When Intersection Ideals of Graphs of Rings are a Divisor graph

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Abstract: Let **R** be a commutative principal ideal ring with unity. In this paper, we classify when the intersection graphs of ideals of a ring **R** $G(\mathbf{R})$, is a divisor graph. We prove that the intersection graphs of ideals of a ring **R** $G(\mathbf{R})$, is a divisor graph if and only if **R** is a local ring or it is a product of two local rings with each of them has one chain of ideals. We also prove that $G(\mathbf{R})$, is a divisor graph if it is a product of two local rings one of them has at most two non-trivial ideals with empty intersection.

Key-Words: The commutative ring, The intersection ideals of graphs of rings **R**, Principal ideal ring, Local ring, Divisor graph.

Received: April 5, 2020. Revised: June 26, 2020. Accepted: August 14, 2020. Published: September 8, 2020.

1 Introduction

In this paper, all rings are be a commutative principal ideal ring with unity and all graphs are permused to be simple.

Let **S** be a nonempty set of positive integers and let $\mathbf{G}_S$ be the graph whose vertices are the element of **S**. Which is two distinct vertices a, b are adjacent if and only if a divides b or b divides a. A graph **G** is called a divisor graph if there is a set of positive integers **S** such that $\mathbf{G} \cong \mathbf{G}_S$. For **S**$= \{1, 2, ..., n\}$, the length of longest path in $\mathbf{G}_S$ is studied in [5, 6, 7]. In [4], divisor graph are investigated. Some results are listed below:

1. No divisor graph contains an induced odd cycle of length 5 or more (proposition 2.1).
2. An induced subgraph of a divisor graph is a divisor graph (proposition 2.2).
3. Complete graphs and bipartite graphs are divisor graphs (proposition 2.5 and Theorem, 2.7).
4. A graph **G** is a divisor graph if and only if there is an orientation **D** of **G** in which every vertex is transmitter, receiver, or transitive (Theorem 3.1).

Divisor graphs are also studied in [2, 3].

Another concept of the undirected simple graph is the concept of intersection graphs of ideals of rings **R**, denoted by **G(R)** which is vertices are in one-to-one correspondence with all nontrivial ideals of **R** and two distinct vertices are joined by an edge if and only if the corresponding ideals of **R** have a nontrivial (nonzero) intersection. Evidently the set of vertices is empty for simple rings. In this case **G(R)** is an empty graph.It is shown that for aye simple graph is an intersection graph, ([8]).
It is exciting to study the intersection graphs **G** when the members of **F** have an algebraic structure. And Bosak [9] in 1964 studied graphs of semigroups. And after that Cskny and Pollk [10] in 1969 studied the intersection graphs of subgroups of a finite group. Zelinka [11] in 1975 proceeded the work on intersection graphs of nontrivial subgroups of finite abelian groups.

Chakrabarty et al. [4] studied intersection graphs of ideals of rings. The intersection graph of ideals of a ring **R**, denoted by **G(R)**, is the undirected simple graph (without loops and multiple edges) whose vertices are in one-to-one correspondence with all nontrivial left ideals of **R** and two distinct vertices are joined by an edge if and only if the corresponding left ideals of **R** have a nontrivial (nonzero) intersection. Clearly the set of vertices is empty for left simple rings. In this case we refer **G(R)** as empty graph. The idea behind presenting the intersection, **G(R)**, of a ring **R** is to study and research the relationship between algebraic properties of the ring **R** and the graph theoretic properties of the graph **G(R)**.

Let **R** be a ring. By **I** and $\mathbf{I}^*$ we mean the set of all ideals of **R** and the set of all nontrivial ideals of **R** respectively. A ring **R** is local if it has a unique maximal ideal.

Let **G** be a graph wich the vertex set is **V(G)**. The complete graph of order n is denoted by K_n, is a graph whose n vertices in which any two distinct vertices are adjacent. A star graph is a graph with a vertex adjacent to all other vertices and has no other edges. Recall that a graph **G** is called connected graph if there is a path between every two distinct vertices. For every pair of distinct vertices x and y of **G**, let $d(x, y)$ be the length of the shortest path from x to y and if there is no such a path we define $d(x, y) = \infty$. The diameter of **G**, $diam(\mathbf{G})$, is the supremum of the set $\{d(x, y) : x$ and y are distinct vertices of $\mathbf{G}\}$.

In our investigation, we start with Lemma.

Lemma 1: A graph that includes the following induced subgraph (Figure 1) is not a divisor graph.

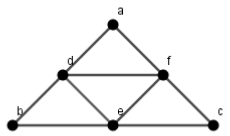

Figure 1: A graph which is not a divisor graph.

Proof: Let **D** be an orientation of the above graph in which every vertex is a receiver, a transmitter or transitive. We may assume without loss of generality that $a \rightarrow e$ in **D**. Thus, we will have the following digraph (Figure 2):

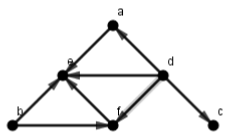

Figure 2: A graph which is not a divisor graph.

To entirety the orientation we must have either $f \rightarrow c$ or $c \rightarrow f$. Both cases are infeasible because in either case the vertex c is neither a receiver nor a transmitter nor transitive.

The following pan out (Figure 3) which was given in [4] as example of a graph which is not divisor graph.

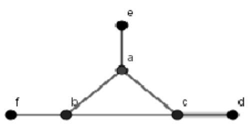

Figure 3: A graph which is not a divisor graph.

2 When $G(\mathbf{R})$ is a divisor graph

In our investigation, we begin with local ring.

Theorem 1: Let **R** be a local ring. Then **G(R)** is a divisor graph.

Proof: Let M be a maximal ideal of **R** since **R** is principal ideal ring then $M = (x)$ is a principal ideal. Thus every ideal is a power of M so, **R** has one chain of ideals of which **G(R)** is a complete graph. Therefore **G(R)** is a divisor graph.

By using Figure 1, we deduce the following Theorem.

Theorem 2: If **R** is a product of 3 non-trivial rings, then **G(R)** is not a divisor graph.

Proof: Assume that $\mathbf{R} \cong \mathbf{R}_1 \times \mathbf{R}_2 \times \mathbf{R}_3$. Then we have the following induced subgraph of **G(R)** (Figure 1). Let $a = \mathbf{R}_1 \times \{0\} \times \{0\}$, $d = \mathbf{R}_1 \times \{0\} \times \mathbf{R}_3$, $b = \{0\} \times \{0\} \times \mathbf{R}_3$, $e = \{0\} \times \mathbf{R}_2 \times \mathbf{R}_3$, $c = \{0\} \times \mathbf{R}_2 \times \{0\}$ and $f = \mathbf{R}_1 \times \mathbf{R}_2 \times \{0\}$. Then **G(R)** is not a divisor graph.

By Theorem 1 and 2 we need to consider the product of two local rings case only to finish our investigation. Our discussion will be based on the fact that the diameters of intersection graph of ideals of rings can not exceed 2 (see [1]). Note that if $\mathbf{R} \cong \mathbf{R}_1 \times \mathbf{R}_2$ is a product of two fields, then $diam(\mathbf{R}) = \infty$ since **G(R)** has only two non-trivial ideals $\{0\} \times \mathbf{R}_2$ and $\mathbf{R}_1 \times \{0\}$ which are non-adjacent. Then **G(R)** is a divisor graph.

Theorem 3: Let $\mathbf{R} \cong \mathbf{R}_1 \times \mathbf{R}_2$ such that $\mathbf{R}_1$ or $\mathbf{R}_2$ has three distinct ideals and intersection between any of them is empty. Then **R** is not a divisor graph.

Proof: Without loss of generality assume that $\mathbf{R}_1$ has three distinct non-trivial ideals say I_1^*, I_2^* and I_3^* such that $I_1^* \cap I_2^* = \{0\}$, $I_1^* \cap I_3^* = \{0\}$ and $I_2^* \cap I_3^* = \{0\}$. Then we have the following induced subgraph of **G(R)** (Figure 3). Let $a = I_3^* \times \mathbf{R}_2$, $c = I_2^* \times \mathbf{R}_2$, $b = I_1^* \times \mathbf{R}_2$, $e = I_3^* \times \{0\}$, $d = I_2^* \times \{0\}$ and $f = I_1^* \times \{0\}$. Then **G(R)** is not a divisor graph.

Theorem 4: Let $\mathbf{R} \cong \mathbf{R}_1 \times \mathbf{R}_2$ such that $\mathbf{R}_1$ and $\mathbf{R}_2$ each of them has two distinct ideals. Then **R** is not a divisor graph.

Proof: Assume that $\mathbf{R} \cong \mathbf{R}_1 \times \mathbf{R}_2$ each of them has two distinct ideals say I_1^*, I_2^* with $I_1^* \cap I_2^* = \{0\}$ and J_1^*, J_2^* with $J_1^* \cap J_2^* = \{0\}$ respectively. Then we have the following induced subgraph of **G(R)** (Figure 3). Let $a = \mathbf{R}_1 \times \{0\}$, $c = I_2^* \times J_2^*$, $b = I_2^* \times J_1^*$, $e = I_1^* \times \{0\}$, $d = \{0\} \times J_2^*$ and $f = \{0\} \times J_1^*$. Then **G(R)** is not a divisor graph.

If **G** is a divisor graph, then there is one -to-one function $f : V(\mathbf{G}) \rightarrow N$ such that v is adjacent to u in **G** if and only if $f(u)$ divides $f(v)$ or $f(v)$ divides $f(u)$. This function is called a divisor labeling of **G**, see [6]. We use labeling functions in the proofs of Theorem 5, 6 and 7.

Theorem 5: Let $\mathbf{R} \cong \mathbf{R}_1 \times \mathbf{R}_2$ such that $\mathbf{R}_1$ and $\mathbf{R}_2$ each of them has a unique minimal ideal. Then **G(R)** is a divisor graph.

Proof: Let $\{I_i;\ i = 1, ..., n\}$ be the set of all non-trivial ideals in $\mathbf{R}_1$ and $\{J_i;\ i = 1, ..., m\}$ be the set of all non-trivial ideals in $\mathbf{R}_2$. Then the set of all non-trivial ideals in **R** are $\{\mathbf{R}_1 \times \{0\},\ \{0\} \times \mathbf{R}_2,\ \mathbf{R}_1 \times M_2,\ M_1 \times \mathbf{R}_2,\ I_i^* \times \{0\},\ \{0\} \times J_i^*,\ I_i^* \times J_i^*,\ \mathbf{R}_1 \times J_i^*,\ I_i^* \times \mathbf{R}_2,\ M_1 \times J_i^*,\ I_i^* \times M_2,\ M_1 \times \{0\},\ \{0\} \times M_2,\ M_1 \times M_2\}$. Let **X** be the set of all non-trivial ideals of the form $\{\mathbf{R}_1 \times M_2,\ I_i^* \times \mathbf{R}_2,\ \mathbf{R}_1 \times J_i^*,\ M_1 \times \mathbf{R}_2,\ I_i^* \times M_2,\ M_1 \times J_i^*,\ M_1 \times M_2\}$. Let $j = \{1,\ ....,\ k\}$ where k is the number of ideal in **X** and $l = \{1,\ 2\}$. Then define the function $f : V(\mathbf{G}) \rightarrow N$

by

$$f(x\times y)=\begin{cases} 2^j & , \ x\times y\in \mathbf{X} \\ 2^k\times 5^l & , \ x\times y\in\{\mathbf{R}_1\times\{0\},\ M_1\times\{0\}\} \\ 2^k\times 5^{2+i} & , \ x\times y=\{I_i^*\times\{0\}\} \\ 2^k\times 3^l & , \ x\times y\in\{\{0\}\times\mathbf{R}_2,\ \{0\}\times M_2\} \\ 2^k\times 3^{i+2} & , \ x\times y=\{\{0\}\times J_i^*\}\end{cases}.$$

Then f is a one to one function such that $(x\times y)\cap(\alpha\times\ \beta)\neq\{0\}$ if and only if $f(x\times y)$ divides $f(\alpha\times\beta)$ or $f(\alpha\times\beta)$ divides $f(x\times y)$. Hence **G(R)** is a divisor graph.

Theorem 6: Let **R** be a product of two local rings, $\mathbf{R}\cong\mathbf{R}_1\times\mathbf{R}_2$ with M_1 and M_2 are two maximal ideals in $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively. Then **G(R)** is a divisor graph.

Proof: We have the following cases:

- **Case 1:** If $\mathbf{R}_1$ has two non-trivial distinct ideals with empty intersection and $\mathbf{R}_2$ has a unique non-trivial ideal.
  In this case let $I_1^*,\ I_2^*$ are two non-trivial ideals in $\mathbf{R}_1$ such that $I_1^*\cap I_2^*=\{0\}$ and J^* is a non-trivial ideal in $\mathbf{R}_2$. Then the set of all ideals are $\{\mathbf{R}_1\times M_2,\ M_1\times\mathbf{R}_2,\ M_1\times M_2,\ \mathbf{R}_1\times\{0\},\ \{0\}\times\mathbf{R}_2,\ I_1^*\times M_2,\ I_2^*\times M_2,\ M_1\times\{0\},\ \{0\}\times M_2,\ I_1^*\times\{0\},\ I_2^*\times\{0\},\ \mathbf{R}_1\times J^*,\ I_1^*\times J^*,\ I_2^*\times J^*,\ \{0\}\times J^*,\ M_1\times J^*\}$. Let **X** be the set of non-trivial ideals of the form $\{\mathbf{R}_1\times M_2,\ I_1^*\times\mathbf{R}_2,\ I_2^*\times\mathbf{R}_2,\ \mathbf{R}_1\times J^*,\ M_1\times\mathbf{R}_2,\ I_1^*\times M_2,\ I_2^*\times M_2,\ M_1\times J^*,\ M_1\times M_2,\ I_1^*\times J^*,\ I_2^*\times J^*\}$. Let $j=\{1,\ ....,\ 8\},l=\{1,\ 2\}$. Then define the function $f:\ V(\mathbf{G})\rightarrow N$ by

$$f(x\times y)=\begin{cases} 2^j & , \ x\times y\in\mathbf{X} \\ 2^{11}\times 5^l\times 7^l & , \ x\times y\in\{\mathbf{R}_1\times\{0\},\ M_1\times\{0\}\} \\ 2^{11}\times 5 & , \ x\times y=\{I_1^*\times\{0\}\} \\ 2^{11}\times 7 & , \ x\times y=\{I_2^*\times\{0\}\} \\ 2^{11}\times 3^l & , \ x\times y\in\{\{0\}\times\mathbf{R}_2,\ \{0\}\times M_2\} \\ 2^{11}\times 3^3 & , \ x\times y=\{\{0\}\times J^*\}\end{cases}.$$

  Then f is a one to one function such that $(x\times y)\cap(\alpha\times\ \beta)\neq\{0\}$ if and only if $f(x\times y)$ divides $f(\alpha\times\beta)$ or $f(\alpha\times\beta)$ divides $f(x\times y)$. Hence **G(R)** is a divisor graph.

- **Case 2:** If $\mathbf{R}_1$ has two non-trivial distinct ideals with empty intersection and $\mathbf{R}_2$ has no non-trivial ideal.
  In this case let $I_1^*,\ I_2^*$ are two non-trivial ideals in $\mathbf{R}_1$ such that $I_1^*\cap I_2^*=\{0\}$. Then the set of all ideals in **R** are $\{\mathbf{R}_1\times M_2,\ M_1\times\mathbf{R}_2,\ M_1\times M_2,\ \mathbf{R}_1\times\{0\},\ \{0\}\times\mathbf{R}_2,\ I_1^*\times M_2,\ I_2^*\times M_2,\ M_1\times\{0\},\ \{0\}\times M_2,\ I_1^*\times\{0\},\ I_2^*\times\{0\},\ I_1^*\times\mathbf{R}_2,\ I_2^*\times\mathbf{R}_2\}$. Let **X** be the set of all non-trivial ideal of the form $\{\mathbf{R}_1\times M_2,\ I_1^*\times\mathbf{R}_2,\ I_2^*\times\mathbf{R}_2,\ M_1\times\mathbf{R}_2,\ I_1^*\times M_2,\ I_2^*\times M_2,\ M_1\times M_2\}$. Let $j=\{1,\ ....,\ 7\}$ and $l=\{1,\ 2\}$. Then define the function $f:\ V(\mathbf{G})\rightarrow N$ by

$$f(x\times y)=\begin{cases} 2^j & , \ x\times y\in\mathbf{X} \\ 2^7\times 5^l\times 7^l & , \ x\times y\in\{\mathbf{R}_1\times\{0\},\ M_1\times\{0\}\} \\ 2^7\times 5 & , \ x\times y=\{I_1^*\times\{0\}\} \\ 2^7\times 7 & , \ x\times y=\{I_2^*\times\{0\}\} \\ 2^7\times 3^l & , \ x\times y\in\{\{0\}\times\mathbf{R}_2,\ \{0\}\times M_2\}\end{cases}.$$

  Then f is a one to one function such that $(x\times y)\cap(\alpha\times\ \beta)\neq\{0\}$ if and only if $f(x\times y)$ divides $f(\alpha\times\beta)$ or $f(\alpha\times\beta)$ divides $f(x\times y)$. Hence **G(R)** is a divisor graph.

- **Case 3:** If $\mathbf{R}_1$ and $\mathbf{R}_2$ each of them has a unique non-trivial ideal.
  In this case let I^* is a non-trivial ideal in $\mathbf{R}_1$ and J^* is a non-trivial ideal in $\mathbf{R}_2$. Then the set of all non-trivial ideals are $\{\mathbf{R}_1\times M_2,\ M_1\times\mathbf{R}_2,\ M_1\times M_2,\ \mathbf{R}_1\times\{0\},\ \{0\}\times\mathbf{R}_2,\ I^*\times M_2,\ M_1\times\{0\},\ \{0\}\times M_2,\ I^*\times\{0\},\ \mathbf{R}_1\times J^*,\ I^*\times J^*,\ \{0\}\times J^*,\ M_1\times J^*,\ I^*\times J^*\}$. Let **X** be the set of non-trivial ideal of the form $\{\mathbf{R}_1\times M_2,\ I^*\times\mathbf{R}_2,\ \mathbf{R}_1\times J^*,\ M_1\times\mathbf{R}_2,\ I^*\times M_2,\ M_1\times J^*,\ M_1\times M_2\}$. Let $j=\{1,\ ....,\ 8\}$ and $l=\{1,\ 2\}$. Then define the function $f:\ V(\mathbf{G})\rightarrow N$ by

$$f(x\times y)=\begin{cases} 2^j & , \ x\times y\in\mathbf{X} \\ 2^8\times 5^l\times 7 & , \ x\times y\in\{\mathbf{R}_1\times\{0\},\ M_1\times\{0\}\} \\ 2^8\times 5 & , \ x\times y=\{I^*\times\{0\}\} \\ 2^8\times 3^l & , \ x\times y\in\{\{0\}\times\mathbf{R}_2,\ \{0\}\times M_2\} \\ 2^8\times 3^3 & , \ x\times y=\{\{0\}\times J^*\}\end{cases}.$$

  Then f is a one to one function such that $(x\times y)\cap(\alpha\times\ \beta)\neq\{0\}$ if and only if $f(x\times y)$ divides $f(\alpha\times\beta)$ or $f(\alpha\times\beta)$ divides $f(x\times y)$. Hence **G(R)** is a divisor graph.

- **Case 4:** If **R** is a product of two local rings with no non-trivial ideal different than maximal ideal for each $\mathbf{R}_1$ and $\mathbf{R}_2$, then the set of all ideals in **R** are $\{\mathbf{R}_1\times\{0\},\ M_1\times\mathbf{R}_2,\ \mathbf{R}_1\times M_2,\ \{0\}\times\mathbf{R}_2,\ M_1\times M_2,\ M_1\times\{0\},\ \{0\}\times M_2\}$. Let **X** be the set of non-trivial ideal of the form $\{\mathbf{R}_1\times M_2,\ M_1\times\mathbf{R}_2,\ M_1\times M_2\}$. Let $j=\{1,\ 2,\ 3\}$ and $l=\{1,\ 2\}$. Then define the function $f:\ V(\mathbf{G})\rightarrow N$ by

$$f(x\times y)=\begin{cases} 2^j & , \ x\times y\in\mathbf{X} \\ 2^3\times 5^l & , \ x\times y\in\{\mathbf{R}_1\times\{0\},\ M_1\times\{0\}\} \\ 2^3\times 3^l & , \ x\times y\in\{\{0\}\times\mathbf{R}_2,\ \{0\}\times M_2\}\end{cases}.$$

  Then f is a one to one function such that $(x\times y)\cap(\alpha\times\ \beta)\neq\{0\}$ if and only if $f(x\times y)$ divides $f(\alpha\times\beta)$ or $f(\alpha\times\beta)$ divides $f(x\times y)$. Hence **G(R)** is a divisor graph.

Theorem 7: Let **R** be a product of two local rings one of them is a field, assume that $\mathbf{R}\cong\mathbf{R}_1\times\mathbf{R}_2$ without loss of generality let $\mathbf{R}_1$ is a local ring with M_1 is a maximal ideal in $\mathbf{R}_1$ and $\mathbf{R}_2$ is a field. Then **G(R)** is a divisor graph.

Proof:

- **Case 1:** If $\mathbf{R}_1$ has two non-trivial distinct ideals with empty intersection.
  In this case let I_1^*, I_2^* are two non-trivial ideals in $\mathbf{R}_1$ such that $I_1^* \cap I_2^* = \{0\}$. Then the set of all ideals are $\{M_1 \times \mathbf{R}_2,\ \mathbf{R}_1 \times \{0\},\ \{0\} \times \mathbf{R}_2,\ M_1 \times \{0\},\ I_1^* \times \{0\},\ I_2^* \times \{0\},\ I_1^* \times \mathbf{R}_2,\ I_2^* \times \mathbf{R}_2\}$. Let $\mathbf{X}$ be the set of all ideal of the form $\{M_1 \times \mathbf{R}_2,\ I_1^* \times \mathbf{R}_2,\ I_2^* \times \mathbf{R}_2\}$. Let $j = \{1,\ 2,\ 3\}$ and $l = \{1,\ 2\}$. Then define the function $f:\ V(\mathbf{G}) \to N$ by

$$f(x \times y) = \begin{cases} 2^j & ,\ x \times y \in \mathbf{X} \\ 2^3 \times 5^l \times 7^l & ,\ x \times y \in \{\mathbf{R}_1 \times \{0\},\ M_1 \times \{0\}\} \\ 2^3 \times 5 & ,\ x \times y = \{I_1^* \times \{0\}\} \\ 2^3 \times 7 & ,\ x \times y = \{I_2^* \times \{0\}\} \\ 2^3 \times 3 & ,\ x \times y \in \{\{0\} \times \mathbf{R}_2\} \end{cases}.$$

  Then f is a one to one function such that $(x \times y) \cap (\alpha \times\ \beta) \neq \{0\}$ if and only if $f(x \times y)$ divides $f(\alpha \times \beta)$ or $f(\alpha \times \beta)$ divides $f(x \times y)$. Hence **G(R)** is a divisor graph.

- **Case 2:** If $\mathbf{R}_1$ has a unique non-trivial ideal.
  Let I^* is a non-trivial ideal in $\mathbf{R}_1$. Then the set of all ideals are $\{M_1 \times \mathbf{R}_2,\ M_1 \times \{0\},\ \mathbf{R}_1 \times \{0\},\ \{0\} \times \mathbf{R}_2,\ I^* \times \mathbf{R}_2,\ I^* \times \{0\}\}$. Let $l = \{1,\ 2\}$, define the function $f:\ V(\mathbf{G}) \to N$ by

$$f(x \times y) = \begin{cases} 2^l & ,\ x \times y \in \{M_1 \times \mathbf{R}_2,\ I^* \times \mathbf{R}_2\} \\ 2^2 \times 5^l & ,\ x \times y \in \{\mathbf{R}_1 \times \{0\},\ M_1 \times \{0\}\} \\ 2^2 \times 5^3 & ,\ x \times y = \{I^* \times \{0\}\} \\ 2^7 \times 3 & ,\ x \times y \in \{\{0\} \times \mathbf{R}_2\} \end{cases}.$$

  Then f is a one to one function such that $(x \times y) \cap (\alpha \times\ \beta) \neq \{0\}$ if and only if $f(x \times y)$ divides $f(\alpha \times \beta)$ or $f(\alpha \times \beta)$ divides $f(x \times y)$. Hence **G(R)** is a divisor graph.

- **Case 3:** If $\mathbf{R}_1$ is a local rings with no non-trivial ideal different than maximal ideal. Then the set of all ideals in $\mathbf{R}$ are $\{\mathbf{R}_1 \times \{0\},\ M_1 \times \mathbf{R}_2,\ \{0\} \times \mathbf{R}_2,\ M_1 \times \{0\}\}$. Let $l = \{1,\ 2\}$ define the function $f:\ V(\mathbf{G}) \to N$ by

$$f(x \times y) = \begin{cases} 2 & ,\ x \times y \in \{M_1 \times \mathbf{R}_2\} \\ 2 \times 5^l & ,\ x \times y \in \{\mathbf{R}_1 \times \{0\},\ M_1 \times \{0\}\} \\ 2 \times 3 & ,\ x \times y \in \{\{0\} \times \mathbf{R}_2\} \end{cases}.$$

  Then f is a one to one function such that $(x \times y) \cap (\alpha \times\ \beta) \neq \{0\}$ if and only if $f(x \times y)$ divides $f(\alpha \times \beta)$ or $f(\alpha \times \beta)$ divides $f(x \times y)$. Hence **G(R)** is a divisor graph.

3 Conclusion and questions

In this paper, we determined when **G(R)** is a divisor graph and we sum up in the following theorem.
Theorem 8: Let **R** be a commutative principal ideal ring with unity. Then **G(R)** is a divisor graph if it is a local ring or is product of two local rings with each of them has one chain of ideals or is product of two local rings one of them has at most two non-trivial ideals with empty intersection.

One can ask the following questions:

(1) Can we generalize the results of this article to the ring is Noetherian or Artinian?
(2) When the complement **G(R)** is a divisor graph?
(3) When Intersection Ideals of Graphs of Rings are Eulerian graph ?

and [13].

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This research was partially supported by University Of Petra. We are thankful for the comments offered by the peer reviewers.

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