Conserved Quantities and Fluxes for Some Nonlinear Evolution Equations

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Abstract: - In this paper, I shall show that the conservation laws structure can be defined for any nonlinear evolution equations which describe surfaces of a constant negative curvature, so that the densities of conservation laws and fluxes can be calculated .

Key-Words: - conservation laws; pseudo spherical surfaces; nonlinear evolution equations.
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1. Introduction

Investigations of solitary waves and solitons began with their discovery by John Scott Russell in 1834, as he rode his horse beside the narrow Union Canal near Edinburgh, Scotland [4]. His subsequent laboratory work, and that of Stokes, Boussinesq, and Rayleigh, further probed the nature of solitary waves (nonlinear waves that do not change shape as they travel), describing them in terms of equations from fluid dynamics [1-8].
The question as to whether equations for water waves allowed for the existence of solitary-wave solutions was finally answered in 1895.
when the Dutch physicist Diederik Johannes Korteweg and his student Gustav de Vries derived an equation that supported the existence of solitary waves, which now bears their names. Despite this early derivation of the Korteweg-de Vries (KdV) equation, it was not until 1960 that any new application of the equation was discovered [9].
In 1965, from detailed numerical study, Zabusky and Kruskal [10] found that stable pulse-like waves could exist in a system described by the KdV equation. A remarkable property of these solitary waves was that they could collide with each other and yet preserve their shapes and speeds after the collision. Solitary waves with that property are called solitons. This discovery created renewed interest in the equations for solitary waves and the special properties of their solutions. New more powerful methods for describing the waves mathematically have been developed, and many equations have been found to have solitary waves and solitons as solutions [11-20].
In the mid 1920‟s, Oskar Klein and Felix Gordon [21-23] derived an equation for a charged particle in an electromagnetic field, using thennew ideas in the realm of quantum theory. From their work several equations addressed in this paper arise. Perhaps the most well-known example, the sine-Gordon equation, has been seen in the propagation of a dislocation in a crystal, in the modulation of wave packets in a moving medium, and in the propagation of magnetic flux in superconductor equations, among other areas of modern research. Solitary waves have been observed in a variety of natural realms: in the atmosphere, in oceans, in plasmas, and possibly in nervous systems of living organisms. Finally, solitons started playing an important technological role in modern telecommunications. Their persistent shape and immunity to distortion make them suitable carriers of long-distance signals [24].
Typically, autonomous evolution equations with translation invariance (such as the KdV equation) have only the three „classical‟ conserved quantities, namely the mass, the momentum, and the energy. However, the KdV equation has infinitely many conserved quantities. The existence of an infinite sequence of conservation laws for a given system of partial differential equations (PDEs) suggests that it is completely integrable, though such a condition is not required [25]. Indeed, there are systems (such as the Burgers equation)
that can be directly integrated, though possess only a finite number of conservation laws [26]. Additionally, conservation laws provide a simple and efficient method

to study both quantitative and qualitative properties of solutions. A comprehensive definition of the term integrable is proving to be elusive. Integrable systems are in some sense exactly solvable and exhibit globally regular solutions for all initial conditions. In contrast, the term nonintegrable is, generally, taken to imply that a system can not be solved exactly and that its solutions can behave in an irregular fashion due to sensitivity to initial conditions [27].
The method of solution, however, is complex. Though many physical problems are modeled by nonlinear evolution equations (NLEEs), the Fourier transform method is insufficient to solve the problem [28]. In fact, the method of solution for NLEEs, the Inverse Scattering Transform (IST), would come from a classical scattering problem of quantum mechanics.
The computational mechanics of the IST are similar to those involved in the Fourier transform (for solving linear equations), except that the final step of solving the IST is "highly nontrivial". In this paper, I present a method for computing conservation laws for systems of nonlinear evolution equations.

2. A Local Polynomial Conservation Law:

The focus of this research and the determination of conservation laws for nonlinear systems of PDEs of the form:

$$u_t = k(u), \quad \text{where} \quad k(u) = k(u, u_t, u_{tx}, \ldots), \quad u = u(x,t) \tag{1}$$

A conservation law for (1) is an equation of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{2}$$

Which is satisfied for all solutions of (1), where p the conserved density, and – (pu) the associated flux, in general are functions of x, t, u and the partial derivatives of u *(with respect to).* D_t denotes the total derivative with respect to t
D_x the total derivative with respect to x .
If p is a polynomial in x derivatives exclusively, then p is
called a local polynomial conserved density.

Example 1: The most famous scalar evolution equation from soliton theory, the Korteweg-de Vries (KdV) equation ,

$$F[u] \equiv u_t - 6uu_x + u_{xxx} = 0 \tag{3}$$

is known to have infinitely many polynomial conservation laws. The first three polynomial conservation laws are given by:

$$(u)_t + (u_{xx} - 3u^2)_x = 0$$
$$(\frac{1}{2}u^2)_t + (uu_{xx} - \frac{1}{2}u_x^2 - 2u^3)_x = 0, \tag{4}$$
$$(\frac{u^3}{3} + \frac{1}{6}u_x^2)_t + (u^2u_{xx} + \frac{1}{3}u_xu_{xxx} - \frac{3}{2}u^4 - 2uu_x^2 - \frac{1}{6}u_{xx}^2)_x = 0.$$

Example 2: the sine – Gordon equation

$$F[u] \equiv u_{xt} - \sin u = 0, \tag{5}$$

is known to have infinitely many polynomial conservation laws. The first three polynomial conservation laws are given by:

$$(1-\cos u)_t + (-\frac{1}{2}u_t^2)_x = 0,$$
$$(\frac{1}{2}u_x^2)_t + (\cos u - 1)_x = 0, \tag{6}$$
$$(\frac{1}{4}u_x^4 - u_{xx}^2)_t + (u_x^2 \cos u)_x = 0.$$

The first two express conservation of momentum and energy, respectively, and are relatively easy to compute by hand. The third one, which is less obvious, requires more work.

3. Conserved Quantities and Fluxes

For Nonlinear – Evolution Equations

The class of equations of pseudo-spherical type (or "describing pseudo-spherical surfaces") was introduced by S.S. Chern and K. Tenenblat [27] in 1986, motivated by the following observation by Sasaki: the solutions of equations integrable by the Ablowitz, Kaup, Newell, and Segur (AKNS) [26] inverse scattering approach can be equipped, whenever their associated linear problems are real, with Riemannian metrics of constant Gaussian curvature equal to .
I recall the definition [16, 21] of a differential equation (DE) that describes a pss. Let M^2 be a two dimensional differentiable manifold with coordinates . A DE for a real function $u(x,t)$ describes a pss if it is a necessary and sufficient condition for the existence of differentiable functions

$$f_{ij}, \qquad 1 \le i \le 3, \quad 1 \le j \le 2, \tag{7}$$

depending on and its derivatives such that the one-forms

$$\omega_1 = f_{11}dx + f_{12}dt, \quad \omega_2 = f_{21}dx + f_{22}dt, \quad \omega_3 = f_{31}dx + f_{32}dt, \quad (8)$$

satisfy the structure equations of a pss, i.e.,

$$d\omega_1 = \omega_3 \wedge \omega_2, \qquad d\omega_2 = \omega_1 \wedge \omega_3, \qquad d\omega_3 = \omega_1 \wedge \omega_2. \quad (9)$$

As a consequence, each solution of the DE provides a local metric on M^2, whose Gaussian curvature is constant, equal to −1. Moreover, the above definition is equivalent to saying that DE for *u* is the integrability condition for the problem [14,26]:

$$d\phi = \Omega\phi, \quad \phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}, \quad (10)$$

where denotes exterior differentiation, is a column vector and the 2 × 2 matrix $\Omega\,(\Omega_{ij}, i, j = 1,2)$ is traceless

$$\Omega = \frac{1}{2}\begin{pmatrix}\omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2\end{pmatrix}.$$

Take

$$\Omega = \begin{pmatrix}\eta dx + Adt & qdx + Bdt \\ rdx + Cdt & -\eta dx - Adt\end{pmatrix} = Sdx + Tdt, \quad (11)$$

from Eqs. (10) and (11), we obtain

$$\phi_x = S\phi, \quad \phi_t = T\phi, \quad (12)$$

where *S* and *T* are two 2 × 2 null-trace matrices

$$S = \begin{pmatrix}\eta & q \\ r & -\eta\end{pmatrix}, \quad (13)$$

$$T = \begin{pmatrix}A & B \\ C & -A\end{pmatrix}. \quad (14)$$

Here ŋ is a parameter, independent of *x* and *t*, while *q* and *r* are functions of *x* and *t*. Now

$$0 = d^2\phi = d\Omega\phi - \Omega \wedge d\phi = (d\Omega - \Omega \wedge \Omega)\phi,$$

which requires the vanishing of the two form

$$\Theta \equiv d\Omega - \Omega \wedge \Omega = 0, \quad (15)$$

or in component form

$$-A_x + qC - rB = 0,$$

$$q_t - 2Aq - B_x + 2\eta B = 0, \quad (16)$$

$$r_t - C_x + 2Ar - 2\eta C = 0,$$

Or

$$-f_{11,t} + f_{12,x} = f_{31}f_{22} - f_{21}f_{32},$$

$$-f_{21,t} + f_{22,x} = f_{11}f_{32} - f_{12}f_{31}, \quad (17)$$

$$-f_{31,t} + f_{32,x} = f_{11}f_{22} - f_{12}f_{21},$$

where

$$q = \frac{1}{2}(f_{11} - f_{31}), \qquad r = \frac{1}{2}(f_{11} + f_{31})$$

$$A = \frac{1}{2}f_{22}, \; B = \frac{1}{2}(f_{12} - f_{32}), \; c = \frac{1}{2}(f_{12} + f_{32}). \quad (18)$$

Chern and Tenenblat [27] obtained Eq. (16) directly from the structure equations (9). By suitably choosing r, A, B and C in (16), I shall obtain various nonlinear evolution equation which q must satisfy. Konno and Wadati introduced the function [28]

$$\Gamma = \frac{\phi_1}{\phi_2}, \quad (19)$$

this function first appeared used and explained in the geometric context of pseudo spherical equations in [11,13], and see also the classical papers by Sasaki and Chern–Tenenblat [27]. Then Eq. (12) is reduced to the Riccati equations:

$$\frac{\partial\Gamma}{\partial x} = \eta\Gamma - r\Gamma^2 + q, \quad (20)$$

$$\frac{\partial\Gamma}{\partial t} = 2A\Gamma - C\Gamma^2 + B. \quad (21)$$

Equations (20) and (21) imply that

$$c\Gamma_x - r\Gamma_t = (cq - rB) + (\eta c - 2Ar)\Gamma. \quad (22)$$

Adding $(r_t T)$ to both sides and using the expression $A_X = qc + rB$ from (24), equation (22) takes the form

$$\frac{\partial}{\partial t}(r\Gamma) = \frac{\partial}{\partial x}(-A + c\Gamma), \quad (23)$$

where (*rT*) are conserved densities and (*-A+ CT*) are fluxes

Example 1: family of equations

$$[u_t - (\alpha g(u) + \beta)u_x]_x = g'(u) \quad (24)$$

$$\text{where} \qquad g' = \frac{\partial g}{\partial u}, \; g'' + \mu g = \theta, \; \xi^2 = \alpha\eta^2 - \mu$$

The differentiable functions fᵢ depending on *u* and its derivatives are

$$\left.\begin{array}{ll} f_{11}=\xi u_x \ , & f_{12}=\xi(\alpha g+\beta)u_x \ , \\ f_{21}=\eta \ , & f_{22}=(\dfrac{\xi^2 g-\theta}{\eta}+\beta\eta) \ , \\ f_{31}=0 \ , & f_{32}=\dfrac{\xi}{\eta}g' \ . \end{array}\right\} \tag{25}$$

Then, from equation (18), I obtain

$$r=\frac{1}{2}(\xi u_x), \qquad A=\frac{1}{2}(\frac{\xi^2 g-\theta}{\eta}+\beta\eta)$$

$$r=\frac{1}{2}(\xi u_x), \qquad A=\frac{1}{2}(\frac{\xi^2 g-\theta}{\eta}+\beta\eta)$$

$$c=\frac{1}{2}\left(\xi(\alpha g+\beta)u_x+\frac{\xi}{\eta}g'\right), \qquad \Gamma=e^{\eta\rho},$$

$$u=-u_0+\frac{4}{\xi}\tanh^{-1}\left(e^{\eta\rho}\right), \ \textit{where} \ \ u_0 \ \textit{is a constant solution}$$

$$\textit{where} \qquad \rho=x+kt \ , \ k=\frac{\xi^2 g(u_0)-\theta}{\eta^2}+\beta \tag{26}$$

Then

The conserved densities = $(r\Gamma)=\frac{1}{2}e^{\eta\rho}\left(\xi[\frac{4}{\xi}\tanh^{-1}(e^{\eta\rho})]\right)_x$.

And

The flux = $(-A+c\Gamma)=-\frac{1}{2}(\frac{\xi^2 g-\theta}{\eta}+\beta\eta)+\frac{1}{2}\left(\xi(\alpha g+\beta)u_x+\frac{\xi}{\eta}g'\right)e^{\eta\rho}$

Example 2:the modified Korteweg-de Vries (MKdV) equation

$$u_t=u_{xxx}+(a+u^2)u_x \tag{27}$$

The differentiable functions fi depending on and its derivatives are

$$\left.\begin{array}{ll} f_{11}=0 \ , & f_{12}=-\eta\sqrt{\dfrac{2}{3}}u_x \ , \\ f_{21}=\eta \ , & f_{22}=(\eta^3+\dfrac{\eta u^2}{3}+a\eta) \ , \\ f_{31}=\sqrt{\dfrac{2}{3}}u \ , & f_{32}=\sqrt{\dfrac{2}{3}}(\eta^2 u+\dfrac{u^3}{3}+u_{xx}+au) \ , \end{array}\right\} \tag{28}$$

Then, from equation (18), I obtain

$$r=\frac{1}{2}\left(\sqrt{\frac{2}{3}}u\right), \qquad A=\frac{1}{2}(\eta^3+\frac{\eta u^2}{3}+a\eta)$$

$$c=\frac{1}{\sqrt{6}}(-\eta u_x+u_{xx}+\frac{u^3}{3}+\eta^2 u+au) \ , \qquad \Gamma=(\frac{\sqrt{6}}{\rho u_0}+1),$$

$$u=u_0+2\sqrt{6}\frac{\partial}{\partial x}[\tan^{-1}(\frac{\sqrt{6}}{\rho u_0}+1)], \ \textit{where} \ \ u_0 \ \textit{is a constant solution}$$

$$\rho=x+kt \ , \ k=\eta^2+\frac{1}{3}u_0^{\ 2}+a$$

Then

The conserved densities

$$=r\Gamma=\frac{1}{\sqrt{6}}(\frac{\sqrt{6}}{\rho u_0}+1)\left(u_0+2\sqrt{6}\frac{\partial}{\partial x}[\tan^{-1}(\frac{\sqrt{6}}{\rho u_0}+1)]\right) \ ,$$

and

The flux =

$$(-A+c\Gamma)=-\frac{1}{2}(\eta^3+\frac{\eta u^2}{3}+a\eta)+\frac{1}{\sqrt{6}}(-\eta u_x+u_{xx}+\frac{u^3}{3}+\eta^2 u+au)(\frac{\sqrt{6}}{\rho u_0}+1)$$

4. Conclusion

This geometrical method is considered for NLEEs which describe pseudo-spherical surfaces: family of equations and the modified Korteweg-de Vries equation. In this paper, I considered the construction of conservation laws to some NLEEs (family of equations and the modified Korteweg-de Vries equation) by inverse scattering method Next the conservation laws is derived for the NLEEs mentioned above are derived in this way.

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