_1)\cup F^-(Y-B_2)]\\ &\subseteq F^+(\mathrm{Cl}^\star(Y-B_1))\cup F^-(\mathrm{Cl}^\star(Y-B_2))\\ &=F^+(Y-\mathrm{Int}^\star(B_1))\cup F^-(Y-\mathrm{Int}^\star(B_2))\\ &=[X-F^-(\mathrm{Int}^\star(B_1))]\cup[X-F^+(\mathrm{Int}^\star(B_2))]\\ &=X-[F^-(\mathrm{Int}^\star(B_1))\cap F^+(\mathrm{Int}^\star(B_2))]\end{aligned}$$

and hence

$$\begin{aligned}&F^-(\mathrm{Int}^\star(B_1))\cap F^+(\mathrm{Int}^\star(B_2))\\ &\subseteq \mathrm{Int}^\star[F^-(B_1)\cap F^+(B_2)].\end{aligned}$$

(6) $\Rightarrow$ (1): Let $x\in X$ and V_1, V_2 be any $\star$-open sets of Y such that $F(x)\subseteq V_1$ and $F(x)\cap V_2\neq\emptyset$. By (6), we have

$$F^+(V_1)\cap F^-(V_2)\subseteq \mathrm{Int}^\star[F^+(V_1)\cap F^-(V_2)].$$

Now, put $U=F^+(V_1)\cap F^-(V_2)$, then U is a $\star$-open set of X containing x such that $F(U)\subseteq V_1$ and

$$F(z)\cap V_2\neq\emptyset$$

for each $z\in U$. This shows that F is $\imath^\star$-continuous. □

4 On characterizations almost $\imath^\star$-continuous multifunctions

In this section, we introduce the concept of almost $\imath^\star$-continuous multifunctions and investigate several characterizations of such multifunctions.

Definition 5. *A multifunction $F:(X,\tau,\mathscr{I})\to(Y,\sigma,\mathscr{J})$ is said to be almost $\imath^\star$-continuous if for each $x\in X$ and each $\star$-open set V_1, V_2 of Y such that $F(x)\in V_1^+\cap V_2^-$, there exists a $\star$-open set U of X containing x such that $F(U)\subseteq \mathit{Int}(\mathit{Cl}^\star(V_1))$ and $F(z)\cap \mathit{Int}(\mathit{Cl}^\star(V_2))\neq\emptyset$ for every $z\in U$.*

Remark 6. *For a multifunction $F:(X,\tau,\mathscr{I})\to(Y,\sigma,\mathscr{J})$, the following implication holds:*

$\imath^\star$-continuity $\Rightarrow$ almost $\imath^\star$-continuity.

The converse of the implication is not true in general. We give example for the implication as follows.

Example 7. *Let $X=\{1,2,3\}$ with a topology $\tau=\{\emptyset,\{1\},\{2\},\{1,2\},X\}$ and an ideal $\mathscr{I}=\{\emptyset,\{3\}\}$. Let $Y=\{a,b,c\}$ with a topology $\sigma=\{\emptyset,\{a\},\{a,b\},Y\}$ and an ideal $\mathscr{J}=\{\emptyset,\{c\}\}$. A multifunction $F:(X,\tau,\mathscr{I})\to(Y,\sigma,\mathscr{J})$ is defined as follows: $F(2)=\{c\}$ and $F(1)=\{a\}$ and $F(3)=\{a,b\}$. Then F is almost $\imath^\star$-continuous but F is not $\imath^\star$-continuous.*

Theorem 8. *For a multifunction $F:(X,\tau,\mathscr{I})\to(Y,\sigma,\mathscr{J})$, the following properties are equivalent:*

(1) F is almost $\imath^\star$-continuous;

(2)

$$F^+(V_1) \cap F^-(V_2) \subseteq \mathit{Int}^\star[F^+(\mathit{Int}(\mathit{Cl}^\star(V_1))) \cap F^-(\mathit{Int}(\mathit{Cl}^\star(V_2)))]$$

for every $\star$-open sets V_1, V_2 of Y;

(3)

$$\mathit{Cl}^\star[F^-(\mathit{Cl}(\mathit{Int}^\star(K_1))) \cup F^+(\mathit{Cl}(\mathit{Int}^\star(K_2)))] \subseteq F^-(K_1) \cup F^+(K_2)$$

for every $\star$-closed sets K_1, K_2 of Y;

(4)

$$\mathit{Cl}^\star[F^-(\mathit{Cl}(\mathit{Int}^\star(\mathit{Cl}^\star(B_1)))) \cup F^+(\mathit{Cl}(\mathit{Int}^\star(\mathit{Cl}^\star(B_2))))] \subseteq F^-(\mathit{Cl}^\star(B_1)) \cup F^+(\mathit{Cl}^\star(B_2))$$

for every subsets B_1, B_2 of Y;

(5)

$$F^+(\mathit{Int}^\star(B_1)) \cap F^-(\mathit{Int}^\star(B_2)) \subseteq \mathit{Int}^\star[F^+(\mathit{Int}(\mathit{Cl}^\star(\mathit{Int}^\star(B_1)))) \cap F^-(\mathit{Int}(\mathit{Cl}^\star(\mathit{Int}^\star(B_2))))]$$

for every subsets B_1, B_2 of Y.

Proof. $(1) \Rightarrow (2)$: Let V_1, V_2 be any $\star$-open sets of Y such that $x \in F^+(V_1) \cap F^-(V_2)$. Then

$$F(x) \in V_1^+ \cap V_2^-$$

and hence there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \mathrm{Int}(\mathrm{Cl}^\star(V_1))$ and

$$F(z) \cap \mathrm{Int}(\mathrm{Cl}^\star(V_2)) \neq \emptyset$$

for each $z \in U$. Thus,

$$U \subseteq F^+(\mathrm{Int}(\mathrm{Cl}^\star(V_1))) \cap F^-(\mathrm{Int}(\mathrm{Cl}^\star(V_2)))$$

and hence

$$x \in \mathrm{Int}^\star[F^+(\mathrm{Int}(\mathrm{Cl}^\star(V_1))) \cap F^-(\mathrm{Int}(\mathrm{Cl}^\star(V_2)))].$$

This shows that

$$F^+(V_1) \cap F^-(V_2) \subseteq \mathrm{Int}^\star[F^+(\mathrm{Int}(\mathrm{Cl}^\star(V_1))) \cap F^-(\mathrm{Int}(\mathrm{Cl}^\star(V_2)))].$$

$(2) \Rightarrow (3)$: Let K_1, K_2 be any $\star$-closed sets of Y. Then $Y - K_1$ and $Y - K_2$ are $\star$-open sets of Y and by (2), we have

$$\begin{aligned} X - [F^-(K_1) \cup F^+(K_2)] &= [X - F^-(K_1)] \cap [X - F^+(K_2)] \\ &= F^+(Y - K_1) \cap F^-(Y - K_2) \\ &\subseteq \mathrm{Int}^\star[F^+(\mathrm{Int}(\mathrm{Cl}^\star(Y - K_1))) \\ &\quad \cap F^-(\mathrm{Int}(\mathrm{Cl}^\star(Y - K_2)))] \\ &= \mathrm{Int}^\star[(X - F^-(\mathrm{Cl}(\mathrm{Int}^\star(K_1)))) \\ &\quad \cap (X - F^+(\mathrm{Cl}(\mathrm{Int}^\star(K_2))))] \\ &= \mathrm{Int}^\star[X - [F^-(\mathrm{Cl}(\mathrm{Int}^\star(K_1))) \\ &\quad \cup F^+(\mathrm{Cl}(\mathrm{Int}^\star(K_2)))]] \\ &= X - \mathrm{Cl}^\star[F^-(\mathrm{Cl}(\mathrm{Int}^\star(K_1))) \\ &\quad \cup F^+(\mathrm{Cl}(\mathrm{Int}^\star(K_2)))] \end{aligned}$$

and hence

$$\mathrm{Cl}^\star[F^-(\mathrm{Cl}(\mathrm{Int}^\star(K_1))) \cup F^+(\mathrm{Cl}(\mathrm{Int}^\star(K_2)))] \subseteq F^-(K_1) \cup F^+(K_2).$$

$(3) \Rightarrow (4)$: Let B_1, B_2 be any subsets of Y. Then $\mathrm{Cl}^\star(B_1)$ and $\mathrm{Cl}^\star(B_2)$ are $\star$-closed in Y and by (3),

$$\mathrm{Cl}^\star[F^-(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(B_1)))) \cup F^+(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(B_2))))] \subseteq F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2)).$$

$(4) \Rightarrow (5)$: Let B_1, B_2 be any subsets of Y. By (4), we have

$$\begin{aligned} &F^-(\mathrm{Int}^\star(B_1)) \cap F^+(\mathrm{Int}^\star(B_2)) \\ &= X - [F^+(\mathrm{Cl}^\star(Y - B_1)) \cup F^-(\mathrm{Cl}^\star(Y - B_2))] \\ &\subseteq X - \mathrm{Cl}^\star[F^+(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(Y - B_1)))) \\ &\quad \cup F^-(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(Y - B_2))))] \\ &= X - \mathrm{Cl}^\star[F^+(Y - \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B_1)))) \\ &\quad \cup F^-(Y - \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B_2))))] \\ &= X - \mathrm{Cl}^\star[(X - F^-(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B_1))))) \\ &\quad \cup (X - F^+(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B_2)))))] \\ &= X - \mathrm{Cl}^\star[X - [F^-(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B_1)))) \\ &\quad \cap F^+(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B_2))))]] \\ &= \mathrm{Int}^\star[F^-(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B_1)))) \\ &\quad \cap F^+(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B_2))))]. \end{aligned}$$

$(5) \Rightarrow (2)$: The proof is obvious.

$(2) \Rightarrow (1)$: Let V_1, V_2 be any $\star$-open sets of Y such that $x \in F^+(V_1) \cap F^-(V_2)$. By (2), we have

$$x \in F^+(V_1) \cap F^-(V_2) \subseteq \mathrm{Int}^\star[F^+(\mathrm{Int}(\mathrm{Cl}^\star(V_1))) \cap F^-(\mathrm{Int}(\mathrm{Cl}^\star(V_2)))].$$

Then, there exists a $\star$-open set U of X such that

$$x \in U \subseteq F^+(\text{Int}(\text{Cl}^\star(V_1))) \cap F^-(\text{Int}(\text{Cl}^\star(V_2))).$$

Thus, $F(U) \subseteq \text{Int}(\text{Cl}^\star(V_1))$ and

$$F(z) \cap \text{Int}(\text{Cl}^\star(V_2)) \neq \emptyset$$

for every $z \in U$. This shows that F is almost $\imath^\star$-continuous. □

5 On characterizations of weakly $\imath^\star$-continuous multifunctions

In this section, we introduce the concept of weakly $\imath^\star$-continuous multifunctions. Moreover, some characterizations of weakly $\imath^\star$-continuous multifunctions are discussed.

Definition 9. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be weakly $\imath^\star$-continuous if for each $x \in X$ and each $\star$-open set V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a $\star$-open set U of X containing x such that $F(U) \subseteq Cl^\star(V_1)$ and $F(z) \cap Cl^\star(V_2) \neq \emptyset$ for every $z \in U$.*

Remark 10. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following implication holds:*

almost $\imath^\star$-continuity $\Rightarrow$ weak $\imath^\star$-continuity.

The converse of the implication is not true in general. We give example for the implication as follows.

Example 11. *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, \{2\}, \{1, 3\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset\}$. Let $Y = \{a, b, c\}$ with a topology $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ and an ideal $\mathscr{J} = \{\emptyset, \{c\}\}$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is defined as follows: $F(1) = \{a\}$, $F(2) = \{b\}$ and $F(3) = \{a, c\}$. Then F is weakly $\imath^\star$-continuous but F is not almost $\imath^\star$-continuous.*

Theorem 12. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is weakly $\imath^\star$-continuous;

(2)

$$F^+(V_1) \cap F^-(V_2) \subseteq Int^\star[F^+(Cl^\star(V_1)) \cap F^-(Cl^\star(V_2))]$$

for every $\star$-open sets V_1, V_2 of Y;

(3)

$$Cl^\star[F^-(Int^\star(K_1)) \cup F^+(Int^\star(K_2))] \subseteq F^-(K_1) \cup F^+(K_2)$$

for every $\star$-closed sets K_1, K_2 of Y;

(4)

$$Cl^\star[F^-(Int^\star(Cl^\star(B_1))) \cup F^+(Int^\star(Cl^\star(B_2)))] \subseteq F^-(Cl^\star(B_1)) \cup F^+(Cl^\star(B_2))$$

for every subsets B_1, B_2 of Y;

(5)

$$F^+(Int^\star(B_1)) \cap F^-(Int^\star(B_2)) \subseteq Int^\star[F^+(Cl^\star(B_1)) \cap F^-(Cl^\star(B_2))]$$

for every subsets B_1, B_2 of Y;

(6)

$$Cl^\star[F^-(V_1) \cup F^+(V_2)] \subseteq F^-(Cl^\star(V_1)) \cup F^+(Cl^\star(V_2))$$

for every $\star$-open sets V_1, V_2 of Y.

Proof. (1) $\Rightarrow$ (2): Let V_1, V_2 be any $\star$-open sets of Y such that $x \in F^+(V_1) \cap F^-(V_2)$. Then

$$F(x) \in V_1^+ \cap V_2^+$$

and hence there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \text{Int}(\text{Cl}^\star(V_1))$ and

$$F(z) \cap \text{Int}(\text{Cl}^\star(V_2)) \neq \emptyset$$

for each $z \in U$. Thus,

$$x \in U \subseteq F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2))$$

and hence $x \in \text{Int}^\star[F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2)))]$. Consequently, we obtain

$$F^+(V_1) \cap F^-(V_2) \subseteq \text{Int}^\star[F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2)))].$$

(2) $\Rightarrow$ (3): Let K_1, K_2 be any $\star$-closed sets of Y. Then $Y - K_1$ and $Y - K_2$ are $\star$-open sets in Y. By (2), we have

$$\begin{aligned}
&X - (F^-(K_1) \cup F^+(K_2)) \\
&= (X - F^-(K_1)) \cap (X - F^+(K_2)) \\
&= F^+(Y - K_1) \cap F^-(Y - K_2) \\
&\subseteq \text{Int}^\star[F^+(\text{Cl}^\star(Y - K_1)) \cap F^-(\text{Cl}^\star(Y - K_2))] \\
&= \text{Int}^\star[(X - F^-(\text{Int}^\star(K_1))) \cap (X - F^+(\text{Int}^\star(K_2)))] \\
&= \text{Int}^\star[X - [F^-(\text{Int}^\star(K_1)) \cup F^+(\text{Int}^\star(K_2))]] \\
&= X - \text{Cl}^\star[F^-(\text{Int}^\star(K_1)) \cup F^+(\text{Int}^\star(K_2))]
\end{aligned}$$

and hence

$$\begin{aligned}&\text{Cl}^\star[F^-(\text{Int}^\star(K_1)) \cup F^+(\text{Int}^\star(K_2))]\\ &\subseteq F^-(K_1) \cup F^+(K_2).\end{aligned}$$

$(3) \Rightarrow (4)$: Let B_1, B_2 be any subsets of Y. Then $\text{Cl}^\star(B_1)$ and $\text{Cl}^\star(B_2)$ are $\star$-closed in Y and by (3),

$$\begin{aligned}&\text{Cl}^\star[F^-(\text{Int}^\star(\text{Cl}^\star(B_1))) \cup F^+(\text{Int}^\star(\text{Cl}^\star(B_2)))]\\ &\subseteq F^-(\text{Cl}^\star(B_1)) \cup F^+(\text{Cl}^\star(B_2)).\end{aligned}$$

$(4) \Rightarrow (5)$: Let B_1, B_2 be any subsets of Y. By (4), we have

$$\begin{aligned}&F^-(\text{Int}^\star(B_1)) \cap F^+(\text{Int}^\star(B_2))\\ &= X - [F^+(\text{Cl}^\star(Y - B_1)) \cup F^-(\text{Cl}^\star(Y - B_2))]\\ &\subseteq X - \text{Cl}^\star[F^+(\text{Int}^\star(\text{Cl}^\star(Y - B_1)))\\ &\quad \cup F^-(\text{Int}^\star(\text{Cl}^\star(Y - B_2)))]\\ &= X - \text{Cl}^\star[F^+(Y - \text{Cl}^\star(\text{Int}^\star(B_1)))\\ &\quad \cup F^-(Y - \text{Cl}^\star(\text{Int}^\star(B_2)))]\\ &= X - \text{Cl}^\star[(X - F^-(\text{Cl}^\star(\text{Int}^\star(B_1))))\\ &\quad \cup (X - F^+(\text{Cl}^\star(\text{Int}^\star(B_2))))]\\ &= X - \text{Cl}^\star[X - [F^-(\text{Int}^\star(\text{Cl}^\star(B_1)))\\ &\quad \cap F^+(\text{Cl}^\star(\text{Int}^\star(B_2)))]]\\ &= \text{Int}^\star[F^-(\text{Cl}^\star(\text{Int}^\star(B_1))) \cap F^+(\text{Cl}^\star(\text{Int}^\star(B_2)))].\end{aligned}$$

Thus,

$$\begin{aligned}&F^+(\text{Int}^\star(B_1)) \cap F^-(\text{Int}^\star(B_2))\\ &\subseteq \text{Int}^\star[F^+(\text{Cl}^\star(B_1)) \cap F^-(\text{Cl}^\star(B_2))].\end{aligned}$$

$(5) \Rightarrow (2)$: This is obvious.

$(2) \Rightarrow (1)$: Let V_1, V_2 be any $\star$-open sets of Y such that $x \in F^+(V_1) \cap F^-(V_2)$. By (2), we have

$$\begin{aligned}&x \in F^+(V_1) \cap F^-(V_2)\\ &\subseteq \text{Int}^\star[F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2)))].\end{aligned}$$

Then, there exists a $\star$-open set U of X such that $x \in U \subseteq F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2))$. Therefore, $F(U) \subseteq \text{Cl}^\star(V_1)$ and $F(z) \cap \text{Cl}^\star(V_2) \neq \emptyset$ for every $z \in U$. This shows that F is weakly $\imath^\star$-continuous.

$(4) \Rightarrow (6)$: Let V_1, V_2 be any $\star$-open sets of Y. By (4), we have

$$\begin{aligned}&\text{Cl}^\star[F^-(V_1) \cup F^+(V_2)]\\ &\subseteq \text{Cl}^\star[F^-(\text{Int}^\star(\text{Cl}^\star(V_1))) \cup F^+(\text{Int}^\star(\text{Cl}^\star(V_2)))]\\ &\subseteq F^-(\text{Cl}^\star(V_1)) \cup F^+(\text{Cl}^\star(V_2)).\end{aligned}$$

$(6) \Rightarrow (2)$: Let V_1, V_2 be any $\star$-open sets of Y. By (6), we have

$$\begin{aligned}&F^+(V_1) \cap F^-(V_2)\\ &\subseteq F^+(\text{Int}^\star(\text{Cl}^\star(V_1))) \cap F^-(\text{Int}^\star(\text{Cl}^\star(V_2)))\\ &= X - [F^-(\text{Cl}^\star(Y - \text{Cl}^\star(V_1)))\\ &\quad \cup F^+(\text{Cl}^\star(Y - \text{Cl}^\star(V_2)))]\\ &\subseteq X - \text{Cl}^\star[F^-(Y - \text{Cl}^\star(V_1)) \cup F^+(Y - \text{Cl}^\star(V_2))]\\ &= \text{Int}^\star[F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2))].\end{aligned}$$

□

Definition 13. [4] *A point x in an ideal topological space $(X, \tau, \mathscr{I})$ is called a $\star_\theta$-cluster point of A if $Cl^\star(U)) \cap A \neq \emptyset$ for every $\star$-open set U of X containing x. The set of all $\star_\theta$-cluster points of A is called the $\star_\theta$-closure of A and is denoted by $\star_\theta Cl(A)$.*

Definition 14. [4] *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called*

(1) $\star_\theta$-closed if $\star_\theta Cl(A) = A$.

(2) $\star_\theta$-open if its complement is $\star_\theta$-closed.

Lemma 15. [4] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) If A is $\star$-open in X, then $Cl^\star(A) = \star_\theta Cl(A)$.

(2) $\star_\theta Cl(A)$ is $\star$-closed in X.

Theorem 16. *For a multifunction $F : (X, \tau, \mathscr{I}) \rightarrow (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is weakly $\imath^\star$-continuous;

(2)

$$\begin{aligned}&Cl^\star[F^-(Int^\star(\star_\theta Cl(B_1))) \cap F^+(Int^\star(\star_\theta Cl(B_2)))]\\ &\subseteq F^-(\star_\theta Cl(B_1)) \cup F^+(\star_\theta Cl(B_2))\end{aligned}$$

for every subsets B_1, B_2 of Y;

(3)

$$\begin{aligned}&Cl^\star[F^-(Int^\star(Cl^\star(B_1))) \cup F^+(Int^\star(Cl^\star(B_2)))]\\ &\subseteq F^-(\star_\theta Cl(B_1)) \cup F^+(\star_\theta Cl(B_2))\end{aligned}$$

for every subsets B_1, B_2 of Y;

(4)

$$\begin{aligned}&Cl^\star[F^-(Int^\star(Cl^\star(V_1))) \cup F^+(Int^\star(Cl^\star(V_2)))]\\ &\subseteq F^-(Cl^\star(V_1)) \cup F^+(Cl^\star(V_2))\end{aligned}$$

for every $\star$-open sets V_1, V_2 of Y;

(5)

$$\textit{Cl}^\star[F^-(\textit{Int}^\star(\textit{Cl}^\star(V_1))) \cup F^+(\textit{Int}^\star(\textit{Cl}^\star(V_2)))] \subseteq F^-(\textit{Cl}^\star(V_1)) \cup F^+(\textit{Cl}^\star(V_2))$$

for every $\mathscr{J}^\star$-preopen sets V_1, V_2 of Y;

(6)

$$\textit{Cl}^\star[F^-(\textit{Int}^\star(K_1)) \cup F^+(\textit{Int}^\star(K_2))] \subseteq F^-(K_1) \cup F^+(K_2)$$

for every R-$\mathscr{J}^\star$-closed sets K_1, K_2 of Y.

Proof. (1) $\Rightarrow$ (2): Let B_1, B_2 be any subset of Y. Then $\star_\theta\mathrm{Cl}(B_1)$ and $\star_\theta\mathrm{Cl}(B_2)$ are $\star$-closed in Y. By Theorem 12, we have

$$\mathrm{Cl}^\star[F^-(\mathrm{Int}^\star(\star_\theta\mathrm{Cl}(B_1))) \cup F^+(\mathrm{Int}^\star(\star_\theta\mathrm{Cl}(B_2)))] \subseteq F^-(\star_\theta\mathrm{Cl}(B_1)) \cup F^+(\star_\theta\mathrm{Cl}(B_2)).$$

(2) $\Rightarrow$ (3): This is obvious since $\mathrm{Cl}^\star(B) \subseteq \star_\theta\mathrm{Cl}(B)$ for every subset B of Y.

(3) $\Rightarrow$ (4): This is obvious since $\mathrm{Cl}^\star(V) = \star_\theta\mathrm{Cl}(V)$ for every $\star$-open set V of Y.

(4) $\Rightarrow$ (5): Let V_1, V_2 be any $\mathscr{J}^\star$-preopen sets of Y. Since $V_i \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(V_i))$, we have

$$\mathrm{Cl}^\star(V_i) = \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(V_i)))$$

for $i = 1, 2$. Now, put $U_i = \mathrm{Int}^\star(\mathrm{Cl}^\star(V_i))$, then U_i is $\star$-open in Y and $\mathrm{Cl}^\star(U_i) = \mathrm{Cl}^\star(V_i)$. Therefore, by (4), we have

$$\mathrm{Cl}^\star[F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V_1))) \cup F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(V_2)))] \subseteq F^-(\mathrm{Cl}^\star(V_1)) \cup \mathrm{Cl}^\star(V_2).$$

(5) $\Rightarrow$ (6): Let K_1, K_2 be any R-$\mathscr{J}^\star$-closed sets of Y. Then $\mathrm{Int}^\star(K_1)$ and $\mathrm{Int}^\star(K_2)$ are $\mathscr{J}^\star$-preopen in Y and by (5),

$$\begin{aligned}&\mathrm{Cl}^\star(F^-[\mathrm{Int}^\star(K_1)) \cup F^+(\mathrm{Int}^\star(K_2))] \\ &= \mathrm{Cl}^\star[F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(K_1)))) \\ &\quad \cup F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(K_2))))] \\ &\subseteq F^-(K_1) \cup F^+(K_2).\end{aligned}$$

(6) $\Rightarrow$ (1): Let V_1, V_2 be any $\star$-open sets of Y. Then $\mathrm{Cl}^\star(V_1)$ and $\mathrm{Cl}^\star(V_1)$ are R-$\mathscr{J}^\star$-closed in Y and by (6), we have

$$\begin{aligned}&\mathrm{Cl}^\star[F^-(V_1) \cup F^+(V_2)] \\ &\subseteq \mathrm{Cl}^\star[F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V_1))) \cup F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(V_2)))] \\ &\subseteq F^-(\mathrm{Cl}^\star(V_1) \cup F^+(\mathrm{Cl}^\star(V_2)).\end{aligned}$$

It follows from Theorem 12 that F is weakly $\imath^\star$-continuous. $\square$

Theorem 17. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is weakly $\imath^\star$-continuous;

(2)

$$\textit{Cl}^\star[F^-(\textit{Int}^\star(\textit{Cl}^\star(V_1))) \cup F^+(\textit{Int}^\star(\textit{Cl}^\star(V_2)))] \subseteq F^-(\textit{Cl}^\star(V_1)) \cup F^+(\textit{Cl}^\star(V_2))$$

for every semi-$\mathscr{J}^\star$-preopen sets V_1, V_2 of Y;

(3)

$$\textit{Cl}^\star[F^-(\textit{Int}^\star(\textit{Cl}^\star(V_1))) \cup F^+(\textit{Int}^\star(\textit{Cl}^\star(V_2)))] \subseteq F^-(\textit{Cl}^\star(V_1)) \cup F^+(\textit{Cl}^\star(V_2))$$

for every semi-$\mathscr{J}^\star$-open sets V_1, V_2 of Y.

Proof. (1) $\Rightarrow$ (2): Let V_1, V_2 be any semi-$\mathscr{J}^\star$-preopen sets of Y. Then, we have

$$V_i \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(V_i)))$$

and $\mathrm{Cl}^\star(V_i) = \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(V_i)))$ for $i = 1, 2$. Since $\mathrm{Cl}^\star(V_1)$ and $\mathrm{Cl}^\star(V_2)$ are R-$\mathscr{J}^\star$-closed in Y and by Theorem 16,

$$\mathrm{Cl}^\star[F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V_1))) \cup F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(V_2)))] \subseteq F^-(\mathrm{Cl}^\star(V_1)) \cup F^+(\mathrm{Cl}^\star(V_2)).$$

(2) $\Rightarrow$ (3): This is obvious since every semi-$\mathscr{J}^\star$-open set is semi-$\mathscr{J}^\star$-preopen.

(3) $\Rightarrow$ (1): Let V_1, V_2 be any semi-$\mathscr{J}^\star$-preopen sets of Y. Then $\mathrm{Cl}^\star(V_1)$ and $\mathrm{Cl}^\star(V_2)$ are R-$\mathscr{J}^\star$-closed sets of Y and hence $\mathrm{Cl}^\star(V_2)$ are semi-$\mathscr{J}^\star$-open in Y. By (3), we have

$$\mathrm{Cl}^\star[F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V_1))) \cup F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(V_2)))] \subseteq F^-(\mathrm{Cl}^\star(V_1)) \cup F^+(\mathrm{Cl}^\star(V_2))$$

and by Theorem 16, F is weakly $\imath^\star$-continuous. $\square$

6 Conclusion

Topology plays an important role in computational topology for geometric design, computer-aided geometric design, engineering design, information systems, quantum physics, high energy physics and mathematical sciences. The notions of openness and continuity are fundamental concepts for the study and investigation in topological spaces. Besides, the study of continuity has been found to be useful in computer science and digital topology. Several investigations related to generalized open sets have been presented and various forms of continuity types have been introduced. Recently, continuity of multifunctions in topological spaces has been researched by many mathematicians. This article deals with the concept of $\imath^{\star}$-continuous multifunctions in ideal topological spaces. Some characterizations of $\imath^{\star}$-continuous multifunctions are obtained. Moreover, the relationships between $\imath^{\star}$-continuity and the other types of continuity for multifunctions are explored. The ideas and results of this article may motivate further research.

Acknowledgements: This research project was financially supported by Mahasarakham University (Fast Track 2021).

References:

[1] M.–E. Abd El-Monsef, E.–F. Lashien and A.–A. Nasef, On $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions, *Kyungpook Math. J.* 32, 1992, pp. 21–30.

[2] A. Açikgöz, T. Noiri and Ş. Yüksel, A decomposition of continuity in ideal topological spaces, *Acta Math. Hungar.* 105, 2004, pp. 27–37.

[3] C. Berge, *Espaces topologiques fonctions multivoques*, Dunod, Paris, 1959.

[4] C. Boonpok, Weak quasi continuity for multifunctions in ideal topological spaces, *Adv. Math., Sci. J.* 9(1), 2020, pp. 339–355.

[5] C. Boonpok, On continuous multifunctions in ideal topological spaces, *Lobachevskii J. Math.* 40, 2019, pp. 24–35.

[6] J. Dontchev, Idealization of Granter-Reilly decomposition theorems, arxiv:math. GN/9901017v1, 1999.

[7] G. Freud, Ein Beitrag zu dem Satze von Cantor und Bendixson, *Acta Math. Acad. Sci. Hungar.* 9, 1958, pp. 333–336.

[8] E. Hatir, A. Keskin and T. Noiri, A note on strong β-$\mathscr{I}$-open sets and strongly β-$\mathscr{I}$-continuous functions, *Acta Math. Hungar.* 108, 2005, pp. 87–94.

[9] E. Hatir and T. Noiri, On semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous functions, *Acta Math. Hungar.* 107, 2005, pp. 345–353.

[10] E. Hatir and T. Noiri, On decompositions of continuity via idealization, *Acta Math. Hungar.* 96, 2002, pp. 341–349.

[11] D. Janković and T.–R. Hamlett, New topologies from old via ideals, *Amer. Math. Monthly* 97, 1990, pp. 295–310.

[12] K. Kuratowski, *Topology*, Vol. I, Academic–Press, New York, 1966.

[13] N. Levine, A Decomposition of continuity in topological spaces, *Amer. Math. Monthly* 68, 1961, pp. 44–46.

[14] T. Noiri and V. Popa, Weakly quasi-continuous multifunctions, *Anal. Univ. Timişoara, Ser. Mat.* 26, 1988, pp. 33–38.

[15] V. Popa and T. Noiri, Properties of weakly precontinuous multifunctions, *Istanbul Univ. Fen. Fa. Mat. Dergisi* 57/58, 1998/1999, pp. 41–52.

[16] V. Popa and T. Noiri, On weakly β-continuous multifunctions, *Bui. St. Univ. "Politehnica", Sre. Mat. Fiz. Timişoara* 45, 2000, pp. 1–16.

[17] D.–A. Rose, Weak continuity and almost continuity, *Internat. J. Math. Math. Sci.* 7, 1984, pp. 311–318.

[18] S. Scheinberg, Topologies which generate a complete measure algebra, *Advances in Math.* 7, 1971, pp. 231–239.

[19] R. Vaidyanathaswamy, The localization theory in set topology, *Proc. Indian Acad. Sci.* 20, 1944, pp. 51–61.