

MULTIVARIATE GENERAL SPATIAL THREE-STAGE LEAST SQUARES FIXED EFFECT PANEL SIMULTANEOUS MODELS AND ESTIMATION OF THEIR PARAMETERS

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Abstract: Simultaneous equation models describe a two-way flow of influence among variables. Simultaneous equation models using panel data, especially for fixed effect where there are spatial autoregressive and spatial errors with exact solutions, still require to be developed. In this paper, we develop the new models that consist of spatial autoregressive and spatial errors. We call it as general spatial. This paper proposes feasible generalized least squares-three-stage least squares (FGLS-3SLS) to find all the estimators with exact solution and the numerical approximation estimators by concentrated log-likelihood formulation with method of forming sequence. All proposed estimators especially for closed-form estimators are proved to be consistent.

Key-Words: spatial autoregressive, spatial error, general spatial, FGLS-3SLS, concentrated log-likelihood, consistent

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1 Introduction

If the model contains spatial influence and the spatial influence comes only through the error terms, we can use spatial error model [1]. Moreover, if the model contains spatial influence and the spatial influence comes only through the dependent, we can use spatial autoregressive model [2]. Now, we develop the paper where there are spatial autoregressive and spatial errors. The new models include two spatials, namely spatial autoregressive and spatial error. We call it general spatial.

System methods are the methods which are much more efficient than the single-equation methods because they use much more informations [3]. Single-equation methods and system methods are two methods which it can be used to find the estimators of parameter in simultaneous equation models [3].

Estimators of three-stage least squares (3SLS) are more robust than other estimators, like full information maximum likelihood (FIML) [4]. Consequently, solution technique by means of 3SLS is much more advantageous than the one by FIML because it is both time saving and cost saving.

In this solution, we still use first-order queen contiguity to find row-standardized spatial weight matrix [5] and Moran Index to examine spatial influence [6-8]. Some papers about estimation of parameter in simultaneous equation models for fixed

effect are revealed in [9], and [10]. But, estimating these parameters had done by simulation.

In this paper, we are motivated to develop simultaneous equation models for fixed effect panel data with one-way error component by means of 3SLS solutions, especially for both spatial correlation among dependent variables and spatial correlation among errors. We call it as general spatial.

The objective of this paper is to obtain the closed-form estimators of parameter models by means of feasible generalized least squares-three-stage least squares (FGLS-3SLS) and the numerical approximation estimators of parameter models by means of concentrated log-likelihood formulation with method of forming sequence. And then, to prove their consistency, especially for closed-form estimators.

2 Models Development

We had an equation by [2], namely

$$\mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{X}_{hj}\boldsymbol{\alpha}_h + \mathbf{Y}_{-hj}\boldsymbol{\beta}_{-h} + \mathbf{1}\gamma_{hj} + \mathbf{u}_{hj}, \quad (1)$$

for $h=1,2,3,\dots,m$, $j=1,2,3,\dots,T$, where \mathbf{y}_{hj} denotes the j th time period h th endogenous vector, \mathbf{X}_{hj} denotes the j th time period h th matrix including (for example k_h) exogenous variables,

\mathbf{Y}_{-hj} denotes the j th time period $-h$ th matrix including endogenous explanatory variables except the j th time period h th endogenous explanatory variables, μ_h denotes the h th mean parameter, \mathbf{a}_h denotes the h th parameters vector of exogenous variables, $\mathbf{\beta}_{-h}$ denotes the $-h$ th parameters vector of endogenous explanatory variables, γ_{hj} denotes the j th time period h th time specific effect parameter, $\mathbf{1}$ denotes the unit vector, \mathbf{u}_{hj} denotes the j th time period h th random error vector assuming mean vector $\mathbf{0}$ and covariance matrix $\sigma_h^2 \mathbf{I}_n$ (homoscedasticity) in which σ_h^2 denotes the unknown h th error variance and \mathbf{I}_n denotes the $n \times n$ identity matrix. There is one restriction, namely $\sum_{j=1}^T \gamma_{hj} = 0$. In this context, we suppose that (1) are over identified.

The next model is general spatial model (GSM) which refers to [11], namely:

$$\begin{aligned} \mathbf{y} &= \mathbf{1}\mu + \mathbf{X}\mathbf{a} + \rho\mathbf{W}\mathbf{y} + \mathbf{u}, \\ \mathbf{u} &= \lambda\mathbf{W}\mathbf{u} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_n), \end{aligned} \quad (2)$$

where \mathbf{y} denotes the endogenous vector, \mathbf{X} denotes the matrix of observations including (for example k) exogenous variables, μ denotes the mean parameter, \mathbf{a} denotes the parameters vector of exogenous variables, ρ and λ denote the spatial autoregressive and the spatial autocorrelation parameters, respectively, \mathbf{W} denotes the row-standardized spatial weight matrix, and \mathbf{u} denotes the spatial autocorrelation of random error vector, and $\boldsymbol{\varepsilon}$ denotes the random error vector assuming normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\sigma^2 \mathbf{I}_n$ in which σ^2 denotes the unknown error variance.

If (1) contains spatial influences and the spatial influences come through the endogenous and the error variables, then we can adopt models in equations (2) and obtain new form equations as follows:

$$\mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{X}_{hj}\mathbf{a}_h + \rho_h\mathbf{W}\mathbf{y}_{hj} + \mathbf{Y}_{-hj}\mathbf{\beta}_{-h} + \mathbf{1}\gamma_{hj} + \mathbf{u}_{hj} \quad (3)$$

$$\mathbf{u}_{hj} = \lambda_h\mathbf{W}\mathbf{u}_{hj} + \boldsymbol{\varepsilon}_{hj}, \quad \boldsymbol{\varepsilon}_{hj} \sim N(\mathbf{0}, \sigma_h^2 \mathbf{I}_n).$$

Equation (3) can be simplified as follows:

$$\mathbf{A}_h\mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{X}_{hj}\mathbf{a}_h + \mathbf{Y}_{-hj}\mathbf{\beta}_{-h} + \mathbf{1}\gamma_{hj} + \mathbf{B}_h^{-1}\boldsymbol{\varepsilon}_{hj}, \quad (4)$$

for $h=1, 2, 3, \dots, m$, $j=1, 2, 3, \dots, T$, where $\mathbf{A}_h = \mathbf{I}_n - \rho_h\mathbf{W}$, $\mathbf{B}_h = \mathbf{I}_n - \lambda_h\mathbf{W}$, ρ_h and λ_h denote the h th spatial autoregressive and the h th spatial autocorrelation parameters, respectively, and \mathbf{u}_{hj}

denotes the j th time period h th spatial autocorrelation of random error vector, and $\boldsymbol{\varepsilon}_{hj}$ denotes the j th time period h th random error vector assuming normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\sigma_h^2 \mathbf{I}_n$. There is one restriction, namely $\sum_{j=1}^T \gamma_{hj} = 0$.

We refer to [12] for the properties of kronecker products, [13] for reparameterization, [3, 4, 14] for 3SLS estimation, [15] for GLS and FGLS, [5] for the use first-order queen contiguity to find the row-standardized spatial weight matrix, [6-8] for examining spatial influences by means of Moran Index and [16] for consistency.

For the solution of (4) by 3SLS, we obtain the following equation:

$$\begin{aligned} \bar{\mathbf{X}}_h^t \mathbf{A}_h \mathbf{y}_{hj} &= \bar{\mathbf{X}}_h^t \mathbf{1} \mu_h + \bar{\mathbf{X}}_h^t \mathbf{X}_{hj} \mathbf{a}_h + \bar{\mathbf{X}}_h^t \mathbf{Y}_{-hj} \mathbf{\beta}_{-h} \\ &+ \bar{\mathbf{X}}_h^t \mathbf{1} \gamma_{hj} + \bar{\mathbf{X}}_h^t \mathbf{B}_h^{-1} \boldsymbol{\varepsilon}_{hj}. \end{aligned} \quad (5)$$

We use average value approach of the matrix of observations [1, 17, 18] because the estimator of the mean is unbiased, consistent, and efficient as revealed by [3, 4, 19]. If we use \mathbf{X}_j^t then the

restriction $\sum_{j=1}^T \gamma_{hj} = 0$ will not be achieved. This is due to \mathbf{X}_j^t having in general, different values of the matrix of observations in every j th time period [1, 7, 17, 18].

We can rewrite (5) to obtain new forms of vectors and matrices as follows:

$$\bar{\mathbf{X}}_{**}^t \mathbf{A}_j \mathbf{y}_j = \bar{\mathbf{X}}_{**}^t \mathbf{G} \boldsymbol{\mu} + \bar{\mathbf{X}}_{**}^t \mathbf{Z}_j \boldsymbol{\theta} + \bar{\mathbf{X}}_{**}^t \mathbf{G} \boldsymbol{\gamma}_j + \bar{\mathbf{X}}_{**}^t \mathbf{B}_* \boldsymbol{\varepsilon}_j, \quad (6)$$

where $\mathbf{Z}_j = [\mathbf{X}_j \quad \vdots \quad \mathbf{Y}_{-j}]$ and $\boldsymbol{\theta}^t = [\boldsymbol{\alpha}^t \quad \vdots \quad \boldsymbol{\beta}_-^t]$

having dimensions $mn \times \left(\sum_{h=1}^m k_h + m(m-1) \right)$ and

$\left(\sum_{h=1}^m k_h + m(m-1) \right) \times 1$, respectively.

Explanation of the vectors and matrices from equations (5)-(6) are $\bar{\mathbf{X}}_{**}$ denotes the $mn \times m \sum_{h=1}^m k_h$ diagonal matrix whose submain diagonal is $\bar{\mathbf{X}}_*$, $\bar{\mathbf{X}}_* = \frac{1}{T} \sum_{j=1}^T \mathbf{X}_j^*$ where \mathbf{X}_j^* denotes the $n \times \sum_{h=1}^m k_h$ matrix including all the exogenous variables in the system, \mathbf{A} denotes the $mn \times mn$ diagonal matrix whose submain diagonal is the $n \times n$ matrix \mathbf{A}_h , \mathbf{y}_j denotes the $mn \times 1$ vector including all of the $n \times 1$ vectors \mathbf{y}_{hj} , \mathbf{G} denotes the $mn \times m$ diagonal

matrix whose submain diagonal is $\mathbf{1}$, \mathbf{B}_* denotes the $mn \times mn$ diagonal matrix whose submain diagonal is the $n \times n$ matrix \mathbf{B}_h^{-1} , $\boldsymbol{\mu}$ denotes the $m \times 1$ vector including all of μ_h , \mathbf{X}_j denotes the $mn \times \sum_{h=1}^m k_h$ diagonal matrix whose submain diagonal is the $n \times k_h$ matrix \mathbf{X}_{hj} , $\boldsymbol{\alpha}$ denotes the $\sum_{h=1}^m k_h \times 1$ vector including all of the $k_h \times 1$ vectors $\boldsymbol{\alpha}_h$, \mathbf{Y}_{-j} denotes the $mn \times m(m-1)$ diagonal matrix whose submain diagonal is the $n \times (m-1)$ matrix \mathbf{Y}_{-hj} , $\boldsymbol{\beta}_-$ denotes the $m(m-1) \times 1$ vector including all of the $(m-1) \times 1$ vectors $\boldsymbol{\beta}_{-h}$, $\boldsymbol{\gamma}_j$ denotes the $m \times 1$ vector including all of γ_{hj} , and $\boldsymbol{\varepsilon}_j$ denotes the $mn \times 1$ vector including all of the $n \times 1$ vectors $\boldsymbol{\varepsilon}_{hj}$, as well as n denotes the sampel size of observations. For $j=1, 2, 3, \dots, T$, the restriction $\sum_{j=1}^T \boldsymbol{\gamma}_j = \mathbf{0}$ is changed $\sum_{j=1}^T \boldsymbol{\gamma}_j = \mathbf{0}$.

3 Estimating the Parameters

Now, we consider the equation (6). Estimation of parameters models are conducted in three stages. At the first-stage, we estimate all the endogenous explanatory variables in the system in every time period. This first-stage is the same as the previous paper [2].

At the second-stage, we estimate parameters of $\mu_h, \boldsymbol{\alpha}_h, \boldsymbol{\beta}_{-h}$, and γ_{hj} to obtain residual estimate of equation (4). Because equation (4) has non-homoscedastic error, we first tranform this equation.

$\text{var}(\mathbf{B}_h^{-1} \boldsymbol{\varepsilon}_{hj}) = \sigma_h^2 (\mathbf{B}_h' \mathbf{B}_h)^{-1} \neq \sigma_h^2 \mathbf{I}_n$. Premultiplying (4) by $\sigma_h^{-1} (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}}$, we obtain

$$\begin{aligned} (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{A}_h \mathbf{y}_{hj} &= (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{1} \mu_h + (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{X}_{hj} \boldsymbol{\alpha}_h \\ &+ (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{Y}_{-hj} \boldsymbol{\beta}_{-h} + (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{1} \gamma_{hj} \quad (7) \\ &+ (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{B}_h^{-1} \boldsymbol{\varepsilon}_{hj}. \end{aligned}$$

We can omit the value of σ_h^{-1} because it is a finite constant. Now, the equation (7) has satisfied a regression model requirement.

We then substitute \mathbf{Y}_{-hj} by $\hat{\mathbf{Y}}_{-hj}$ in (7), where $\mathbf{Y}_{-hj} = \hat{\mathbf{Y}}_{-hj} + \hat{\mathbf{V}}_{-hj}$, where $\hat{\mathbf{V}}_{-hj}$ is residual estimate from the first stage, and obtain new equations as follows:

$$\begin{aligned} (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{A}_h \mathbf{y}_{hj} &= (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{1} \mu_h + (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{Z}_{hj} \boldsymbol{\theta}_h \quad (8) \\ &+ (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{1} \gamma_{hj} + \mathbf{u}_{hj}^*, \end{aligned}$$

where $\mathbf{Z}_{hj} = [\mathbf{X}_{hj} \quad \hat{\mathbf{Y}}_{-hj}]$ and $\boldsymbol{\theta}_h = [\boldsymbol{\alpha}_h' \quad \boldsymbol{\beta}_{-h}']$ having dimensions $n \times (k_h + m - 1)$ and $1 \times (k_h + m - 1)$, respectively, and \mathbf{u}_{hj}^* denotes the composite random error with $\mathbf{u}_{hj}^* = (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \hat{\mathbf{V}}_{-hj} \boldsymbol{\beta}_{-h} + (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{B}_h^{-1} \boldsymbol{\varepsilon}_{hj}$.

The right-hand side matrix of equation (8) is less than full rank. But, we can not use $n \times n$ dimensional transformation matrix \mathbf{Q} directly, in which $\mathbf{Q}\mathbf{1} = \mathbf{0}$, to find the estimator of $\boldsymbol{\theta}_h$. We remind again that $\mathbf{Q} = \mathbf{I}_n - \frac{1}{n} \mathbf{1}\mathbf{1}'$ is symmetrical and idempotent matrices.

We need to reparameterize equation (8), namely:

$$\begin{aligned} (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{A}_h \mathbf{y}_{hj} &= (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{1} \mu_{hj} + (\mathbf{B}_h' \mathbf{B}_h)^{\frac{1}{2}} \mathbf{Z}_{hj} \boldsymbol{\theta}_h \quad (9) \\ &+ \mathbf{u}_{hj}^*, \end{aligned}$$

where $\mu_{hj} = \mu_h + \gamma_{hj}$.

By GLS Solution, we first get the estimator of μ_{hj} , namely

$$\hat{\mu}_{hj} = (\mathbf{1}' \mathbf{B}_h' \mathbf{B}_h \mathbf{1})^{-1} \mathbf{1}' \mathbf{B}_h' \mathbf{B}_h (\mathbf{A}_h \mathbf{y}_{hj} - \mathbf{Z}_{hj} \boldsymbol{\theta}_h) \quad (10)$$

We then use (10) to find estimator of $\boldsymbol{\theta}_h$ and by GLS solution, we obtain

$$\begin{aligned} \hat{\boldsymbol{\theta}}_h &= \left[\sum_{j=1}^T \mathbf{Z}_{hj}' \mathbf{B}_h' \mathbf{B}_h [\mathbf{1} \mathbf{b}_h' - \mathbf{I}_n] \mathbf{Z}_{hj} \right]^{-1} \quad (11) \\ &\times \sum_{j=1}^T \mathbf{Z}_{hj}' \mathbf{B}_h' \mathbf{B}_h [\mathbf{1} \mathbf{b}_h' - \mathbf{I}_n] \mathbf{A}_h \mathbf{y}_{hj}, \end{aligned}$$

where $\mathbf{b}_h' = (\mathbf{1}' \mathbf{B}_h' \mathbf{B}_h \mathbf{1})^{-1} \mathbf{1}' \mathbf{B}_h' \mathbf{B}_h$ having dimension $1 \times n$.

We recall to equation (8) and use the GLS solution, we obtain the estimators of μ_h and γ_{hj} , namely

$$\hat{\mu}_h = \frac{1}{T} \mathbf{b}_h' \sum_{j=1}^T (\mathbf{A}_h \mathbf{y}_{hj} - \mathbf{Z}_{hj} \hat{\boldsymbol{\theta}}_h), \quad (12)$$

and

$$\hat{\gamma}_{hj} = \mathbf{b}_h' (\mathbf{A}_h \mathbf{y}_{hj} - \mathbf{1} \hat{\mu}_h - \mathbf{Z}_{hj} \hat{\boldsymbol{\theta}}_h). \quad (13)$$

respectively.

From (11) to (13), we can estimate \mathbf{u}_{hj}^* as follows

$$\hat{\mathbf{u}}_{hj}^* = \mathbf{A}_h \mathbf{y}_{hj} - \mathbf{1}(\hat{\mu}_h + \hat{\gamma}_{hj}) - \mathbf{Z}_{hj} \hat{\boldsymbol{\theta}}_h = \mathbf{A}_h \mathbf{y}_{hj} - \hat{\mathbf{a}}_{hj}. \quad (14)$$

Matrices of \mathbf{A}_h and \mathbf{B}_h contain ρ_h and λ_h . In case ρ_h and λ_h are not known, we can estimate it by means of concentrated log-likelihood.

We pay attention to equation (4). Premultiplying its both sides by \mathbf{B}_h , we obtain

$$\mathbf{B}_h \mathbf{A}_h \mathbf{y}_{hj} = \mathbf{B}_h \mathbf{1} \mu_h + \mathbf{B}_h \mathbf{X}_{hj} \boldsymbol{\alpha}_h + \mathbf{B}_h \mathbf{Y}_{-hj} \boldsymbol{\beta}_{-h} + \mathbf{B}_h \mathbf{1} \gamma_{hj} + \boldsymbol{\varepsilon}_{hj}, \quad (15)$$

By equation (15), the likelihood function of $\boldsymbol{\varepsilon}_{hj}$, $j=1,2,3,\dots,T$, denoted by L_h is as follows:

$$L_h = \prod_{j=1}^T (2\pi\sigma_h^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma_h^2} \boldsymbol{\varepsilon}_{hj}^t \boldsymbol{\varepsilon}_{hj}\right), \quad \text{and by}$$

Jacobian transformation, we obtain the natural logarithm of L_h as follow:

$$\ln L_h = -\frac{nT}{2} \ln(2\pi\sigma_h^2) - \frac{1}{2\sigma_h^2} \sum_{j=1}^T (\mathbf{B}_h \mathbf{A}_h \mathbf{y}_{hj} - \mathbf{B}_h \mathbf{a}_{hj})^t \times (\mathbf{B}_h \mathbf{A}_h \mathbf{y}_{hj} - \mathbf{B}_h \mathbf{a}_{hj}) + T \ln \|\mathbf{B}_h\| + T \ln \|\mathbf{A}_h\|,$$

where $\|\mathbf{A}_h\|$ and $\|\mathbf{B}_h\|$ are the absolute of the determinants of \mathbf{A}_h and of \mathbf{B}_h , respectively.

We take derivative for σ_h^2 . Setting this derivative equal to zero, we obtain the estimator of σ_h^2 , namely:

$$\hat{\sigma}_h^2 = \frac{1}{nT} \sum_{j=1}^T (\mathbf{B}_h \mathbf{A}_h \mathbf{y}_{hj} - \mathbf{B}_h \mathbf{a}_{hj})^t (\mathbf{B}_h \mathbf{A}_h \mathbf{y}_{hj} - \mathbf{B}_h \mathbf{a}_{hj}). \quad (16)$$

By (18), we obtain concentrated log-likelihood as follows:

$$\ln L_h^{con} = C - \frac{nT}{2} \ln \left(\frac{1}{nT} \sum_{j=1}^T (\mathbf{B}_h \mathbf{A}_h \mathbf{y}_{hj} - \mathbf{B}_h \mathbf{a}_{hj})^t \times (\mathbf{B}_h \mathbf{A}_h \mathbf{y}_{hj} - \mathbf{B}_h \mathbf{a}_{hj}) \right) + T \ln \|\mathbf{B}_h\| + T \ln \|\mathbf{A}_h\|, \quad (17)$$

where $C = -\frac{nT}{2} \ln(2\pi) - \frac{nT}{2}$.

Let \mathbf{W} have eigenvalues $\omega_1, \omega_2, \dots, \omega_n$. The acceptable spatial autoregressive and spatial autocorrelation parameters are $\frac{1}{\omega_{\text{minimum}}} < \rho_h (\lambda_h) < 1$

[20]. We use numerical method for $\ln L_h^{con}$ to find estimators of ρ_h and λ_h , namely method of forming sequence of ρ_h and λ_h by means of R program [1, 7, 17, 18]. Its procedure is as follows:

1. We make sequences values of ρ_h and λ_h , respectively, namely

$\rho_h = \text{seq}(\text{start value, end value, increasing})$, and

$\lambda_h = \text{seq}(\text{start value, end value, increasing})$.

2. For every \mathbf{y}_{hj} and \mathbf{a}_{hj} , $h=1,2,3,\dots,m$, we insert values of ρ_h and λ_h in (17). Because the values of \mathbf{a}_{hj} are unknown, we use the estimator, $\hat{\mathbf{a}}_{hj}$, where $\hat{\mathbf{a}}_{hj} = \mathbf{1}(\hat{\mu}_h + \hat{\gamma}_{hj}) + \mathbf{Z}_{hj} \hat{\boldsymbol{\theta}}_h$, with $\mathbf{Z}_{hj} = \begin{bmatrix} \mathbf{X}_{hj} \\ \vdots \\ \hat{\mathbf{Y}}_{-hj} \end{bmatrix}$.
3. Finding the values of ρ_h and λ_h that gives the largest $\ln L_h^{con}$.

Based on the estimate ρ_h and λ_h , the equations (11) to (13) can be rewritten as follows:

$$\hat{\boldsymbol{\theta}}_h = \left[\sum_{j=1}^T \mathbf{Z}_{hj}^t \hat{\mathbf{B}}_h' \hat{\mathbf{B}}_h [\mathbf{1b}'_h - \mathbf{I}_n] \mathbf{Z}_{hj} \right]^{-1} \times \sum_{j=1}^T \mathbf{Z}_{hj}^t \hat{\mathbf{B}}_h' \hat{\mathbf{B}}_h [\mathbf{1b}'_h - \mathbf{I}_n] \hat{\mathbf{A}}_h \mathbf{y}_{hj}, \quad (18)$$

$$\hat{\mu}_h = \frac{1}{T} \mathbf{b}'_h \sum_{j=1}^T (\hat{\mathbf{A}}_h \mathbf{y}_{hj} - \mathbf{Z}_{hj} \hat{\boldsymbol{\theta}}_h), \quad (19)$$

and

$$\hat{\gamma}_{hj} = \mathbf{b}'_h (\hat{\mathbf{A}}_h \mathbf{y}_{hj} - \mathbf{1} \hat{\mu}_h - \mathbf{Z}_{hj} \hat{\boldsymbol{\theta}}_h), \quad (20)$$

respectively, where

$$\hat{\mathbf{A}}_h = \mathbf{I}_n - \hat{\rho}_h \mathbf{W}, \quad \hat{\mathbf{B}}_h = \mathbf{I}_n - \hat{\lambda}_h \mathbf{W}, \quad \text{and}$$

$$\mathbf{b}'_h = (\mathbf{1}' \hat{\mathbf{B}}_h' \hat{\mathbf{B}}_h \mathbf{1})^{-1} \mathbf{1}' \hat{\mathbf{B}}_h' \hat{\mathbf{B}}_h.$$

The furthermore, equation (14) can be rewritten as follows:

$$\hat{\mathbf{u}}_{hj}^* = \hat{\mathbf{A}}_h \mathbf{y}_{hj} - \mathbf{1}(\hat{\mu}_h + \hat{\gamma}_{hj}) - \mathbf{Z}_{hj} \hat{\boldsymbol{\theta}}_h = \hat{\mathbf{A}}_h \mathbf{y}_{hj} - \hat{\mathbf{a}}_{hj}. \quad (21)$$

We then use (21) and (16) to find the estimated covariance matrix of the estimator $\hat{\mathbf{u}}_{hj}^*$, namely

$$\hat{\boldsymbol{\Sigma}} = \begin{bmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} & \hat{\sigma}_{13} & \cdots & \hat{\sigma}_{1m} \\ \hat{\sigma}_{21} & \hat{\sigma}_2^2 & \hat{\sigma}_{23} & \cdots & \hat{\sigma}_{2m} \\ \hat{\sigma}_{31} & \hat{\sigma}_{32} & \hat{\sigma}_3^2 & \cdots & \hat{\sigma}_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_{m1} & \hat{\sigma}_{m2} & \hat{\sigma}_{m3} & \cdots & \hat{\sigma}_m^2 \end{bmatrix}, \quad \hat{\sigma}_h^2 = \hat{\sigma}_{hh^*} \text{ if } h = h^*$$

with

$$\hat{\sigma}_{hh^*} = \frac{1}{T(n-1) - (k+m-1)} \sum_{j=1}^T \hat{\mathbf{u}}_{hj}^* \hat{\mathbf{B}}_h' \hat{\mathbf{B}}_h \hat{\mathbf{u}}_{hj^*}^*,$$

where $\hat{\sigma}_h^2$ denotes the h th estimated error variance, $\hat{\sigma}_{hh^*}$ denotes the h^* th and the h th estimated error covariance, and $\hat{\boldsymbol{\Sigma}}$ denotes $m \times m$ estimated covariance matrix. We change the denominator of (16) so that it becomes an unbiased estimator [18].

From (6), we have an error covariance matrix, namely $\text{var}(\bar{\mathbf{X}}_{**}'\hat{\mathbf{B}}_*\boldsymbol{\varepsilon}_j) = \bar{\mathbf{X}}_{**}'\hat{\mathbf{B}}_*\text{var}(\boldsymbol{\varepsilon}_j)\hat{\mathbf{B}}_*'\bar{\mathbf{X}}_{**} = \boldsymbol{\Sigma}_{\#}$.

This covariance shows that the random errors are heteroscedastic, where $\text{var}(\boldsymbol{\varepsilon}_j) = E(\boldsymbol{\varepsilon}_j\boldsymbol{\varepsilon}_j')$ for

$$h = h^* = 1, 2, 3, \dots, m,$$

$$\boldsymbol{\varepsilon}_j^t = [\boldsymbol{\varepsilon}_{1j}^t \quad \boldsymbol{\varepsilon}_{2j}^t \quad \boldsymbol{\varepsilon}_{3j}^t \quad \dots \quad \boldsymbol{\varepsilon}_{mj}^t],$$

$$\boldsymbol{\varepsilon}_{hj}^t = [\boldsymbol{\varepsilon}_{h1j} \quad \boldsymbol{\varepsilon}_{h2j} \quad \boldsymbol{\varepsilon}_{h3j} \quad \dots \quad \boldsymbol{\varepsilon}_{hmj}],$$

in which we assumed that

$$E(\boldsymbol{\varepsilon}_{hij}\boldsymbol{\varepsilon}_{h^*i^*j}) = \begin{cases} \sigma_{hh^*} & \text{if } i = i^* \\ 0 & \text{if } i \neq i^* \end{cases}$$

so that $E(\boldsymbol{\varepsilon}_{hj}\boldsymbol{\varepsilon}_{h^*j}^t) = \sigma_{hh^*}\mathbf{I}_n$. We obtain

$$\text{var}(\boldsymbol{\varepsilon}_j) = \boldsymbol{\Sigma} \otimes \mathbf{I}_n \text{ with } mn \times mn \text{ as its dimension.}$$

Consequently, $\boldsymbol{\Sigma}_{\#} = \bar{\mathbf{X}}_{**}'\hat{\mathbf{B}}_*(\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\hat{\mathbf{B}}_*'\bar{\mathbf{X}}_{**}$ which is $m \sum_{h=1}^m k_h \times m \sum_{h=1}^m k_h$ symmetrical matrix. If $\boldsymbol{\Sigma}$ is

unknown then we can use its estimator. If we use its estimator then $\hat{\boldsymbol{\Sigma}}_{\#} = \bar{\mathbf{X}}_{**}'\hat{\mathbf{B}}_*(\hat{\boldsymbol{\Sigma}} \otimes \mathbf{I}_n)\hat{\mathbf{B}}_*'\bar{\mathbf{X}}_{**}$.

In the above results, we see that the error variance in equation (6) is not constant and the matrix in the right-hand side is less than full rank. For the last-stage, we overcome those problems again by means of reparameterization and GLS. The estimators are as follows:

$$\hat{\boldsymbol{\theta}} = \left[\sum_{j=1}^T \mathbf{Z}_j' \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right]^{-1} \sum_{j=1}^T \mathbf{Z}_j' \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \hat{\mathbf{A}}\mathbf{y}_j, \quad (22)$$

$$\hat{\boldsymbol{\mu}} = \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \mathbf{G}' \hat{\mathbf{H}}^* \sum_{j=1}^T (\hat{\mathbf{A}}\mathbf{y}_j - \mathbf{Z}_j \hat{\boldsymbol{\theta}}), \quad (23)$$

$$\hat{\boldsymbol{\gamma}}_j = \left[\mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \mathbf{G}' \hat{\mathbf{H}}^* (\hat{\mathbf{A}}\mathbf{y}_j - \mathbf{G}\hat{\boldsymbol{\mu}} - \mathbf{Z}_j \hat{\boldsymbol{\theta}}). \quad (24)$$

where

$$\hat{\mathbf{H}}^* = \bar{\mathbf{X}}_{**}' \boldsymbol{\Sigma}_{\#}^{-1} \bar{\mathbf{X}}_{**}$$

$$\hat{\mathbf{M}}^* = \mathbf{G} \left[\mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \mathbf{G}' \hat{\mathbf{H}}^* - \mathbf{I}_{mn}.$$

They have dimensions $mn \times mn$, respectively.

In this paper, the estimators of $\boldsymbol{\theta}, \boldsymbol{\alpha}$, and $\boldsymbol{\gamma}_j$ are called the *estimators* of feasible generalized least squares-multivariate general spatial three-stage least squares fixed effect panel simultaneous models (FGLS-MGS3SLSFPEPSM).

4 Properties of Estimators

Theorem (Consistency). *If*

$$\bar{\mathbf{X}}_{**}' \mathbf{A}\mathbf{y}_j = \bar{\mathbf{X}}_{**}' \mathbf{G}\boldsymbol{\mu} + \bar{\mathbf{X}}_{**}' \mathbf{Z}_j \boldsymbol{\theta} + \bar{\mathbf{X}}_{**}' \mathbf{G}\boldsymbol{\gamma}_j + \bar{\mathbf{X}}_{**}' \mathbf{B}_* \boldsymbol{\varepsilon}_j$$

as defined in (6), then $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\mu}}$, and $\hat{\boldsymbol{\gamma}}_j$ are consistent estimators.

Proof. Recall (6). This can be rewritten as $\mathbf{A}\mathbf{y}_j = \mathbf{G}\boldsymbol{\mu} + \mathbf{Z}_j \boldsymbol{\theta} + \mathbf{G}\boldsymbol{\gamma}_j + \mathbf{B}_* \boldsymbol{\varepsilon}_j$. However, we use the

estimate $\hat{\rho}_h$ and $\hat{\lambda}_h$. The equation (6) can be

rewritten as $\hat{\mathbf{A}}\mathbf{y}_j = \mathbf{G}\boldsymbol{\mu} + \mathbf{Z}_j \boldsymbol{\theta} + \mathbf{G}\boldsymbol{\gamma}_j + \hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j$.

Estimators of equation (6) are as follows:

$$\begin{aligned} \hat{\boldsymbol{\theta}} &= \left[\sum_{j=1}^T \mathbf{Z}_j' \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right]^{-1} \sum_{j=1}^T \mathbf{Z}_j' \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \hat{\mathbf{A}}\mathbf{y}_j \\ &= \boldsymbol{\theta} + \left[\sum_{j=1}^T \mathbf{Z}_j' \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right]^{-1} \left[\sum_{j=1}^T \mathbf{Z}_j' \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right], \end{aligned}$$

where $\hat{\mathbf{M}}^* \mathbf{G} = \mathbf{0}$,

$$\begin{aligned} \hat{\boldsymbol{\mu}} &= \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \mathbf{G}' \hat{\mathbf{H}}^* \sum_{j=1}^T (\hat{\mathbf{A}}\mathbf{y}_j - \mathbf{Z}_j \hat{\boldsymbol{\theta}}) \\ &= \boldsymbol{\mu} + \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \left(\sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{Z}_j (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right), \end{aligned}$$

where $\sum_{j=1}^T \boldsymbol{\gamma}_j = \mathbf{0}$, and

$$\begin{aligned} \hat{\boldsymbol{\gamma}}_j &= \left[\mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \mathbf{G}' \hat{\mathbf{H}}^* (\hat{\mathbf{A}}\mathbf{y}_j - \mathbf{G}\hat{\boldsymbol{\mu}} - \mathbf{Z}_j \hat{\boldsymbol{\theta}}) \\ &= (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) + \left[\mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{Z}_j (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \boldsymbol{\gamma}_j \\ &\quad + \left[\mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \mathbf{G}' \hat{\mathbf{H}}^* \hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j. \end{aligned}$$

We refer to [3, 4, 14, 16, 21-23]. Asymptotic expectation and variance of $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\mu}}$, and $\hat{\boldsymbol{\gamma}}_j$ are as follows:

$$\begin{aligned} \bar{E}\{\hat{\boldsymbol{\theta}}\} &= \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} E\{\hat{\boldsymbol{\theta}}\} = \boldsymbol{\theta} + \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} E \left\{ \left[\frac{1}{nT} \sum_{j=1}^T \mathbf{Z}_j' \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right]^{-1} \right. \\ &\quad \left. \times \left[\frac{1}{nT} \sum_{j=1}^T \mathbf{Z}_j' \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right] \right\} \\ &= \boldsymbol{\theta} + \left[\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} \mathbf{K} \right]^{-1} \left[\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} \sum_{j=1}^T \mathbf{Z}_j' \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \hat{\mathbf{B}}_* \times \mathbf{0} \right] \\ &= \boldsymbol{\theta} + \left[\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \bar{\mathbf{K}} \right]^{-1} \times \mathbf{0} = \boldsymbol{\theta} + [\bar{\mathbf{K}}]^{-1} \times \mathbf{0} \\ &= \boldsymbol{\theta}, \end{aligned}$$

where \mathbf{K} and $\bar{\mathbf{K}}$ are constant nonsingular matrices.

$$\begin{aligned} \text{asy. var} \{ \hat{\theta} \} &= \text{asy. var} \left\{ \left[\sum_{j=1}^T \mathbf{Z}'_j \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right]^{-1} \mathbf{Z}'_j \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right\} \\ &= \left[\sum_{j=1}^T \mathbf{Z}'_j \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right]^{-1} \left[\sum_{j=1}^T \mathbf{Z}'_j \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \hat{\mathbf{B}}_* (\boldsymbol{\Sigma} \otimes \mathbf{I}_n) \right. \\ &\quad \left. \times \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right] \left[\sum_{j=1}^T \mathbf{Z}'_j \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right]^{-1}, \end{aligned}$$

where $\hat{\mathbf{H}}^*$ and $\hat{\mathbf{H}}^* \hat{\mathbf{M}}^*$ are symmetrical. Now,

$$\begin{aligned} \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \text{asy. var} \{ \hat{\theta} \} &= \left[\sum_{j=1}^T \mathbf{Z}'_j \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right]^{-1} \left[\sum_{j=1}^T \mathbf{Z}'_j \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \hat{\mathbf{B}}_* \right. \\ &\quad \left. \times \left\{ \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} (\boldsymbol{\Sigma} \otimes \mathbf{I}_n) \right\} \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right] \\ &\quad \times \left[\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} \sum_{j=1}^T \mathbf{Z}'_j \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right]^{-1} \end{aligned}$$

$$\begin{aligned} \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \text{asy. var} \{ \hat{\theta} \} &= [\mathbf{K}]^{-1} \left[\sum_{j=1}^T \mathbf{Z}'_j \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \hat{\mathbf{B}}_* \times \mathbf{0} \right. \\ &\quad \left. \times \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right] \left[\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \bar{\mathbf{K}} \right]^{-1} \\ &= \mathbf{K}^{-1} \times \mathbf{0} \times [\bar{\mathbf{K}}]^{-1} = \mathbf{0}, \end{aligned}$$

This shows that $\hat{\theta}$ is asymptotically unbiased estimator. If $n \rightarrow \infty$ or $T \rightarrow \infty$ or both of $n \rightarrow \infty$ and $T \rightarrow \infty$, then $\text{asy. var} \{ \hat{\theta} \} \rightarrow \mathbf{0}$. Therefore, $\hat{\theta}$ is a consistent estimator. Next,

$$\begin{aligned} \bar{E} \{ \hat{\mu} \} &= \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} E \{ \hat{\mu} \} \\ &= \boldsymbol{\mu} + \left(\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} \left[\frac{1}{n} \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \right) \left(\sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{Z}_j \right. \\ &\quad \left. \times \left(\boldsymbol{\theta} - \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \{ \hat{\theta} \} \right) + \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \hat{\mathbf{B}}_* E \{ \boldsymbol{\varepsilon}_j \} \right) \\ &= \boldsymbol{\mu} + \left(\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} \left[\frac{1}{n} \mathbf{K}_1 \right]^{-1} \right) \left(\sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{Z}_j (\boldsymbol{\theta} - \boldsymbol{\theta}) \right. \\ &\quad \left. + \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \hat{\mathbf{B}}_* \times \mathbf{0} \right) \\ &= \boldsymbol{\mu} + \left(\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} [\bar{\mathbf{K}}_1]^{-1} \right) \times (\mathbf{0} + \mathbf{0}) = \boldsymbol{\mu}, \end{aligned}$$

where \mathbf{K}_1 and $\bar{\mathbf{K}}_1$ are constant nonsingular matrices. We have

$$\begin{aligned} \text{asy. var} \{ \hat{\mu} \} &= \text{asy. var} \left\{ \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{Z}_j \hat{\theta} \right\} \\ &\quad + \text{asy. var} \left\{ \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right\}, \end{aligned}$$

$$\begin{aligned} \text{asy. var} \left\{ \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{Z}_j \hat{\theta} \right\} &= \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \\ &\quad \times \left[\sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{Z}_j \text{asy. var} \{ \hat{\theta} \} \mathbf{Z}'_j \hat{\mathbf{H}}^* \mathbf{G} \right] \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1}, \end{aligned}$$

$$\begin{aligned} \text{asy. var} \left\{ \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right\} &= \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \\ &\quad \times \left[T \mathbf{G}' \hat{\mathbf{H}}^* \hat{\mathbf{B}}_* (\boldsymbol{\Sigma} \otimes \mathbf{I}_n) \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \mathbf{G} \right] \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \\ &= \left[\mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \left[\mathbf{G}' \hat{\mathbf{H}}^* \hat{\mathbf{B}}_* (\boldsymbol{\Sigma} \otimes \mathbf{I}_n) \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \mathbf{G} \right] \\ &\quad \times \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1}. \end{aligned}$$

$$\begin{aligned} \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \text{asy. var} \left\{ \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{Z}_j \hat{\theta} \right\} \\ &= \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} \left[\frac{1}{n} \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \left[\sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{Z}_j \left(\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \text{asy. var} \{ \hat{\theta} \} \right) \right. \\ &\quad \left. \times \mathbf{Z}'_j \hat{\mathbf{H}}^* \mathbf{G} \right] \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} \left[\frac{1}{n} \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \\ &= \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} [\bar{\mathbf{K}}_1]^{-1} \left[\sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{Z}_j \times \mathbf{0} \times \mathbf{Z}'_j \hat{\mathbf{H}}^* \mathbf{G} \right] \\ &\quad \times \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} [\bar{\mathbf{K}}_1]^{-1} \\ &= \mathbf{0} \times \mathbf{0} \times \mathbf{0} = \mathbf{0}. \end{aligned}$$

$$\begin{aligned} \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \text{asy. var} \left\{ \left[T \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \sum_{j=1}^T \mathbf{G}' \hat{\mathbf{H}}^* \hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right\} \\ &= \left[\mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \left[\mathbf{G}' \hat{\mathbf{H}}^* \hat{\mathbf{B}}_* \left\{ \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} (\boldsymbol{\Sigma} \otimes \mathbf{I}_n) \right\} \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \mathbf{G} \right] \\ &\quad \times \left[\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{n} \mathbf{G}' \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \\ &= [\mathbf{K}_1]^{-1} \left[\mathbf{G}' \hat{\mathbf{H}}^* \hat{\mathbf{B}}_* \times \mathbf{0} \times \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \mathbf{G} \right] \left[\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \bar{\mathbf{K}}_1 \right]^{-1} \\ &= \mathbf{K}_1^{-1} \times \mathbf{0} \times [\bar{\mathbf{K}}_1]^{-1} = \mathbf{0}. \end{aligned}$$

Consequently, $\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \text{asy. var} \{ \hat{\mu} \} = \mathbf{0}$.

This shows that $\hat{\mu}$ is asymptotically unbiased estimator. If $n \rightarrow \infty$ or $T \rightarrow \infty$ or both of $n \rightarrow \infty$ and $T \rightarrow \infty$, then $\text{asy. var} \{ \hat{\mu} \} \rightarrow \mathbf{0}$. Therefore, $\hat{\mu}$ is a consistent estimator. Now,

$$\begin{aligned} \bar{E}\{\hat{\gamma}_j\} &= \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} E\{\hat{\gamma}_j\} \\ &= \left(\boldsymbol{\mu} - \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} E\{\hat{\boldsymbol{\mu}}\} \right) + [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{Z}_j \\ &\quad \times \left(\boldsymbol{\theta} - \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} E\{\hat{\boldsymbol{\theta}}\} \right) + \gamma_j + \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{n} \left[\frac{1}{n} \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G} \right]^{-1} \\ &\quad \times \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* E\{\boldsymbol{\varepsilon}_j\} \end{aligned}$$

$$\begin{aligned} \bar{E}\{\hat{\gamma}_j\} &= (\boldsymbol{\mu} - \boldsymbol{\mu}) + [\mathbf{K}_1]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{Z}_j (\boldsymbol{\theta} - \boldsymbol{\theta}) + \gamma_j \\ &\quad + \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{n} [\bar{\mathbf{K}}_1]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* \times \mathbf{0} \\ &= \gamma_j. \end{aligned}$$

$$\begin{aligned} \text{asy. var}\{\hat{\gamma}_j\} &= \text{asy. var}\left\{ (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) + [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \right. \\ &\quad \times \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{Z}_j (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \gamma_j \\ &\quad \left. + [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right\} \\ &= \text{asy. var}\{\hat{\boldsymbol{\mu}}\} \\ &\quad + \text{asy. var}\left\{ [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{Z}_j \hat{\boldsymbol{\theta}} \right\} \\ &\quad + \text{asy. var}\left\{ [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right\}. \end{aligned}$$

$$\begin{aligned} \text{asy. var}\left\{ [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{Z}_j \hat{\boldsymbol{\theta}} \right\} \\ &= [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{Z}_j \text{asy. var}\{\hat{\boldsymbol{\theta}}\} \\ &\quad \times \mathbf{Z}'_j \hat{\mathbf{H}}^* \mathbf{G} [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1}, \end{aligned}$$

$$\begin{aligned} \text{asy. var}\left\{ [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right\} \\ &= [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* (\boldsymbol{\Sigma} \otimes \mathbf{I}_n) \\ &\quad \times \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \mathbf{G} [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1}. \end{aligned}$$

$$\begin{aligned} \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \text{asy. var}\left\{ [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{Z}_j \hat{\boldsymbol{\theta}} \right\} \\ &= [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{Z}_j \left(\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \text{asy. var}\{\hat{\boldsymbol{\theta}}\} \right) \\ &\quad \times \mathbf{Z}'_j \hat{\mathbf{H}}^* \mathbf{G} [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \\ &= [\mathbf{K}_1]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{Z}_j \times \mathbf{0} \times \mathbf{Z}'_j \hat{\mathbf{H}}^* \mathbf{G} [\mathbf{K}_1]^{-1} \\ &= \mathbf{0}, \end{aligned}$$

$$\begin{aligned} \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \text{asy. var}\left\{ [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right\} \\ &= [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* \left\{ \lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{nT} (\boldsymbol{\Sigma} \otimes \mathbf{I}_n) \right\} \\ &\quad \times \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \mathbf{G} \left[\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{n} \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G} \right]^{-1} \\ &= [\mathbf{K}_1]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* \times \mathbf{0} \times \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \mathbf{G} \left[\lim_{\substack{n \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{T} \bar{\mathbf{K}}_1 \right]^{-1} \\ &= \mathbf{0} \times [\mathbf{0}]^{-1} = \infty \text{ (infinite)}, \end{aligned}$$

therefore, convergency be satisfied only if $n \rightarrow \infty$, namely

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{asy. var}\left\{ [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* \boldsymbol{\varepsilon}_j \right\} \\ &= [\mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G}]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} (\boldsymbol{\Sigma} \otimes \mathbf{I}_n) \right\} \\ &\quad \times \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \mathbf{G} \left[\lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{G}'\hat{\mathbf{H}}^*\mathbf{G} \right]^{-1} \\ &= [\mathbf{K}_1]^{-1} \mathbf{G}'\hat{\mathbf{H}}^*\hat{\mathbf{B}}_* \times \mathbf{0} \times \hat{\mathbf{B}}'_* \hat{\mathbf{H}}^* \mathbf{G} \left[\lim_{n \rightarrow \infty} \bar{\mathbf{K}}_1 \right]^{-1} \\ &= \mathbf{K}_1^{-1} \times \mathbf{0} \times [\bar{\mathbf{K}}_1]^{-1} = \mathbf{0}. \end{aligned}$$

Consequently, $\lim_{n \rightarrow \infty} \text{asy. var}\{\hat{\gamma}_j\} = \mathbf{0}$.

This shows that $\hat{\gamma}_j$ is asymptotically unbiased estimator. If $n \rightarrow \infty$, then $\text{asy. var}\{\hat{\gamma}_j\} \rightarrow \mathbf{0}$. Therefore, $\hat{\gamma}_j$ is a consistent estimator.

5 Illustration

Suppose there are three endogenous variables y_1, y_2, y_3 and six exogenous variables $x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}$ observed for two time periods and the number of observation being 10 locations. We use illustration of data, locations, and row-standardized spatial weight matrix as it was presented in [2]. The equation models are as follows:

$$\begin{aligned} y_{1ij} &= \mu_1 + \alpha_{11}x_{11ij} + \alpha_{12}x_{12ij} + \rho_1 \mathbf{w}'_i \mathbf{y}_{1j} + \beta_{12}y_{2ij} \\ &\quad + \beta_{13}y_{3ij} + \gamma_{1j} + u_{1ij} \\ y_{2ij} &= \mu_2 + \alpha_{21}x_{21ij} + \alpha_{22}x_{22ij} + \rho_2 \mathbf{w}'_i \mathbf{y}_{2j} + \beta_{21}y_{1ij} \\ &\quad + \beta_{23}y_{3ij} + \gamma_{2j} + u_{2ij} \\ y_{3ij} &= \mu_3 + \alpha_{31}x_{31ij} + \alpha_{32}x_{32ij} + \rho_3 \mathbf{w}'_i \mathbf{y}_{3j} + \beta_{31}y_{1ij} \quad (25) \\ &\quad + \beta_{32}y_{2ij} + \gamma_{3j} + u_{3ij}, \end{aligned}$$

$$u_{1ij} = \lambda_1 \mathbf{w}'_i \mathbf{u}_{1j} + \varepsilon_{1ij}, \quad \varepsilon_{1ij} \sim N(0, \sigma_1^2),$$

$$u_{2ij} = \lambda_2 \mathbf{w}'_i \mathbf{u}_{2j} + \varepsilon_{2ij}, \quad \varepsilon_{2ij} \sim N(0, \sigma_2^2),$$

$$u_{3ij} = \lambda_3 \mathbf{w}'_i \mathbf{u}_{3j} + \varepsilon_{3ij}, \quad \varepsilon_{3ij} \sim N(0, \sigma_3^2),$$

where

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1,10} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2,10} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3,10} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{10,1} & w_{10,2} & w_{10,3} & \cdots & w_{10,10} \end{bmatrix} = \begin{bmatrix} \mathbf{w}'_1 \\ \mathbf{w}'_2 \\ \mathbf{w}'_3 \\ \vdots \\ \mathbf{w}'_{10} \end{bmatrix}$$

$$= [\mathbf{w}'_i], \quad i = 1, 2, 3, \dots, 10,$$

The formulation of Moran Index is as follows:

$$I_{hj} = \frac{\sum_{i=1}^{10} \sum_{i'=1}^{10} w_{ii'} (y_{hij} - \bar{y}_{hj})(y_{hi'j} - \bar{y}_{hj})}{\sum_{i=1}^{10} (y_{hij} - \bar{y}_{hj})^2} = \frac{\mathbf{y}_{hj}^* \mathbf{W} \mathbf{y}_{hj}^*}{\mathbf{y}_{hj}^* \mathbf{y}_{hj}^*},$$

for $h=1, 2, 3, j=1, 2,$

where $\bar{y}_{hj} = \frac{1}{10} \sum_{i=1}^{10} y_{hij}$ and $\mathbf{y}_{hj}^* = \mathbf{y}_{hj} - \bar{y}_{hj} \mathbf{1}$.

If there is at least one $I_{hj} > E(I)$, then we conclude that there is a spatial influence for the equation models.

$$\bar{y}_{11} = 15.80; \quad \bar{y}_{21} = 28.30; \quad \bar{y}_{31} = 24.90;$$

$$\bar{y}_{22} = 28.40; \quad \bar{y}_{32} = 25.90;$$

$$I_{11} = -0.2442; \quad I_{21} = 0.0539; \quad I_{31} = 0.4586;$$

$$I_{12} = -0.2317; \quad I_{22} = -0.0878; \quad I_{32} = -0.1078;$$

$$\text{and } E(I_{hj}) = E(I) = \frac{-1}{n-1} = \frac{-1}{10-1} = -0.1111.$$

Based on the above result, by means of R Program version 3.6.1, we obtain that there is a spatial influence for the equation models.

We then continue to estimate parameters by means of FGLS-3SLS. For the first-stage, we estimate all the endogenous explanatory variables in the system in every time period and the results are presented in Table 1.

For the second-stage we estimate $\Sigma_{\#}$. But, we first estimate both of spatial autoregressive and spatial autocorrelation by means of equation (17). By \mathbf{W} matrix, we have the acceptable spatial autoregressive and spatial autocorrelation parameters are $-1.6242 < \rho_h (\lambda_h) < 1$. By method of forming sequence both of ρ_h and λ_h with increasing 0.01 we obtain

1. $\rho_h = \text{seq}(-1.6142, 0.99, 0.01)$ and $\lambda_h = \text{seq}(-1.6142, 0.99, 0.01)$.

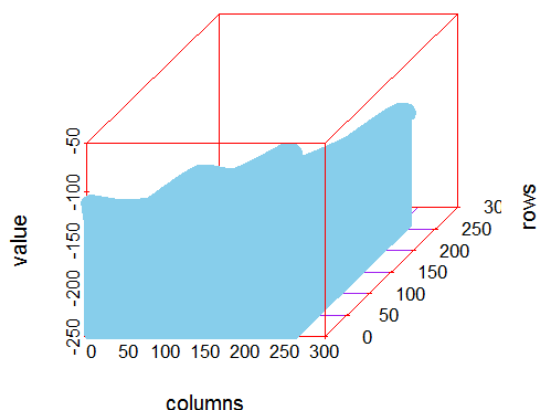
2. For every \mathbf{y}_{hj} and $\mathbf{a}_{hj}, h=1, 2, 3,$ we insert combination of all possible values of ρ_h and λ_h to (17). Because \mathbf{a}_{hj} is unknown, we use the estimate $\hat{\mathbf{a}}_{hj}$, where $\hat{\mathbf{a}}_{hj} = \mathbf{1}(\hat{\mu}_h + \hat{\gamma}_{hj}) + \mathbf{Z}_{hj} \hat{\theta}_h$, with $\mathbf{Z}_{hj} = [\mathbf{X}_{hj} \quad \vdots \quad \hat{\mathbf{Y}}_{-hj}]$.
3. We obtain $\hat{\rho}_1 = -1.6142, \quad \hat{\lambda}_1 = 0.9258, \quad \hat{\rho}_2 = -1.6142, \quad \hat{\lambda}_2 = -0.2242, \quad \hat{\rho}_3 = -1.5742,$ and $\hat{\lambda}_3 = 0.4658$ those give the largest $\ln L_1^{con}, \ln L_2^{con}$ and $\ln L_3^{con}$, respectively.

Table 1 Estimated values for endogenous explanatory variables

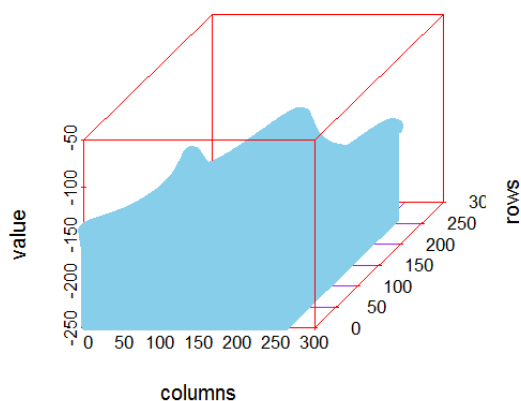
Time	Loca- tion	Endogenous explanatory variables		
		y1- estimate	y2- estimate	y3- estimate
1	1	16.5625	26.5828	21.7290
	2	15.0373	28.5890	25.1588
	3	16.1904	27.6672	20.7955
	4	12.3775	26.0621	23.9145
	5	16.1804	28.3403	22.0819
	6	17.2918	26.9959	24.8593
	7	18.7007	29.1060	26.5129
	8	12.3543	31.5231	28.7592
	9	16.4805	31.4345	30.8012
	10	16.8246	26.6991	24.3877
2	1	15.5100	25.9073	23.1069
	2	17.3247	27.0314	24.7638
	3	15.8433	26.3597	22.8089
	4	13.3019	25.4562	21.0773
	5	17.3259	29.7492	27.8872
	6	18.1930	30.2621	28.3106
	7	18.0785	31.1379	29.8797
	8	14.7653	32.0924	30.1569
	9	14.9671	29.3371	27.3870
	10	18.6902	26.6667	23.6217

By sequences of ρ_h and λ_h with increasing 0.01, we can also make graphs among the values of rho, lambda, and the values of concentrated log-likelihood as presented in Fig. 1.

Graph of function lnLcon1 of rho1 and lambda1



Graph of function lnLcon2 of rho2 and lambda2



Graph of function lnLcon3 of rho3 and lambda3

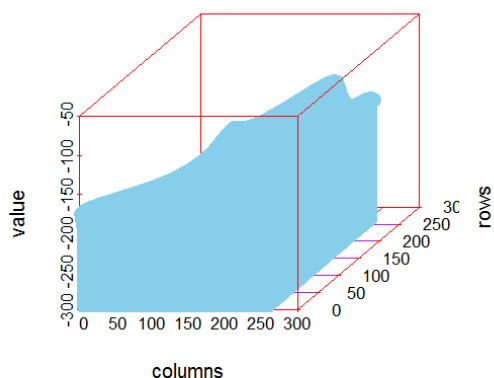


Fig. 1: Graphs of function of rho and lambda

As we need to know in Fig. 1, rows are the points of rho that its values are seq(-1.6142, 0.99, 0.01), columns are the points of lambda that its values are seq(-1.6142, 0.99, 0.01), and value is the value of lnLcon. There are 261 points both of rho and lambda. Its points are 1,2,3, until 261.

Table 2 Estimate values for residual errors

Time	Loca- tion	Residual errors		
		u1- estimate	u2- estimate	u3- estimate
1	1	530.9617	3.6731	-20.1653
	2	634.4891	123.6771	-30.2136
	3	708.4265	-83.8497	20.1640
	4	821.6256	69.4616	11.7260
	5	677.8279	-63.3284	14.2880
	6	612.9724	93.3877	78.6841
	7	612.4407	151.2321	62.3801
	8	570.8471	245.4605	-133.6357
	9	609.1129	280.6660	10.3691
	10	560.6974	109.3295	57.4756
2	1	1,021.2697	-122.8737	58.6738
	2	1,005.5465	-99.2300	60.9696
	3	973.8582	-236.6423	73.0186
	4	945.0748	-280.0563	73.3787
	5	748.1006	33.4929	-10.1951
	6	846.8442	-25.4341	16.2969
	7	828.7046	73.9246	-26.9565
	8	801.7122	79.0934	-50.7021
	9	927.9174	-78.3885	-50.9108
	10	932.3442	-215.1956	104.0915

From (18) to (20), we obtain

$$\hat{\mu}_1 = 59.8488; \hat{\mu}_2 = 35.7021; \hat{\mu}_3 = 30.2172;$$

$$\hat{\gamma}_{11} = -4.7589; \hat{\gamma}_{12} = 4.7589;$$

$$\hat{\gamma}_{21} = 3.0530; \hat{\gamma}_{22} = -3.0530;$$

$$\hat{\gamma}_{31} = 1.2893; \hat{\gamma}_{32} = -1.2893.$$

$$\hat{\theta}_1 = \begin{bmatrix} \hat{\alpha}_1 \\ \dots \\ \hat{\beta}_{-1} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{11} \\ \hat{\alpha}_{12} \\ \dots \\ \hat{\beta}_{12} \\ \hat{\beta}_{13} \end{bmatrix} = \begin{bmatrix} 7.5482 \\ -36.8974 \\ \dots \\ 34.8812 \\ 3.1676 \end{bmatrix};$$

$$\hat{\theta}_2 = \begin{bmatrix} \hat{\alpha}_2 \\ \dots \\ \hat{\beta}_{-2} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{21} \\ \hat{\alpha}_{22} \\ \hat{\beta}_{21} \\ \hat{\beta}_{23} \end{bmatrix} = \begin{bmatrix} 7.9886 \\ 10.6995 \\ 6.2486 \\ -44.5616 \end{bmatrix};$$

$$\hat{\theta}_3 = \begin{bmatrix} \hat{\alpha}_3 \\ \dots \\ \hat{\beta}_{-3} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{31} \\ \hat{\alpha}_{32} \\ \hat{\beta}_{31} \\ \hat{\beta}_{32} \end{bmatrix} = \begin{bmatrix} 8.2808 \\ -7.9064 \\ -23.8113 \\ 16.0997 \end{bmatrix};$$

Next, from (21) we obtain the estimate values for residual errors being presented in Tabel 2. We then use the estimate values for residual errors (in Table 2) to find $\hat{\Sigma}$ as follow:

$$\hat{\Sigma} = \begin{bmatrix} 16,407.7430 & -13,327.4840 & 3,586.3450 \\ -13,327.4840 & 55,037.3740 & -6,806.3970 \\ 3,586.3450 & -6,806.3970 & 8,201.1720 \end{bmatrix},$$

and we obtain

$$\hat{\Sigma}_{\#} = \begin{bmatrix} 95,015,762,019 & 112,938,200,311 \\ 112,938,200,311 & 134,249,432,552 \\ 95,961,464,592 & 114,062,507,032 \\ \vdots & \vdots \\ 3,116,649,203 & 3,704,593,381 \\ 95,961,464,592 & \dots & 3,116,649,203 \\ 114,062,507,032 & \dots & 3,704,593,381 \\ 96,916,769,558 & \dots & 3,147,439,542 \\ \vdots & \ddots & \vdots \\ 3,147,439,542 & \dots & 1,100,342,293 \end{bmatrix},$$

where $\hat{\Sigma}_{\#}$ is the estimator of covariance matrix.

For the last-stage, we estimate the parameters of equation models (25). From (22) to (24), we obtain

$$\hat{\theta} = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{31} \\ \alpha_{32} \\ \beta_{12} \\ \beta_{13} \\ \beta_{21} \\ \beta_{23} \\ \beta_{31} \\ \beta_{32} \end{bmatrix} = \begin{bmatrix} 0.1967 \\ -0.2607 \\ 0.2356 \\ -0.0666 \\ 0.0036 \\ 0.3796 \\ 0.3797 \\ -0.3355 \\ 0.3474 \\ 1.1750 \\ 0.0009 \\ 1.5913 \end{bmatrix}; \hat{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 47.7816 \\ 30.2635 \\ -3.0592 \end{bmatrix}$$

$$\hat{\gamma}_1 = \begin{bmatrix} \gamma_{11} \\ \gamma_{21} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} -4.4505 \\ 1.9577 \\ 1.5276 \end{bmatrix}; \hat{\gamma}_2 = \begin{bmatrix} \gamma_{12} \\ \gamma_{22} \\ \gamma_{32} \end{bmatrix} = \begin{bmatrix} 4.4505 \\ -1.9577 \\ -1.5276 \end{bmatrix};$$

and the estimators equation models (25) are

$$\hat{y}_{1i1} = 47.7816 + 0.1967x_{11i1} - 0.2607x_{12i1} - 1.6142w'_i y_{11} + 0.3797y_{2i1} - 0.3355y_{3i1} - 4.4505 + \hat{u}_{1i1}$$

$$\hat{y}_{2i1} = 30.2635 + 0.2356x_{21i1} - 0.0666x_{22i1} - 1.6142w'_i y_{21} + 0.3474y_{1i1} + 1.1750y_{3i1} + 1.9577 + \hat{u}_{2i1}$$

$$\hat{y}_{3i1} = -3.0592 + 0.0036x_{31i1} + 0.3796x_{32i1} - 1.5742w'_i y_{31} + 0.0009y_{1i1} + 1.5913y_{2i1} + 1.5276 + \hat{u}_{3i1},$$

$$\hat{u}_{1i1} = 0.9258w'_i \hat{u}_{11}$$

$$\hat{u}_{2i1} = -0.2242w'_i \hat{u}_{21}$$

$$\hat{u}_{3i1} = 0.4658w'_i \hat{u}_{31},$$

$$\hat{y}_{1i2} = 47.7816 + 0.1967x_{11i2} - 0.2607x_{12i2} - 1.6142w'_i y_{12} + 0.3797y_{2i2} - 0.3355y_{3i2} + 4.4505 + \hat{u}_{1i2}$$

$$\hat{y}_{2i2} = 30.2635 + 0.2356x_{21i2} - 0.0666x_{22i2} - 1.6142w'_i y_{22} + 0.3474y_{1i2} + 1.1750y_{3i2} - 1.9577 + \hat{u}_{2i2}$$

$$\hat{y}_{3i2} = -3.0592 + 0.0036x_{31i2} + 0.3796x_{32i2} - 1.5742w'_i y_{32} + 0.0009y_{1i2} + 1.5913y_{2i2} - 1.5276 + \hat{u}_{3i2},$$

$$\hat{u}_{1i2} = 0.9258w'_i \hat{u}_{12}$$

$$\hat{u}_{2i2} = -0.2242w'_i \hat{u}_{22}$$

$$\hat{u}_{3i2} = 0.4658w'_i \hat{u}_{32},$$

where \hat{u}_{ij} are the estimate values for residual errors as given in Table 2.

6 Conclusion

In this paper, we are motivated to develop simultaneous equation models for fixed effect panel data with one-way error component by means of 3SLS solutions, especially for general spatial.

The numerical approximation estimators of parameter models are obtained by means of concentrated log-likelihood formulation with method of forming sequence. In this paper, we use the increasing values 0.01.

The closed-form estimators are obtained by means of feasible generalized least squares-three-

stage least squares (FGLS-3SLS) and they are called the *estimators* of feasible generalized least squares-multivariate general spatial three-stage least squares fixed effect panel simultaneous models (FGLS-MGS3SLSFESM). All estimators are consistent estimators.

There is one limitation of this paper, we still use the numerical approximation to find the estimators of spatial autoregressive and spatial autocorrelation. In future research, we encourage to find the closed-form estimators of spatial autoregressive and spatial autocorrelation. In addition, to develop models not only for fixed effect but also both fixed effect and random effect (mixed models).

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