Application of Non-polynomial Splines to Solving Differential Equations

I.G.BUROVA
Department of Computational Mathematics
St. Petersburg State University
7/9 Universitetskaya nab. St. Petersburg, 199034
RUSSIA

Abstract: - The application of the local polynomial and non-polynomial to the construction of methods for numerically solving the heat conduction problem is discussed. The non-polynomial splines are used here to approximate the partial derivatives. Formulas for numerical differentiation based on the application of the non-polynomial splines of the fourth order of approximation are constructed. Particular attention is paid to polynomial, trigonometric, exponential, polynomial-trigonometric and polynomial-exponential splines. This approach allows us to construct explicit and implicit difference schemes. The main focus of the paper is on implicit difference scheme. New approximations with splines of the Lagrange and Hermite type with new properties are obtained. These approximations take into account the first and second derivatives of the function being approximated. Numerical examples are given.

Key-Words: - heat conduction problem, exponential splines, polynomial splines, trigonometric splines, polynomial-trigonometric splines, polynomial-exponential splines

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1 Introduction

Everyone knows the formula for the polynomial interpolation. In 1901 Runge, and in 1916 Bernstein found examples of functions on a uniform grid of nodes on [-1,1]. The Runge function is $f(x) = \frac{1}{1+25x^2}$. The Bernstein function is $f(x) = |x|$. These examples (the Runge function and the Bernstein function) show that on a uniform grid of nodes built on the interval [-1,1], with an increase in the number of nodes, the relation is valid:

$$\lim_{n\to\infty} \max_{-1\le x\le 1} |f(x) - P_n(x)| = \infty.$$

Here $P_n(x)$ is the interpolation polynomial that interpolates $f(x)$ in the nodes. Thus, the interpolation polynomial is not very good for interpolation on the uniform grid of nodes when the number of nodes is large. Spline approximations have a significant advantage over interpolation polynomials.

Currently, there are a wide variety of splines. Everyone knows and often uses the difference method to solve partial differential equations. When constructing solutions, spline approximations are often used (see, [1]-[7]). Among the variety of books on splines, we should first of all mention De Boor's book. Among the variety of splines researchers prefer to use the polynomial splines, mostly the B-splines (see, [4]-[6]). In paper [6], two types of basis functions are considered: B-spline and expo-rational B-spline combined with Bernstein polynomials. In paper [7], the polyharmonic splines with added polynomials defined in a 2D plane are used.

The construction of B -splines involves solving a system of linear algebraic equations. Often the number of unknowns is large, so the system of equations has a large dimension. The matrix of this system of equations turns out to be tridiagonal, therefore this circumstance facilitates the task (although there are various methods on how to solve a system with a tridiagonal matrix).

Nowadays, various types of splines are known. Spline construction techniques are extremely diverse. The approximations obtained using different splines also differ in properties. The splines with a local interpolation basis are of particular interest. These splines interpolate the function at the grid points. To construct the splines, we do not need to solve a system of equations. The splines are constructed separately on each grid interval in the form of a linear combination of basis splines and the function values at the grid nodes.

We will use the technique for constructing splines proposed by Professor S.G. Mikhlin of St. Petersburg State University. The technique proposed by him involves the construction of polynomial splines in the form of a product of the values of a function and derivatives of this function at the grid nodes by the corresponding basis functions. Professor Mikhlin considered non-zero level spline approximations. The level of the spline is the number of the derivatives of the function that (derivatives) are used to construct a spline. The length of the support of each basis spline is two grid intervals. Professor S.G. Mikhlin paid much attention to local polynomial splines of the Hermite type. Polynomial basic splines on each grid interval can be obtained by solving a system of equations (S.G. Mikhlin called this system, a system of fundamental relations). Non-polynomial local splines are constructed similarly to polynomial ones. Professor Yu.K. Demjanovich pays much attention to non-polynomial splines. The approximation constructed using these splines has the property that it is infinitely differentiable within each grid interval. At each nodal point, the approximation is only continuous. Nevertheless, within each interval, the approximation is differentiable, and by continuity it is possible to determine the value of the derivative on the left (or right) at a grid point of arbitrary order.

When constructing an approximation, it is important to take into account the behaviour of the first and second derivatives of the function. We will construct continuous splines of the fourth order of approximation, which interpolate the function at the grid nodes, and also take into account the behaviour of the first and second derivatives of the function. By applying spline approximations of the non-zero level, we improve the quality of the approximation. Using non-polynomial splines also improves the quality of the approximation. In this paper, both polynomial and non-polynomial splines of the fourth order of approximation will be constructed, taking into account the behaviour of the first and second derivatives of the function.

We construct the solution separately on each grid interval. The length of the support of each basis spline is one or two grid intervals. Our focus will also be on continuous interpolation splines.

The construction and application of polynomial and non-polynomial splines were considered by the author earlier in papers [8]-[13]. The construction of difference schemes using some local interpolation non-polynomial splines was considered in papers [12]-[13].

Here we will consider the application of other local interpolation non-polynomial splines and compare the results with those obtained earlier. In addition, we will dwell on the theoretical foundations in more detail. Thus, features of the use of the local interpolation polynomial and non-polynomial will be discussed in this paper. In this paper the application of the trigonometrical splines for constructing the numerical method for solving the heat problem is discussed. The properties of these splines and the theorems of the approximation were presented in papers [8]-[11]. This paper will also discuss the construction of a Lagrangian-type approximation that takes into account the behaviour of the first and second derivatives of a function.

It should be noted that the problem of improving the quality of approximation is very important and interesting. In this regard, we should note that finding points of importance in an interpolation problem were considered in papers [14]-[15].

2 The Difference Methods

The difference methods for solving the heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu + f,\ u = u(x,t),$$

where $a = a(x,t) > 0,\ b = b(x,t),\ c = c(x,t)$, $0 \le x \le 1,\ 0 \le t \le T$, are well known and widely used. In what follows, we will assume that a mesh with step h along the space axis and τ along the time axis is constructed in the rectangular area. Let us put $a = 1,\ b = 0,\ c = 0$. Denote $u_{jk} = u(jh, k\tau)$. Among the most commonly used schemes are two: the explicit and the implicit schemes. The explicit scheme is used to solve the Cauchy problem:

$$u|_{t=0} = u_0(x), \qquad x \in [0,1],$$

$$\frac{u_{jk+1} - u_{jk}}{\tau} = \frac{u_{j-1k} - 2u_{jk} + u_{j+1k}}{h^2} + f(jh, k\tau).$$

The explicit scheme has significant restrictions on the ratio of steps in time and space: $\tau \le h^2/2$.

The implicit scheme is used for solving the mixed problem: $u|_{t=0} = u_0(x)$, $u|_{x=0} = \varphi_0(t)$, $u|_{x=1} = \varphi_1(t)$.

$$\frac{u_{jk+1} - u_{jk}}{\tau} = \frac{u_{j-1k+1} - 2u_{jk+1} + u_{j+1k+1}}{h^2} + f(jh, (k+1)\tau).$$

On each layer, it is necessary to solve a system of linear algebraic equations with a tridiagonal matrix. As it is known, the implicit scheme is stable with respect to rounding errors. The purpose of this paper is to construct difference schemes based on non-polynomial spline approximations and to discuss the stability. Now let us consider in more detail the approximation of $\frac{\partial^2 u}{\partial x^2}$.

2 The Approximation of the Derivatives

First of all, we note that the above formulas for numerical differentiation can be easily obtained using the theory of constructing local interpolation polynomial splines.

Let m, n be integer. Let $\{x_j\}$ be a set of ascending ordered nodes and function $u(x)$ be such that $u \in C^4([x_0, x_n])$. Let us use the approximation $U^{MP}(x)$ of the function $u(x)$ on the grid interval $[x_j, x_{j+1}]$ with the cubic polynomial splines

$$U^{MP}(x) = u(x_{j-1})g_{j-1} + u(x_j)g_j + u(x_{j+1})g_{j+1}$$

$$+u(x_{j+2})g_{j+2}.$$

Here the basis splines g_i, $i = j-1, j, j+1, j+2$, are as follows:

$$g_{j-1} = \frac{(x-x_j)(x-x_{j+1})(x-x_{j+2})}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})(x_{j-1}-x_{j+2})},$$

$$g_j = \frac{(x-x_{j-1})(x-x_{j+1})(x-x_{j+2})}{(x_j-x_{j-1})(x_j-x_{j+1})(x_j-x_{j+2})},$$

$$g_{j+1} = \frac{(x-x_{j-1})(x-x_j)(x-x_{j+2})}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)(x_{j+1}-x_{j+2})},$$

$$g_{j+2} = \frac{(x-x_{j-1})(x-x_{j+1})(x-x_j)}{(x_{j+2}-x_{j-1})(x_{j+2}-x_{j+1})(x_{j+2}-x_j)}.$$

Theorem 1. The following approximation estimate is valid:

$$|u(x) - U^{MP}(x)| \leq 0.02344 h^4 \parallel u^{(4)} \parallel_{[x_j, x_{j+1}]},$$
$$x \in [x_j, x_{j+1}) .$$

Proof. Using the Hermite interpolation remainder theorem, we obtain

$$u(x) - U^{MP}(x) = \frac{u^{(4)}(\theta)}{4!}(x-x_j)(x-x_{j-1})(x-x_{j+1})(x-x_{j+2}),$$

where $\theta = \theta(x) \in [x_j, x_{j+1}]$.

If the grid is uniform with step h, then $x_{j+1} = x_j + h$. When $x \in [x_j, x_{j+1})$ we put $x = x_j + th, t \in [0,1]$. Thus, we obtain the estimate

$$|u(x_j + th) - U^{MP}(x_j + th)| \leq$$

$$\frac{h^4}{4!} \max_{[x_j, x_{j+1}]} |u^{(4)}| \max_{t \in [0,1]} |t(t+1)(t-1)(t-2)|.$$

Let us find the maximum of the expression

$$|t(t+1)(t-1)(t-2)|.$$

It is not difficult to see that

$$\max_{t \in [0,1]} |t(t+1)(t-1)(t-2)| \leq 0.5625.$$

Thus, we obtain:

$$|u - U^{MP}| \leq h^4 \frac{\max_{[x_j, x_{j+1}]} |u^{(4)}|}{4!} 0.5625.$$

The proof is complete.

Thus, the estimation of the approximation can be written in the form: $|u - U^{MP}] \leq Ch^4, C > 0$.

Now let us construct a new approximation of the function u using the obtained basis functions.

Our aim is to construct the approximation which uses only the values of the function u in the nodes. The grid of nodes is uniform with step h therefore, we can use next formulas for approximation the derivatives:

$$u'(x_j) = \frac{-2u(x_{j-1}) - 3u(x_j) + 6u(x_{j+1}) - u(x_{j+2})}{6h} + O(h^3),$$

$$u''(x_j) = \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} + O(h^2).$$

Denote

$$v_1 = \frac{-2u(x_{j-1}) - 3u(x_j) + 6u(x_{j+1}) - u(x_{j+2})}{6\,h},$$

$$v_2 = \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2}.$$

Now we get the approximation $V(x)$ in the form:

$$V(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x)$$
$$+v_1 w_{j,1}(x) + v_2 w_{j,2}(x).$$

Therefore, we obtain the relation:

$$V(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) +$$

$$\frac{-2u(x_{j-1})-3u(x_j)+6u(x_{j+1})-u(x_{j+2})}{6h}w_{j,1}(x)$$

$$+\frac{u(x_{j-1})-2u(x_j)+u(x_{j+1})}{h^2}w_{j,2}(x).$$

The approximation uses only the values of the function in the nodes and the old basis splines. It can be written in the form:

$$Q^p(x)=u(x_j)W_j^p(x)+u(x_{j+1})W_{j+1}^p(x)$$

$$+u(x_{j-1})W_{j-1}^p(x)+\ u(x_{j+2})W_{j+2}^p(x),$$

where

$$W_j^p(x)=w_{j,0}(x)-\frac{w_{j,1}(x)}{2h}-2w_{j,2}(x)/h^2,$$

$$W_{j+1}^p(x)=w_{j+1,0}(x)+\frac{w_{j,1}(x)}{h}+w_{j,2}(x)/h^2,$$

$$W_{j-1}^p(x)=-\frac{w_{j,1}(x)}{3h}+w_{j,2}(x)/h^2,$$

$$W_{j+2}^p(x)=-w_{j,2}(x)/(6h^2).$$

When applying the approximation $U^{MP}(x)$ of the function $u(x)$, we receive the formula:

$$(U^{MP}(x))''=u(x_{j-1})g''_{j-1}(x)+u(x_j)g''_j(x)+u(x_{j+1})g''_{j+1}(x)+u(x_{j+2})g''_{j+2}(x),$$

where $g''_j(x_j)=-\frac{2}{h^2}$, $g''_{j+1}(x_j)=\frac{1}{h^2}$, $g''_{j-1}(x_j)=\frac{1}{h^2}$, $g''_{j+2}(x_j)$=0, x_{j+1}-$x_j=h$.

The approximation $U^{MP}(x)$ is based on the interval $[x_j,x_{j+1}]$. This approximation $U^{MP}(x)$ is continuous on the interval $[x_0,x_n]$. The approximation is continuously differentiable on the interval (x_j,x_{j+1}). We are interested in the second derivatives at the point x_j. We consider a one-sided limit of the function $(U^{MP}(x))''$, $x\in(x_j,x_{j+1})$ as x approaches the point x_j from the right. Thus, we obtain $(U^{MP})''(x_j+)$.

Since we are only interested in points from the interval $[x_j,x_{j+1})$, in what follows, this limit is denoted as $(U^{MP})''(x_j)$.

Thus, at the point x_i, we get the formulas that are presented on this slide:

$$g''_j(x_j)=-\frac{2}{h^2},\ g''_{j+1}(x_j)=\frac{1}{h^2},$$

$$g''_{j-1}(x_j)=\frac{1}{h^2},\ g''_{j+2}(x_j)=0.$$

Applying these formulae, we obtain the well-known formula for numerical differentiation and it can be used to approximate the partial derivative above. It can be shown that the error in approximating the second derivative is the next:

$$\left|\left(U^{MP}(x_j)\right)''-u''(x_j)\right|\le Kh^2, K>0.$$

We will now consider the approximation of derivatives using the trigonometric splines. Let $u\in C^4([x_0,x_n])$. There are many possibilities to construct a local approximation with trigonometric splines. When applying the approximation $U^{MT}(x)$ of the function $u(x)$ on the grid interval $[x_j,x_{j+1}]$ with the trigonometric spline

$$U^{MT}(x)=u(x_{j-1})g_{j-1}+u(x_j)g_j+u(x_{j+1})g_{j+1}+u(x_{j+2})g_{j+2},$$

where

$$g_{j-1}=A_{j-1}/B_{j-1},$$

$$A_{j-1}=\sin\left(\frac{x}{2}-\frac{x_j}{2}\right)\sin\left(\frac{x}{2}-\frac{x_{j+1}}{2}\right)\sin\left(\frac{x}{2}-\frac{x_{j+2}}{2}\right),$$

$$B_{j-1}=\sin\left(\frac{x_{j-1}}{2}-\frac{x_j}{2}\right)\sin\left(\frac{x_{j-1}}{2}-\frac{x_{j+1}}{2}\right)\sin\left(\frac{x_{j-1}}{2}-\frac{x_{j+2}}{2}\right),$$

$$g_j=A_j/B_j,$$

$$A_j=\sin\left(\frac{x}{2}-\frac{x_{j-1}}{2}\right)\sin\left(\frac{x}{2}-\frac{x_{j+1}}{2}\right)\sin\left(\frac{x}{2}-\frac{x_{j+2}}{2}\right),$$

$$B_j=\sin\left(\frac{x_j}{2}-\frac{x_{j-1}}{2}\right)\sin\left(\frac{x_j}{2}-\frac{x_{j+1}}{2}\right)\sin\left(\frac{x_j}{2}-\frac{x_{j+2}}{2}\right),$$

$$g_{j+1}=A_{j+1}/B_{j+1},$$

$$A_{j+1}=\sin\left(\frac{x}{2}-\frac{x_{j-1}}{2}\right)\sin\left(\frac{x}{2}-\frac{x_j}{2}\right)\sin\left(\frac{x}{2}-\frac{x_{j+2}}{2}\right),$$

$$B_{j+1} = \sin\left(\frac{x_{j+1}}{2} - \frac{x_{j-1}}{2}\right)\sin\left(\frac{x_{j+1}}{2} - \frac{x_j}{2}\right)\sin\left(\frac{x_{j+1}}{2} - \frac{x_{j+2}}{2}\right),$$

$$g_{j+2} = A_{j+2}/B_{j+2},$$

$$A_{j+2} = \sin\left(\frac{x}{2} - \frac{x_{j-1}}{2}\right)\sin\left(\frac{x}{2} - \frac{x_{j+1}}{2}\right)\sin\left(\frac{x}{2} - \frac{x_j}{2}\right),$$

$$B_{j+2} = \sin\left(\frac{x_{j+2}}{2} - \frac{x_{j-1}}{2}\right)\sin\left(\frac{x_{j+2}}{2} - \frac{x_{j+1}}{2}\right)\sin\left(\frac{x_{j+2}}{2} - \frac{x_j}{2}\right),$$

we receive the formula:

$$\left(U^{MT}(x)\right)'' = u(x_{j-1}){g''}_{j-1}(x) + u(x_j){g''}_j(x) + u(x_{j+1}){g''}_{j+1}(x) + u(x_{j+2}){g''}_{j+2}(x).$$

When $x_{j-1} = x_j - h$, $x_{j+1} = x_j + h$, $x_{j+2} = x_j + 2h$, it is not difficult to obtain the formula:

$$\left(U^{MT}(x_j)\right)'' = u(x_{j-1}){g''}_{j-1}(x_j) + u(x_j){g''}_j(x_j) + u(x_{j+1}){g''}_{j+1}(x_j) + u(x_{j+2}){g''}_{j+2}(x_j),$$

where

$${g''}_j(x_j) = -\frac{3}{4} - \frac{\cos^2\left(\frac{h}{2}\right)}{2\sin^2\left(\frac{h}{2}\right)},$$

$${g''}_{j-1} = \frac{\cos(h/2)}{2\sin\left(\frac{h}{2}\right)\sin\left(\frac{3h}{2}\right)} + \frac{\cos(h)}{2\sin(h)\sin\left(\frac{3h}{2}\right)},$$

$${g''}_{j+1}(x_j) = \frac{\cos(h/2)}{2\,sin^2\left(\frac{h}{2}\right)} - \frac{\cos(h)}{2\sin(\mathrm{h})\sin\left(\frac{h}{2}\right)}.$$

It can be shown that the error in approximating the second derivative is

$$\left|\left(U^{MT}(x_j)\right)'' - u''(x_j)\right| \le Kh^2, K > 0.$$

Another possibility for the approximation of the second derivative will be discussed in Section 3. Here we note, then if $h \to 0$ we can obtain the relations:

$${g''}_j(x_j) = -2h^{-2} - \frac{5}{12} - \frac{h^2}{120} + O(h^4),$$

$${g''}_{j+1}(x_j) = {g''}_{j-1}(x_j) = h^{-2} + \frac{5}{24} + \frac{53h^2}{1920} + O(h^4).$$

Example 1. We apply the polynomial and the trigonometric approximations of derivatives to solve the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t).$$

Suppose that the exact analytical solution to the equation is $u(x,t) = t\left(\sin(2x) + \cos\left(\frac{x}{2}\right)\right)$.

We find a solution in the domain: $0 \le t \le T$, $x \in [0,1]$, under boundary conditions:$u|_{t=0} = u_0(x)$, $x \in [0,1], u|_{x=0} = \varphi_0(t)$, $u|_{x=1} = \varphi_1(t)$. $T = 0.08$, $\tau = 0.08/m$, $h = 1/n$. We use the Maple program to calculate when *Digits*=15. First, we apply the traditional polynomial approximation of the partial derivatives. Applying an implicit scheme for solving the heat equation, we solve a system of linear algebraic equations on each layer. Fig. 1 shows the plot of the difference between the values of the exact solution and the values of the grid function at the grid nodes, where $n = 20, m = 30$. The graph of the difference in absolute value between the values of the grid function and the values of the exact solution at the grid nodes on the last layer is shown in Fig. 2.

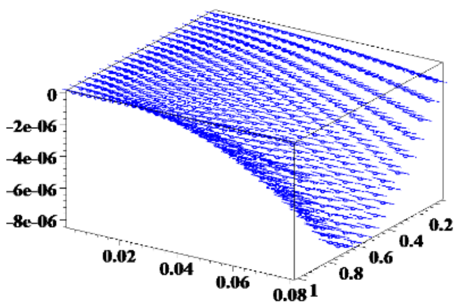

Fig.1. The plot of the difference between the values of the exact solution and the values of the grid function at the grid nodes (the use of the polynomial splines).

Now we apply the approximation of the second partial derivative using the trigonometric splines. Applying an implicit scheme for solving the heat equation, we solve a system of linear algebraic equations on each layer. Fig.3 shows the plot of the error between the values of the exact solution and the values of the grid function at the grid nodes, where $n = 20, m = 30$.

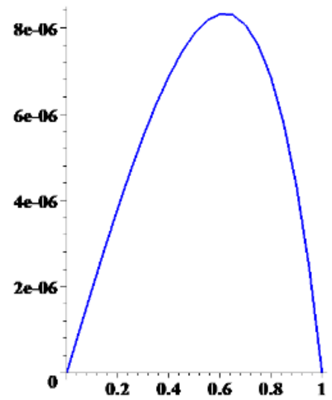

Fig.2. The graph of the difference in absolute value between the values of the grid function and the values of the exact solution at the grid nodes on the last layer (the use of the polynomial splines).

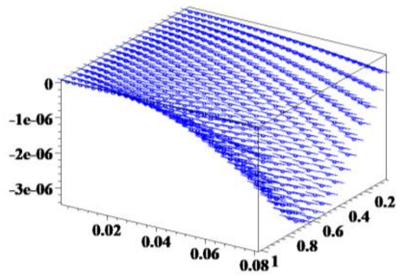

Fig.3. The plot of the difference between the values of the exact solution and the values of the grid function at the grid nodes (the use of the trigonometric splines).

The graph of the difference in absolute value between the values of the grid function and the values of the exact solution at the grid nodes on the last layer is shown in Fig.4.

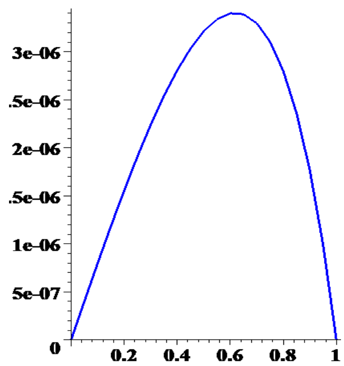

Fig.4. The plot of the difference between the values of the exact solution and the values of the grid function at the grid nodes on the last layer (the use of the trigonometric splines).

This example shows that in some cases the approximation of partial derivatives by trigonometric functions can create result in a smaller error.

Now consider the application of exponential splines. The exponential splines can be constructed in a such way that $U^{ME}(x) = u(x), u(x) = \exp(x), \exp(-x), 1, x,$ for $x \in [x_j, x_{j+1}]$. We can use the next approximation of the second derivative:

$$\left(U^{ME}(x)\right)''_{x_j} = u(x_{j-1})g''_{j-1}(x_j) + u(x_j)g''_j(x_j) + u(x_{j+1})g''_{j+1}(x_j) + u(x_{j+2})g''_{j+2}(x_j),$$

where $g''_{j+2} = 0$,

$$g''_{j-1} = \frac{\exp(h)}{(\exp(h)-1)^2}, \; g''_{j+1} = \frac{\exp(h)}{(\exp(h)-1)^2},$$

$$g''_j = \frac{-2\exp(h)}{(\exp(h)-1)^2}.$$

We will discuss the error of the approximation with these exponential splines later, in Section 3. Here we note, then if $h \to 0$ we can obtain the relations:

$$g''_j(x_j) = -2h^{-2} + \frac{1}{6} - \frac{h^2}{120} + O(h^4),$$

$$g''_{j+1}(x_j) = g''_{j-1}(x_j) = h^{-2} - \frac{1}{12} + \frac{h^2}{240} + O(h^4).$$

The formula for the approximation of the function with the exponential splines in the interval $[x_j, x_{j+1}]$ can be written as follows:

$$U^{ME}(x_j + th) = u(x_{j-1})w_{j-1} + u(x_j)w_j + u(x_{j+1})w_{j+1} + u(x_{j+2})w_{j+2},$$

where $t \in [0,1]$.

The basis splines are the following:

$w_j(x_j + th) = N_j / M_j$, where

$$N_j = \exp(th)(t\exp(4h) + \exp(4h) + t\exp(3h) - t\exp(h) - t - 1) + \exp(h)(\exp(h) + 2) + \exp(2th)(-\exp(2h) - 2\exp(3h)),$$
$$M_j = \exp(th)\,(\exp(h) - 1)^3(\exp(h) + 1),$$

$w_{j+1}(x_j + th) = N_{j+1} / M_{j+1}$, where

$$N_{j+1} = \exp(\mathrm{h} + \mathrm{th})\,(1 + t - \exp(2h) - t\exp(2h)) + \exp(3h + 2th) - \exp(h),$$
$$M_{j+1} = \exp(th)(\exp(4h) - 2\exp(3h) + 2\exp(h) - 1);$$

$w_{j-2}(x_j+th)=N_{j-2}/M_{j-2}$, where

$$N_{j-2}=\exp(h+th)\,(t\exp(2h)-t)-\exp(2h+2th)+\exp(2h),$$

$$M_{j-2}=M_{j+1};$$

$w_{j-1}(x_j+th)=N_{j-1}/M_{j-1}$, where

$$N_{j-1}=\exp(th)(t+t\exp(h)+\exp(h)-t\exp(3h)-\exp(3h)-t\exp(4h))-\exp(h)(1+2\exp(h))+\exp(2th)(\exp(3h)+2\exp(2h)),$$

$$M_{j-1}=M_{j+1}.$$

Example 2. We find a solution

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+(-a^2t+1)\exp(-ax),$$

$$0\le x\le 1,\qquad 0\le t\le T,$$

in the domain: $0\le t\le T$, $x\in[0,1]$, at initial and boundary conditions:

$$u(x,0)=0,\ \ u(0,t)=t,$$
$$u(1,t)=t\exp(-a), a=1.$$

We have $T=0.08$, $\tau=0.08/m$, $h=1/n$. First, we use the exponential splines. Applying an implicit scheme for solving the heat equation, we solve a system of linear algebraic equations on each layer.

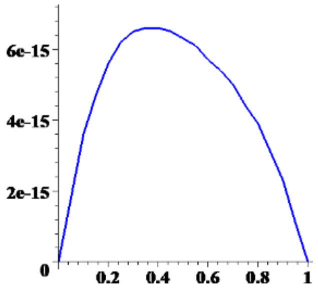

Fig.5. The graph of the difference between the values of the grid function and the values of the exact solution at the grid nodes on the last layer (the use of the exponential splines).

Fig. 6 shows the plot of the error between the values of the exact solution and the values of the grid function at the grid nodes, where $n=20, m=30$. The graph of the difference in absolute value between the values of the grid function and the values of the exact solution at the grid nodes on the last layer is shown in Fig. 5. Here *Digits*=15.

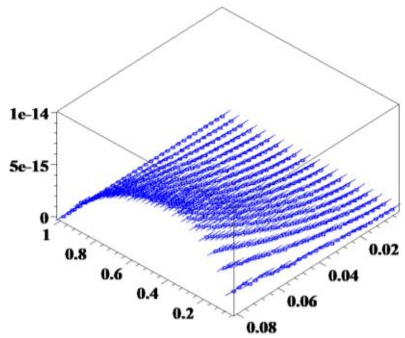

Fig.6. The plot of the difference between the values of the exact solution and the values of the grid function at the grid nodes (the use of the exponential splines).

Now we use the polynomial splines. Applying an implicit scheme for solving the heat equation, we solve a system of linear algebraic equations on each layer. Fig. 8 shows the plot of the difference between the values of the exact solution and the values of the grid function at the grid nodes, where $n=20, m=30$. The graph of the difference in absolute value between the values of the grid function and the values of the exact solution at the grid nodes on the last layer is shown in Fig. 7. Here *Digits=15*.

Now let us use the trigonometric splines. Applying an implicit scheme for solving the heat equation, we solve a system of linear algebraic equations on each layer. Fig. 10 shows the plot of the difference between the values of the exact solution and the values of the grid function at the grid nodes, where $n=20, m=30$. The graph of the difference in absolute value between the values of the grid function and the values of the exact solution at the grid nodes on the last layer is shown in Fig. 9. Here *Digits*=15.

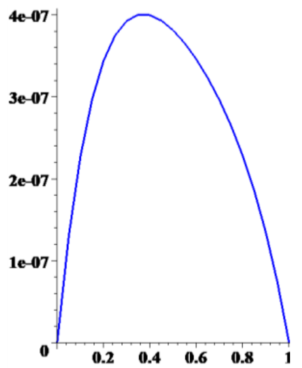

Fig.7. The graph of the difference between the values of the grid function and the values of the exact solution at the grid nodes on the last layer (the use of the polynomial splines).

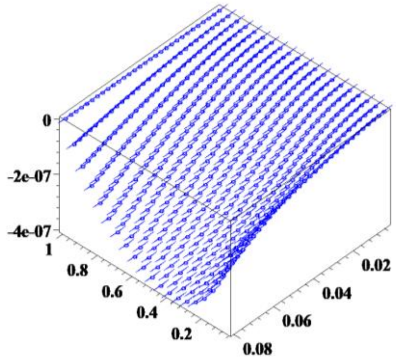

Fig.8. The plot of the error between the values of the exact solution and the values of the grid function at the grid nodes (the use of the polynomial splines)

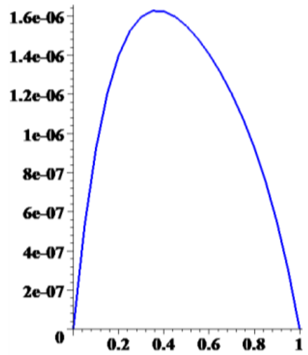

Fig.9. The graph of the difference between the values of the grid function and the values of the exact solution at the grid nodes on the last layer (the use of the trigonometric splines).

As it was written above, the exponential splines are constructed in the way that $U^{ME}(x) = u(x)$, $u(x) = \exp(x), \exp(-x), 1, x$, for $x \in [x_j, x_{j+1}]$. This explains the good quality of the approximate solution obtained using exponential splines of the problem from the example.

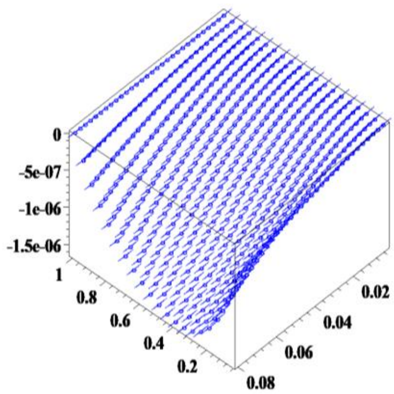

Fig.10. The plot of the errors between the values of the exact solution and the values of the grid function at the grid nodes (the use of the trigonometric splines).

The trigonometric splines in the Example 2 do not have a very good result. Let us apply the polynomial-trigonometric splines which we obtain from the conditions: $U^{MPE}(x) = u(x), u(x) = \sin(x), \cos(x), 1, x$, for $x \in [x_j, x_{j+1}]$. In this case we have the formula for the second derivative:

$$\left(U^{MTP}(x)\right)''_{x_j} = u(x_{j-1})g''_{j-1}(x_j) + u(x_j)g''_j(x_j) + u(x_{j+1})g''_{j+1}(x_j) + u(x_{j+2})g''_{j+2}(x_j),$$

where $g''_{j+2} = 0$, $g''_{j-1} = \frac{-1}{2(\cos(h)-1)}$,

$$g''_{j+1} = \frac{-1}{2(\cos(h)-1)}, g''_j = \frac{1}{\cos(h)-1}.$$

Here we note, then if $h \to 0$ we can obtain the relations:

$$g''_j(x_j) = -2h^{-2} - \frac{1}{6} - \frac{h^2}{120} + O(h^3),$$

$$g''_{j+1}(x_j) = g''_{j-1}(x_i) = h^{-2} + \frac{1}{12} + \frac{h^2}{240} + O(h^3).$$

Applying an implicit scheme for solving the heat equation, we solve a system of linear algebraic equations on each layer. The graph of the difference in absolute value between the values of the grid function and the values of the exact solution at the grid nodes on the last layer is shown in Fig. 11.

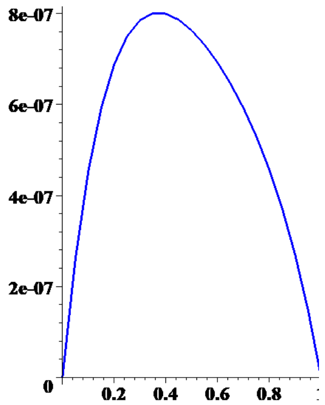

Fig.11. The graph of the difference between the values of the grid function and the values of the exact solution at the grid nodes (the use of new polynomial-trigonometric splines).

Fig. 12 shows the plot of the difference between the values of the exact solution and the values of the grid function at the grid nodes, where $n = 20, m = 30$ (the use of new polynomial-trigonometric splines).

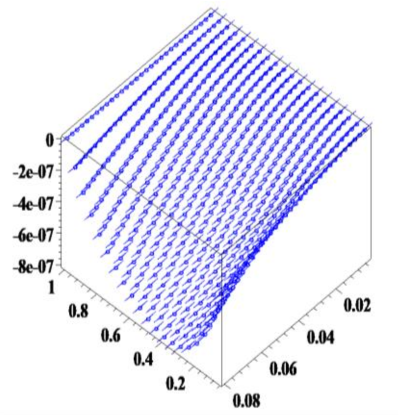

Fig.12. The plot of the errors between the values of the exact solution and the values of the grid function at the grid nodes.

4 The Approximation and Stability

As it is known, when we have to solve the Cauchy problem or a mixed problem, we need to make an approximation of the partial differential equation with the required order of approximation, after that we have to investigate the resulting scheme for stability. In addition, the system of equations should be investigated for solvability, which is formed when using the implicit solution method.

Papers [8-11] show how to obtain an error estimate for the splines that are discussed in this paper. Let us have a uniform grid of nodes with a step h. Using the method published in [9] it is easy to show that the next estimate of the error of approximation is valid: $|U - u| \leq Kh^4$. Therefore, to estimate the approximation of the second derivative, we obtain the inequality $|U'' - u''| \leq Kh^2$. Note that the corresponding interpolation polynomial, trigonometric, or exponential splines are convenient and expedient to use for calculating an approximate solution between the values of the grid function and for visualizing the result. It is very important that in the process of calculations according to the chosen scheme, various calculation errors, including rounding errors, do not accumulate.

To investigate the stability of an explicit scheme, we can apply the von Neumann stability condition. For the stability of the Cauchy problem with respect to the initial data we look for a solution to a homogeneous problem (when $f(x_j, t_{k+1}) = 0$) in the form: $u_{jk} = \lambda^k \exp(Ija)$, where I is the imaginary unit, a is real. For the explicit scheme we get $\tau \leq h^2/2$. We do not discuss the details in this paper.

In the case of the implicit scheme we have the difference equation:

$$\frac{\lambda - 1}{\tau} = \lambda({g''}_{j+1}\exp(Ia) + {g''}_{j-1}\exp(-Ia) + {g''}_j).$$

Our aim is to find out for which τ and h the following inequality will satisfy $|\lambda| \leq 1 + c\tau$ (von Neumann stability), when $c = const$ does not depend on τ and h. Using the equality $\exp(Ia) - 2 + \exp(-Ia) = -4\sin^2(a/2)$, we get for the trigonometric case

$$|\lambda| = \left|\frac{1}{1 - \tau {g''}_j + 2\tau\left(2\sin^2\left(\frac{a}{2}\right) - 1\right){g''}_{j+1}}\right| \leq 1 + c\tau,$$

when $c = 0.0565$. It is not difficult to see that the inequality $|\lambda| \leq 1 + c\tau$ holds for any correlation between τ and h.

Applying the maximum principle it can be shown that the implicit schema is stable for any τ and h. Similarly, stability is considered for exponential splines. The problem of constructing a convergent difference scheme is divided into two. The first problem is to construct a difference scheme that approximates the differential problem on the solution. The second task is to check the stability of the constructed difference scheme.

First of all, we recall some definitions. We have to solve the boundary value problem $Lu = f$ in domain D with the border Γ. Let $D_h = \{M_h\}$ be the set of nodes in $D \cup \Gamma$. Let $u = u(x,t)$ be the solution of the problem. Let the function $u^{(h)}$ be defined only in the set of nodes, so it will be called the mesh function. It is well-known that instead of solving the problem $Lu = f$, we solve the difference scheme $L_h u^{(h)} = f^{(h)}$. Let U_h be the linear normed space with the elements $u^{(h)}$. Let F_h be the linear normed space with the elements $f^{(h)}$. Let $\|\cdot\|_{U_h}$, $\|\cdot\|_{F_h}$ be the norms in the spaces U_h, F_h: $\| u^{(h)} \|_{U_h} = \max_{j,k} |u_{jk}|$,

$$\| f^{(h)} \|_{F_h} = \max(\max_j |u_0(jh)|, \max_k |\varphi_0(k\tau)|, \max_k |\varphi_1(k\tau)|, \max_{j,k} |f(jh, k\tau)|).$$

By the definition of stability, the solution of the difference scheme must satisfy the condition

$$\| u^{(h)} \|_{U_h} \leq K \| f^{(h)} \|_{F_h}$$

for any $f^{(h)}$. We examine for the stability, the implicit difference scheme that was constructed with the polynomial-trigonometric or with the polynomial-exponential splines:

$$\frac{u_{j,k+1}-u_{j,k}}{\tau} = u_{j-1,k+1}g''_{j-1} + u_{j+1,k+1}g''_{j+1} + u_{j,k+1}g''_{j} + f(x_j, t_{k+1}).$$

Multiply both sides of the difference equation by $-\tau$. We get

$$\tau\left(u_{j-1,k+1}g''_{j-1} + u_{j+1,k+1}g''_{j+1} + u_{j,k+1}g''_{j}\right) - u_{jk+1} = -\tau f(x_j, t_{k+1}) - u_{jk}.$$

We choose from all the values $u_{j,k+1}$ which in absolute value equals to $|u_{j,k+1}|$. Such a value, whose index j takes the smallest value $j=j^*$. Suppose $j^* \neq 0, j^* \neq$ n (otherwise the proof is obvious). Let us write the equation corresponding to this value:

$$\tau\left(u_{j^*-1,k+1}g''_{j^*-1} + u_{j^*+1,k+1}g''_{j^*+1} + u_{j^*,k+1}g''_{j^*}\right) - u_{j^*k+1} = -\tau f(x_{j^*}, t_{k+1}) - u_{j^*k}.$$

Let $u_{j^*,k+1} > 0$. Consider the right side of the equation above:

$$\tau\left(u_{j^*-1,k+1}g''_{j^*-1} + u_{j^*+1,k+1}g''_{j^*+1} + u_{j^*,k+1}g''_{j^*}\right) - u_{j^*k+1}$$
$$= \tau g''_{j^*+1}\left(u_{j^*+1,k+1} - u_{j^*,k+1}\right) + \tau g''_{j^*-1}\left(u_{j^*-1,k+1} - u_{j^*,k+1}\right)$$
$$+\tau u_{j^*,k+1}\left(g''_{j^*-1} + g''_{j^*+1} + g''_{j^*}\right) - u_{j^*,k+1} \leq -u_{j^*,k+1}.$$

Therefore $-u_{j^*,k+1} \geq -\tau f(x_{j^*}, t_{k+1}) - u_{j^*k}$.
Hence,

$$\max_j |u_{j,k+1}| = u_{j^*,k+1} \leq |\tau f(x_{j^*}, t_{k+1}) - u_{j^*k}|$$
$$\leq \max_j |u_{j,k+1}| + \tau \max_{j,k} |f(x_j, t_{k+1})|.$$

Using this inequality, it is easy to obtain the following inequality $\| u^{(h)} \|_{U_h} \leq K \| f^{(h)} \|_{F_h}$.

Thus, for any τ and h, the stability condition is satisfied for the difference scheme. Thus, the computational scheme is stable. Since the difference scheme also approximates the problem, the solution of the difference scheme converges to the solution of the problem when the polynomial, the trigonometric, the exponential, the polynomial-exponential and the polynomial-trigonometric were used.

4 A Visualization of the Result

In this section, we discuss the use of local polynomial and non-polynomial splines of the forth order of approximation to visualize the result. As a result of solving the problem, we obtained the values of the function at the grid nodes. These values were obtained with an error of $O(h^2)$. It is required to connect the values of the function using local splines of the desired approximation order. In the previous sections, we considered the application of polynomial and non-polynomial splines of the fourth order of approximation. In this section, we will construct new splines of the fourth order of approximation. These splines will take into account the behaviour of the first and second derivatives of the function. At first, we construct the spline so that the function values, as well as the first and second derivatives, are interpolated. In this case, it is necessary to use not only the values of the function at the grid nodes, but also the values of the first and second derivatives of this function at the grid nodes. Further, we use the formulas of numerical differentiation in order not to apply the derivatives of the function. We use the approximation with an error $O(h^3)$ and the approximation with an error $O(h^2)$. As a result, we get a spline approximation that uses only the function values at the grid points.

In this section, as in the previous sections, let a, b be real, and function $u(x)$ be such that $u \in C^4[a,b]$. Let x_j be a set of nodes such that

$$a = x_0 < \cdots < x_{j-1} < x_j < x_{j+1} < \cdots < x_n = b.$$

Now let us consider the question of constructing a continuous spline U, such that its support contains four intervals, the spline interpolates the function $u(x)$ in the nodes, as well as the first and second derivatives. Thus, it is assumed that the relations are valid:

$$U(x_j) = u(x_j), \qquad U(x_{j+1}) = u(x_{j+1}),$$
$$U'(x_j) = u'(x_j), \qquad U''(x_j) = u''(x_j).$$

We will construct an approximation of the function $u(x)$, $x \in [x_j, x_{j+1})$ in the form:

$$U(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) + u'(x_j)w_{j,1}(x) + u''(x_j)w_{j,2}(x).$$

The first and the second derivatives of this approximation will be discontinued at the nodes.

We find the basis functions $w_{j,i}(x)$ by solving the system of equations (approximation relations):

$$U(x) = u(x), \;\; u = x^i, \qquad i = 0, 1, 2, 3.$$

Let $h = x_{j+1} - x_j$. When $x \in [x_j, x_{j+1})$ we can put $x = x_j + th$, $t \in [0,1)$. The value of the determinant of the system is $2h^3$.
Now we obtain the formulas of the basis functions:
$w_{j,0}(x_j + th) = 1 - t^3$, $w_{j+1,0}(x_j + th) = t^3$,
$w_{j,1}(x_j + th) = th - ht^3$,
$w_{j,2}(x_j + th) = t^2h^2/2\text{-}t^2h^3/2$.
Figures 13-16 show graphs of basis functions $w_{j,i}$.

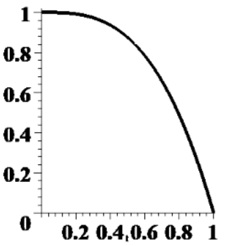

Fig.13. The plot of the basis function $w_{j,0}$.

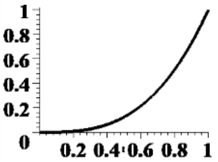

Fig.14. The plot of the basis function $w_{j+1,0}$.

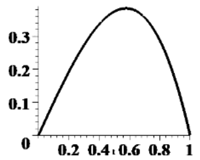

Fig.15. The plot of the basis function $w_{j,1}$.

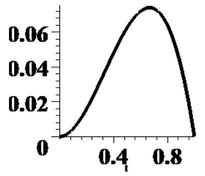

Fig.16. The plot of the basis function $w_{j,2}$.

The following statement is true.

Theorem 2. Let function $u(x)$ be such that $u \in C^4[a, b]$. We approximate the function $u(x)$ on the interval $[x_j, x_{j+1})$ using the expression:

$$U(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) + u'(x_j)w_{j,1}(x) + u''(x_j)w_{j,2}(x),$$

where
$w_{j,0}(x_j + th) = 1 - t^3$, $w_{j+1,0}(x_j + th) = t^3$,
$w_{j,1}(x_j + th) = th - ht^3$,
$w_{j,2}(x_j + th) = t^2h^2/2\text{-}t^2h^3/2$,
$t \in [0,1)$, $\quad h = x_{j+1} - x_j$.

Then the following approximation estimate is valid:

$$|u(x) - U(x)| \le 0.0043945h^4 \parallel u^{(4)} \parallel_{[x_j, x_{j+1}]}, \quad x \in [x_j, x_{j+1}) .$$

Proof. Using the Hermite interpolation remainder theorem, we obtain

$$u(x) - U(x) = \frac{u^{(4)}(\theta)}{4!}(x - x_j)^3(x - x_{j+1}),$$

where $\theta = \theta(x) \in [x_j, x_{j+1}]$.

If the grid is uniform with step h, then $x_{j+1} = x_j + h$. When $x \in [x_j, x_{j+1})$ we put $x = x_j + th, t \in [0,1)$. Thus, we obtain the estimate

$$|u(x_j + th) - U(x_j + th)| \le$$

$$\frac{h^4}{4!} \max_{[x_j, x_{j+1}]} |u^{(4)}| \max_{t \in [0,1]} |(t)^3(t - 1)|.$$

Let us find the maximum of the expression $|(t)^3(t - 1)|$.

It is not difficult to see that $\max_{t \in [0,1]} |(t)^3(t - 1)| \le 0.10547$. Thus, we obtain:

$$|u - U| \le h^4 \frac{\max_{[x_j, x_{j+1}]} |u^{(4)}|}{4!} 0.10547.$$

The proof is complete.

The estimation of the approximation can be written in the form: $|u - U] \le Ch^4$.

Now let us construct a new approximation of the function *u* using the obtained basis functions.

Our aim is to construct the approximation which uses only the values of the function u in the nodes. The grid of nodes is uniform with step h therefore, we can use next formulas for approximation the derivatives:

$$u'(x_j) = \frac{-2u(x_{j-1}) - 3u(x_j) + 6u(x_{j+1}) - u(x_{j+2})}{6h} + O(h^3),$$

$$u''(x_j) = \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} + O(h^2).$$

Denote

$$v_1 = \frac{-2u(x_{j-1}) - 3u(x_j) + 6u(x_{j+1}) - u(x_{j+2})}{6\,h},$$

$$v_2 = \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2}.$$

Now we get the approximation $V(x)$ in the form:

$$Q^p(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) + v_1 w_{j,1}(x) + v_2 w_{j,2}(x).$$

Therefore, we obtain the relation:

$$Q^p(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) + \frac{-2u(x_{j-1}) - 3u(x_j) + 6u(x_{j+1}) - u(x_{j+2})}{6h} w_{j,1}(x) + \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} w_{j,2}(x).$$

The approximation uses only the values of the function in the nodes and the old basis splines. It can be written in the form:

$$Q^p(x) = u(x_j)W_j^p(x) + u(x_{j+1})W_{j+1}^p(x) + u(x_{j-1})W_{j-1}^p(x) + u(x_{j+2})W_{j+2}^p(x),$$

where

$$W_j^p(x) = w_{j,0}(x) - \frac{3w_{j,1}(x)}{6h} - 2w_{j,2}(x)/h^2,$$

$$W_{j+1}^p(x)(x) = w_{j+1,0}(x) + \frac{6w_{j,1}(x)}{6h} + \frac{w_{j,2}(x)}{h^2},$$

$$W_{j-1}^p(x) = -\frac{2w_{j,1}(x)}{6h} + \frac{w_{j,2}(x)}{h^2},$$

$$W_{j+2}^p(x) = -\frac{w_{j,1}(x)}{6h}.$$

When $x = x_j + th$, we get for $t \in [0,1)$:

$$W_{j+2}^p(x_j + th) = t(t-1)(t+1)/6,$$

$$W_{j+1}^p(x_j + th) = -t(t-2)(t+1)/2,$$

$$W_j^p(x_j + th) = (t-1)(t-2)(t+1)/2,$$

$$W_{j-1}^p(x_j + th) = -t(t-2)(t-1)/6.$$

Thus, these basis splines are just the same as are the cubic polynomial splines g_{i+2}, g_{i+1}, g_i, g_{i-1}, and the approximation $Q^p(x)$ is just the same as $U^{MP}(x)$: $Q^p(x) = U^{MP}(x)$.

This approximation is considered only on the interval $[x_j, x_{j+1})$. Considering similar expressions on adjacent intervals, we can write out a formula for the basis spline $W_j(x)$, $x \in [x_{j-2}, x_{j+2})$.. This basis spline on the equidistant set of nodes can be written as a piecewise given polynomial.

$$W_j^p = -\frac{(t-1)(t+1)(t+2)}{2}, \quad t \in [-1,0],$$
$$W_j^p = \frac{(t-1)(t+1)(t-2)}{2}, \quad t \in [0,1],$$
$$W_j^p = -\frac{(t-1)(t-2)(t-3)}{6}, \quad t \in [1,2],$$
$$W_j^p = \frac{(t+1)(t+2)(t+3)}{6}, \quad t \in [-2,-1].$$

The plot of the basis spline $W_j^p(t)$ is shown in Fig.17.

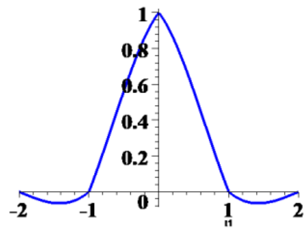

Fig.17. The plot of the basis function $W_j^p(t)$

Now let us compare two types of interpolation by the polynomial splines of the fourth order of approximation: cubic polynomial spline of the second level and the cubic spline when the derivatives were replaced using formulas of numerical differentiation. The interval $[-1,1]$ was considered, on which a uniform grid was constructed wit step $h = 0.1$. Table 1 presents the actual errors of approximation with the polynomial cubic splines $U(x)$ and the cubic splines $Q^p(x)$. Table 2 presents the theoretical errors of approximation with the polynomial cubic splines $U(x)$ and the cubic splines $Q^p(x)$

Figures 18-19 show the errors of approximation of the function $\sin(5x)$ when the interval $[-1,1]$ was considered, $n = 5$, and $h = 2/5$. Figure 18 shows the error of approximation when splines $U(x)$ were used, Figure 19 shows the error of approximation when splines $Q^p(x)$ were used,

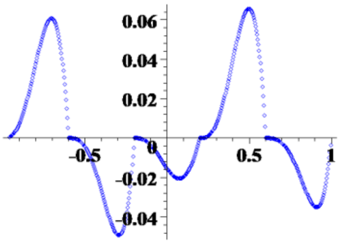

Fig.18. The plot of the error of approximation of function $\sin(5x)$, obtained with $U(x)$.

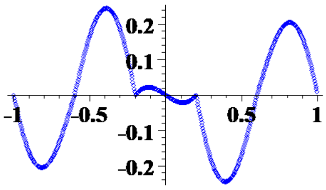

Fig.19. The plot of the error of approximation of the function $\sin(5x)$, obtained with $Q^p(x)$

Table 1. The actual errors of approximation with cubic ($U(x)$) and the cubic splines ($Q^p(x)$).

$F(x)$	$U(x)$	$Q^p(x)$
$\frac{1}{1+25x^2}$	0.001662	0.009529
$\sin(5x)$	0.0002430	0.001411

Table 2. The theoretical errors of approximation with cubic ($U(x)$) and the cubic splines ($Q^p(x)$).

$F(x)$	$U(x)$	$Q^p(x)$
$\frac{1}{1+25x^2}$	0.006592	0.03516
$\sin(5x)$	0.0002747	0.001465

The above examples show that the constructed splines of the second level can have an advantage over the usual polynomial cubic splines.

As it is known, using non-polynomial splines also improves the quality of the approximation. Now, in this section, non-polynomial splines of the fourth order of approximation will be constructed. These splines will take into account the behaviour of the first and second derivatives of the function.

Thus, we will construct the spline so that it interpolates the function, and the first and second derivatives of this function. In addition, we will construct this spline so that it is continuous. Let the function u be such that $u \in C^4[a,b]$. Let a uniform grid of nodes $\{x_j\}$ with a step h be constructed on the interval $[a,b]$. Similar to how it was done in the polynomial case, we will construct an approximation $U(x)$, $x \in [x_j, x_{j+1})$ in the form:

$$U(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) + u'(x_j)w_{j,1}(x) + u''(x_j)w_{j,2}(x).$$

Suppose that the supports of the basis functions are as follows: $supp\, w_{j,0} \in [x_{j-1}, x_{j+1}]$, $supp\, w_{j,1} \in [x_j, x_{j+1}]$, $supp\, w_{j,2} \in [x_j, x_{j+1}]$.

In the case of constructing non-polynomial splines of the second height of this type, we need to solve a system of equations of the form

$$\varphi_i(x_j)w_{j,0}(x) + \varphi_i(x_{j+1})w_{j+1,0}(x) + \varphi_i'(x_j)w_{j,1}(x) + \varphi_i''(x_j)w_{j,2}(x) = \varphi_i(x),$$
$$i = 0,1,2,3, \quad x \in [x_j, x_{j+1}).$$

We assume that the functions φ_i form a Chebyshov system, and the determinant of this system of equations differs from zero. If the basis splines are such that $\varphi_i = x^i$, then we come to the polynomial case considered in the previous section. Now we consider the polynomial-trigonometrical case when $\varphi_i = x^i, i = 0,1, \varphi_2 = \sin(x), \varphi_3 = \cos(x)$. The value of the determinant of the system of equations, when $h = x_{j+1} - x_j$, is equal to $\sin(h) - h$.

Let $x \in [x_j, x_{j+1})$. On the interval $[x_j, x_{j+1})$ the equality $x = x_j + t\,h$, $t \in [0,1)$, $h = x_{j+1} - x_j$, takes place. Thus, the basis splines take the form:

$$w_{j,0}(x_j + th) = \frac{\sin(h) - \sin(th) + h(t-1)}{\sin(h) - h},$$

$$w_{j+1,0}(x_j + th) = \frac{\sin(th) - th}{\sin(h) - h},$$

$$w_{j,1}(x_j + th) = \frac{\sin(h)\,th - h\,\sin(th)}{\sin(h) - h},$$

$$w_{j,2}(x_j + th) = \frac{h\,(\cos(t\,h) - t\cos(h) - 1 + t)}{\sin(h) - h} + \frac{\sin(th - h) - \sin(th) + \sin(h)}{\sin(h) - h}.$$

The plots of the basis functions $w_{j,i}(x)$ on the interval $[x_j, x_{j+1})$, $x_j = 0$, $x_{j+1} = 1$, when are given in Figures 20-23.

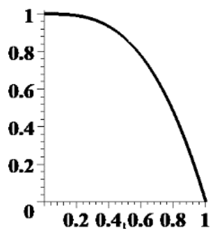

Fig.20. The plot of the basis function $w_{j,0}, h = 1$.

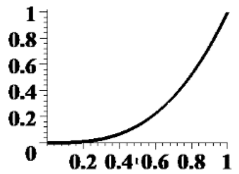

Fig.21.The plot of the basis function $w_{j+1,0}, h = 1$.

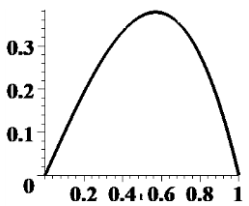

Fig.22. The plot of the basis function $w_{j,1}, h = 1$.

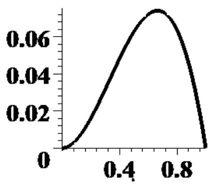

Fig.23. The plot of the basis function $w_{j,2}, h = 1$

On the adjacent interval $[x_{j-1}, x_j)$, which is to the left of the interval $[x_j, x_{j+1})$, we determine the basis functions by solving the system of equations. Thus, we solve a system of equations of the form:

$$\varphi_i(x_{j-1})w_{j-1,0}(x) + \varphi_i(x_j)w_{j,0}(x) + \varphi_i'(x_{j-1})w_{j-1,1}(x) + \varphi_i''(x_j)w_{j-1,2}(x) = \varphi_i(x),$$
$$i = 0, 1, 2, 3, \quad x \in [x_{j-1}, x_j).$$

Combining the results obtained for the basis spline $w_{j,0}(x)$, we obtain the formulas:

$$w_{j,0}(x) = \frac{\sin(h) - h + th - \sin(t\,h)}{\sin(h) - h}, \quad x \in [x_j, x_{j+1}),$$

$$w_{j,0}(x) = \frac{\sin(t\,h) - th}{\sin(h) - h}, \quad x \in [x_{j-1}, x_j).$$

Combining these two possibilities, we get the basic function $w_{j,0}(x)$ on the interval $[x_{j-1}, x_{j+1})$.

The plot of the basis function $w_{j,0}(x)$ on the interval $[x_{j-1}, x_{j+1})$ is given in Fig. 24.

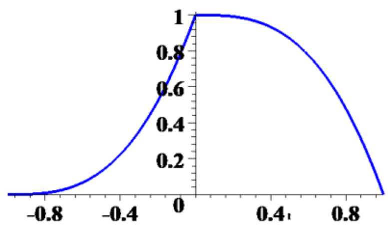

Fig.24. The plot of the basis function $w_{j,0}(x)$ on the interval $[x_{j-1}, x_{j+1})$.

We denote by $V(x)$, $x \in [a, b)$, a piecewise function given on each interval $[x_j, x_{j+1})$ as follows

$$V^{PT}(x) = U(x), \quad U(x) \in [x_j, x_{j+1}),$$
$$j = 0, 1, \dots, n-1.$$

Table 3 shows the results of numerical experiments. Let us construct a uniform grid of nodes on the interval $[-1,1]$ with the step $h = 2/n$. Let the values of the function $u(x)$ and its first and second derivatives are given at the grid nodes. The actual errors of approximation with the polynomial-trigonometric splines od the second level are given in Table 3.

We denote $R = \max_{[-1,1]} |u - V^{PT}|$.

Table 3. The actual errors of approximation with the polynomial-trigonometric splines od the second level

$u(x)$	$R, n = 20$	$R, n = 200$
$\frac{1}{1+25x^2}$	0.001655	$0.5274 \cdot 10^{-6}$
$\sin(5x)$	0.0002333	$0.3464 \cdot 10^{-7}$

The plot of the error of the approximation of the function $1/(1 + 25x^2)$ when $n = 20$, is given in Fig.25.

The plot of the error of the approximation of the function $1/(1 + 25x^2)$ when $n = 40$, is given in Fig.26.

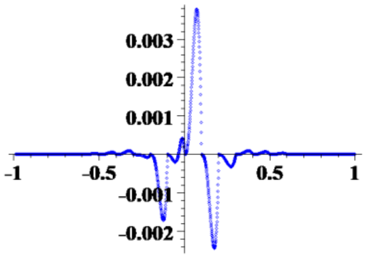

Fig.25. The plot of the error of the approximation of the function $1/(1+25x^2)$. Here $n=20$.

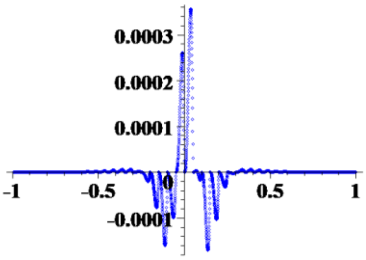

Fig.26. The plot of the error of the approximation of the function $1/(1+25x^2)$. Here $n=40$.

The plot of the error of the approximation of the function $\sin(5x)$ when $n=20$, is given in Fig.27.

The plot of the error of the approximation of the function $\sin(5x)$ when $n=40$, is given in Fig.28.

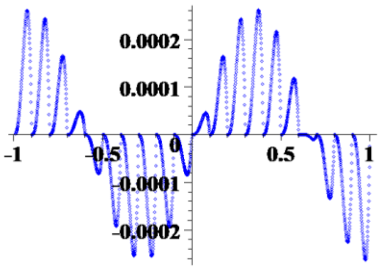

Fig.27. The plot of the error of the approximation of the function $\sin(5x)$. Here $n=20$.

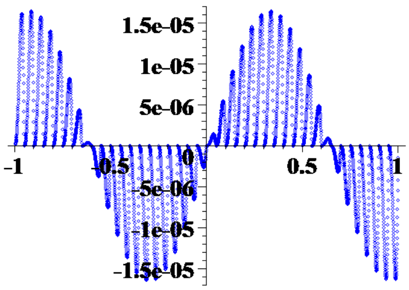

Fig.28. The plot of the error of the approximation of the function $\sin(5x)$. Here $n=40$.

Now our task is to construct an approximation that uses only the values of the function at the grid points. To do this, we replace the first and second derivatives of the function at the grid nodes using the formulas for numerical differentiation. To construct formulas for numerical differentiation, we will use a spline which has been constructed by using the system of functions: $1, \sin(x), \cos(x), x$. Differentiating this spline twice, we obtain the formula for the second derivative at the point x_j:

$$u''(x_j) = \frac{-U_{j-1}+2U_j-U_{j+1}}{2(\cos(h)-1)} + O(h^2).$$

The expression for the first derivative at the point mm gives the equality

$$u'(x_j) = \frac{-A_j\,U_j}{2} + \frac{A_{j+1}U_{j+1}}{2} - \frac{A_{j-1}\,U_{j-1}}{2} - \frac{A_{j+2}\,U_{j+2}}{2} + O(h^3),$$

where

$$A_j = \frac{(\sin(h)-h)(2\cos(h)+1)}{h\sin(h)(\cos(h)-1)},$$

$$A_{j+1} = \frac{2\sin(h)\cos(h)+\sin(h)-h\cos(h)-2h}{h\sin(h)(\cos(h)-1)}.$$

$$A_{j-1} = \frac{h\cos(h)-\sin(h)}{h\sin(h)(\cos(h)-1)},$$

$$A_{j+2} = \frac{\sin(h)-h}{h\sin(h)(\cos(h)-1)}.$$

Now we replace the values of the first and second derivatives of the function through the numerical differentiation formulas in the expression for the spline approximation. In this case, we obtain an approximation of the form

$$V(x_j+th) = C_j(t)U_j + C_{j+1}(t)U_{j+1} + C_{j-1}(t)U_{j-1} + C_{j+2}(t)U_{j+2},$$

where $t\in[0,1)$, and the basis functions are as follows:

$$C_{j+1} = (\sin(t\,h)-t\,h)/(\sin(h)-h) + (\sin(2h)+\sin(h)-h\cos(h)-2h)(h\,t\sin(h) - h\sin(th)))/((2h\sin(h))(\cos(h) - 1)(\sin(h)-h)) -(h\cos(th)-t\,h\cos(h)+\sin(th-h)-h+th - \sin(th)+\sin(h))/((\sin(h)-h)(2\cos(h)-2)),$$

$$C_j = (th-h+\sin(h)-\sin(th))/(\sin(h)-h) -(2\cos(h)+1)(t\,h\sin(h) -h\sin(t\,h))/(2h\sin(h)\,(\cos(h)-1)) +2(h\cos(t\,h)-t\,h\cos(h)+\sin(t\,h-h)-h +t\,h+\sin(h)-\sin(th)) /((\sin(h)-h)(2\cos(h)-2)),$$

$$C_{j-1} = -(h\cos(h)-\sin(h))(t\,h\sin(h) -h\sin(th))$$

$$/(2h\sin(h)(\cos(h)-1)(\sin(h)-h)) - (h\cos(th)-th\cos(h) + \sin(th-h) - h + th - \sin(th) + \sin(h))/((\sin(h)-h)(2\cos(h)-2)),$$

$$C_{j+2} = -\frac{(t\,h\,\sin(h) - h\sin(th))}{2\,h\sin(h)(\cos(h)-1)}.$$

The plot of the basis functions $C_k, k = j, j-1, j+1, j+2$, when $h = 1$, are given in Fig 29.

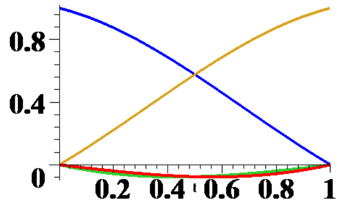

Fig. 29. The plots of the basis functions C_k, $k = j, j-1, j+1, j+2$, when $h = 1$.

It is not difficult to obtain the formula for the basic spline C_j. We will join the endpoints of the spline values at the grid nodes by continuity. Then the support of this basic spline C_j will occupy four neighbouring grid intervals: supp $C_j = [x_{j-2}, x_{j+2}]$. This basis spline can be given by an alternative given function:

$$W_j = A_{j0} + A_{j1} + A_{j2},$$

where

$$A_{j0} = \frac{th - h + \sin(h) - \sin(th)}{\sin(h) - h}$$

$$A_{j1} = -(2\cos(h)+1)(t\,h\sin(h) - h\sin(th))/(2h\sin(h)(\cos(h)-1)),$$

$$A_{j2} = 2(h\cos(th) - t\,h\cos(h) + \sin(th-h) - h + t\,h + \sin(h) - \sin(th))/((\sin(h) - h)(2\cos(h)-2)),$$

and when $t \in [0,1]$

$$W_j = B_{j0} + B_{j1} + B_{j2},$$

where

$$B_{j0} = ((\sin((t+1)h)) - (t+1)h))/(\sin(h) - h),$$

$$B_{j1} = (\sin(2h) + \sin(h)) - h\cos(h) - 2h)((t+1)h\sin(h) - h\sin((t+1)h))/(2h\sin(h)(\cos(h)-1)(\sin(h)-h)),$$

$$B_{j2} = -(h\cos((t+1)h) - h(t+1)\cos(h) + \sin(h(t+1) - h) - h + h(t+1) - \sin((t+1)h) + \sin(h))/((2\cos(h)-2)(\sin(h)-h)),$$

and when $t \in [-1,0]$,

$$W_j = D_{j0} + D_{j1},$$

where

$$D_{j0} = -((h\cos(h) - \sin(h))(h(t-1)\sin(h) - h\sin(h(t-1)))/(2\,h\sin(h)(\cos(h) - 1)(\sin(h)-h)),$$

$$D_{j1} = -(h\cos(h(t-1)) - h(t-1)\cos(h) + \sin((t-1)h - h) - h + h(t-1) - \sin((t-1)h) + \sin(h))/((2\cos(h)-2)(\sin(h)-h)),$$

and when $t \in [1,2]$,

$$W_j = -\left((t+2)h\sin(h) - h\sin((t+2)h)\right)/(2h\sin(h)(\cos(h)-1))$$

when $t \in [-2,-1]$.

The plot of the basis function C_j is given in Fig. 30.

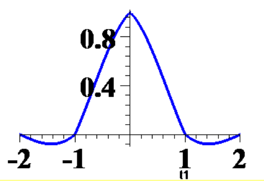

Fig.30. The plot of the basis function C_j, when $h = 1$.

It is easy to see that there are relations between the polynomial and newly constructed splines:

$$W_j = -\frac{(t-1)(t+1)(t+2)}{2} + O(h^2), \quad t \in [-1,0]$$

$$W_j = \frac{(t-1)(t+1)(t-2)}{2} + O\left(h^2\right), \quad t \in [0,1],$$

$$W_j = -\frac{(t-1)(t-2)(t-3)}{6} + O\left(h^2\right), \quad t \in [1,2],$$

$$W_j = \frac{(t+1)(t+2)(t+3)}{6} + O\left(h^2\right), \quad t \in [-2,-1],$$

In addition, the following relations hold:

$$W_{j+1} = -\frac{t(t+1)(t-2)}{2} + O(h^2), \quad t \in [0,1],$$

$$W_j = \frac{(t-1)(t+1)(t-2)}{2} + O(h^2), \quad t \in [0,1],$$

$$W_{j-1} = -\frac{t(t-1)(t-2)}{6} + O(h^2), \quad t \in [0,1],$$

$$W_{j+2} = \frac{t(t+1)(t-1)}{6} + O(h^2), \quad t \in [0,1].$$

Thus, in this paper we have constructed the spline, of the fourth order of approximation, which interpolates the function, and its first and second derivatives at the grid nodes. This spline uses the values of the function and its first and second derivatives. Further, applying the formulas of numerical differentiation and the obtained spline, we constructed a continuous spline, also of the fourth order of approximation. This spline uses only the values of the function at the grid nodes and takes into account the behaviour of the first and second derivatives of the function being approximated.

Remark. In the proposed method, the same rules should be preserved as in the traditional method. It is necessary not to forget about the unremovable error of numerical differentiation and not to select too small a grid step.

5 Conclusion

We considered alternative implicit difference schemes for solving the heat equation. For the construction, polynomial, trigonometric, exponential, polynomial-exponential and polynomial-trigonometric splines were used. The theoretical errors of approximation are obtained. The resulting difference schemes turned out to be stable; therefore, the proposed schemes turned out to be convergent to the solution of the problem.

In this paper, polynomial and non-polynomial splines of the fourth order of approximation are constructed. They take into account the behaviour of the first and second derivatives of the function.

Using non-polynomial splines also improves the quality of the approximation. Having analyzed the results of numerical experiments presented in the article, we note that when constructing a numerical solution, polynomial-trigonometric splines give a smaller error.

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