The bundle of simultaneously diagonalizable n-tuples of matrices

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Abstract: In this paper, a review of the simultaneous diagonalization of n-tuples of matrices for its applications in sciences is presented. For example, in quantum mechanics, position and momentum operators do not have a shared base that can represent the states of the system because they not commute, which is why switching operators form a key element of quantum physics since they define quantities that are compatible, that is, defined simultaneously. We are going to study this kind of family of linear operators using geometric constructions such as the principal bundles and associating them with a cohomology class measuring the deviation of the local product structure from theglobalproductstructure.

Key–Words: Diagonalization, simultaneous diagonalization, bundles

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1 Introduction

Let $\mathfrak{M}$ be the manifold of m-tuples of n-order real matrices $T = (X_1, \dots, X_m)$

The simultaneous diagonalization of two real symmetric matrices has long been of interest and largely studied, (see [5], for example).

In this paper, a review about the simultaneous diagonalization of n-tuples of matrices for its applications in sciences in particular for the case of traceless matrices, [7], [8]. For example they appear founding when we must give the instanton solution of Yang-Mills field presented in an octonion form, and it can be represented by triples of traceless matrices, [1], [6], [13]. Another application of simultaneous diagonalization is found when studying, for example, thermal transmissivity, whose study is different depending on whether the interaction matrices diagonalize simultaneously, [10].

In the space of n-square real matrices, it is well known that the subset of diagonalizable matrices is generic in the, then any no diagonalizable matrix can be diagonalized by a small perturbation of its entries. This property cannot be generalized to the case of simultaneous diagonalization of an m-tuple of n-order real square matrices. For that the simultaneous diagonalization is studied under different points of view as for example analysing the spectra of families of m-tuples of matrices, [8] by means Arnold tools, [3].

When someone is interested in distinguishing one subset from another within a differentiable variety, a good tool may be to try to identify it from the zeros of bundle sections built on the variety, and, then, the characteristic classes allow to identify its obstructions. In this particular setup the interest is about the set of the m-tuples of simultaneously diagonalizable real matrices, some results about families of simulatneously diagonalization can be found in [7], [8].

Principal bundles [9], have significant applications in different mathematical areas as topology and differential geometry, in special bundles given by a Lie group action. They have also applications in physics, concretely they form part of the basic framework of gauge theories, [14], and quantum theory, [4].

The cohomology is a topological invariant of a smooth variety, and it is an algebraic tool, which is a certain algebraic structure extracted from a differentiable variety, that allows us to distinguish whether or not two varieties are homeomorphic, [12].

In this work we are going to talk about invariant polynomials, which are a classic tool that allows a detailed study of the characteristic classes for bundles. A study for the case of the set of the square complex matrices can be found in [2].

2 Preliminaries

Basic properties

Definition 1. *Let* $T = (X_1, \dots, X_m), T' = (Y_1, \dots, Y_m)$ *be two* m*-tuples of matrices. Then,* T *is simultaneous similar to* T' *if and only if there exists*

$P \in \mathfrak{G} = Gl(n; \mathbb{R})$ *such that*

$$\begin{aligned} T' &= (Y_1, \ldots, Y_m) \\ &= (PX_1P^{-1}, \ldots, PX_mP^{-1}) \\ &= PTP^{-1}. \end{aligned} \tag{1}$$

The coefficients of characteristic polynomial $\det(\lambda I - (X_1 + xX_2 + \ldots + x^{m-1}X_m))$ are invariant under this equivalence relation

We are interested on the simultaneous diagonalizable m-tuples.

Definition 2. *The m-tuples of matrices $T = (X_1, \ldots, X_m)$ is simultaneously diagonalizable if and only if there exist an equivalent m-tuple formed by diagonal matrices.*

We will denote by $\mathfrak{D}$ be the manifold of m-tuples of n-order simultaneously diagonalizable real matrices.

Necessary conditions for simultaneous diagonalizable m-tuples are, (see [7]):

Proposition 3. *Let $T = (X_1, \ldots, X_m)$ be a simultaneous diagonalizable m-tuple. Then all matrices X_i must be diagonalizable. (The reciprocal is false).*

Proposition 4. *Let $T = (X_1, \ldots, X_m)$ be a simultaneous diagonalizable m-tuple. Then $X_iX_j = X_jX_i$.*

Theorem 5. *Let $T = (X_1, \ldots, X_m)$ be a m-tuple of commuting n-order square matrices and suppose that the matrix X_i for some i is diagonalizable with simple eigenvalues ($\lambda_j \neq \lambda_k$ for all $j \neq k$, $k, j = 1, \ldots n$). Then T is a m-tuple of simultaneously diagonalizable matrices*

Theorem 6. *Let $T = (X_1, \ldots, X_m)$ be a m-tuple of commuting and diagonalizable n-order square matrices. Then, they diagonalize simultaneously.*

Theorem 7. *Let $T = (X_1, \ldots, X_m)$ be a m-tuple of n-order square matrices and suppose that all matrices X_i are diagonalizable, then a necessary and sufficient condition for simultaneous diagonalization is there exist a basis $\{v_1, \ldots, v_n\}$ of $v \in \mathbb{C}^n$ such that*

$$v_j \in \cap_{i=1}^m \mathrm{Ker}\,(X_i - \lambda_i^j),$$

where $\lambda_j^i \in Spec\, X_i = \{\lambda_1^i, \ldots, \lambda_n^i\}$.

Taking $P = \begin{pmatrix} v_1^t & \ldots & v_n^t \end{pmatrix}^{-1}$ we have

$$PX_iP^{-1} = D_i.$$

(See [7], for more information).

2.1 Fiber Bundles

Following Husmoller [9], fiber bundle is a structure (E, B, π, F), where E, B, and F are topological spaces called the total space, base space of the bundle, and the fiber respectively, and $\pi : E \to B$ is a continuous surjection called the bundle projection, satisfying the following local triviality condition: for every $x \in E$, there is an open neighborhood $U \subset B$ of $\pi(x)$ (called a trivializing neighborhood) such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ in such a way that the following diagram should commute:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\pi_1"'] \\ U & \end{tikzcd}$$

where $\pi_1 : U \times F \to U$ is the natural projection and $\varphi : \pi^{-1}(U) \to U \times F$ is a homeomorphism. The set of all $\{(U_i, \varphi_i)\}$ is called a local trivialization of the bundle.

Thus for any $p \in B$, $\pi^{-1}(\{p\})$ is homeomorphic to F and is called the fiber over p.

A trivial example of bundle is the one given by

$$(B \times F, \pi, B, F)$$

where $\pi : B \times F \longrightarrow B$ is the projection on the first factor, in this case the fibers are $\{p\} \times F$ for all $p \in B$.

A fiber bundle (E', B', π', F'), is a subbundle of (E, B, π, F) provided E' is a subspace of E, B' is a subspace of B, and π' is the restriction of π to E', $\pi' = \pi_{E'} : E' \longrightarrow B'$,

In the special case where the fiber is a group G, the fiber bundle is called principal bundle. In this case any fiber $\pi^{-1}(b)$ is a space isomorphic to G. More specifically, G acts freely without fixed point on the fibers.

In the case where E, B and F are smooth manifolds and all the functions above are required to be smooth maps, the fiber bundle is called a smooth fiber bundle.

It is possible to induce bu

Let $\pi : E \longrightarrow B$ be a fiber bundle with fiber F and let $f : B' \longrightarrow B$ be a continuous map. We can deduce a fiber bundle over E' in the following manner

$$f^*E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\} \subseteq B' \times E$$

and equip it with the subspace topology and the projection map $\pi' : f^*E \longrightarrow B'$ defined as the projection onto the first factor:

$$\pi'(b', e) = b'$$

Defining f' so that the following diagram is commutative

$$\begin{array}{ccc} f^*E & \xrightarrow{f'} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

we have that (f^*E, B', π') is a fiber bundle so that the fibers on $b \in B$ correspond to the fibers on $f^{-1}(b)$.

We consider an important concept on fiber bundles that is the of cross section notion.

Definition 8. *A cross section of a bundle (E, B, π, F) is a map $s : B \longrightarrow E$ such that $\pi s = I_B$. In other words, a cross section is a map $s : B \longrightarrow E$ such that $s(b) \in \pi^{-l}(b)$, the fibre over b, for each $b \in B$.*

Let (E', B, π', F') be a subbundle of (E, B, π, F), and let s be a cross section of (E, B, π, F). Then s is a cross section of (E', B, π', F') if and only if $s(b) \in E'$ for each $b \in B$.

One the main goals studying cross sections is to account for the existence or non-existence of global sections. When there are some problem to construct a global section, one says that there are an obstruction.

3 Bundle of n-tuples of matrices given by a Lie group action

Let $\mathfrak{M}$ be the smooth manifold of m-tuples of n-order real matrices $T = (X_1, \dots, X_m)$.

The equivalence relation defined in 1, can be seen as the action of a Lie group $\mathfrak{G}$ over $\mathfrak{M}$ in the following manner:

Let us consider the following map

$$\begin{aligned} \alpha : \mathfrak{G} \times \mathfrak{M} &\longrightarrow \mathfrak{M} \\ (P, T) &\longrightarrow PTP^{-1} = (PX_1P^{-1}, \dots, PX_mP^{-1}) \end{aligned}$$

that verifies

i) If $I \in \mathfrak{G}$ is the identity element, then $\alpha(I, T) = T$ for all $T \in \mathfrak{M}$.

ii) If P_1 and P_2 are in $\mathfrak{G}$, then $\alpha(P_1, \alpha(P_2, T)) = \alpha(P_1P_2, T)$ for all $T \in \mathfrak{M}$.
$\alpha(P_1, \alpha(P_2, T)) = \alpha(P_1, P_2TP_2^{-1}) = P_1P_2TP_2^{-1}P_1^{-1} = (P_1P_2)T(P_1P_2)^{-1} = \alpha(P_1P_2, T)$

So, the map α defines an action of $\mathfrak{G}$ over $\mathfrak{M}$.

Analogously we can define an action of $\mathfrak{G}$ over $\mathfrak{G} \times \mathfrak{M}$ in the following manner:

$$\begin{aligned} \beta : \mathfrak{G} \times (\mathfrak{G} \times \mathfrak{M}) &\longrightarrow \mathfrak{G} \times \mathfrak{M} \\ (Q, (P, T)) &\longrightarrow (PQ^{-1}, \alpha(Q^{-1}, T)). \end{aligned}$$

Proposition 9. *The $\mathfrak{G}$-action β is free, transitive and its orbits are diffeomorphic to $\mathfrak{G}$*

Proof. Suppose that $\beta(Q, (P, T)) = (P, T)$, so

$$\begin{aligned} \beta(Q, (P, T)) &= (PQ^{-1}, \alpha(Q^{-1}, T)) \\ &= (PQ^{-1}, Q^{-1}TQ) \\ &= (P, T) \end{aligned}$$

then, $PQ^{-1} = P$ and $Q^{-1}TQ = T$ and $Q = I$.

$$\begin{aligned} \beta(R, \beta(Q, (P, T)) &= \beta(R, (PQ^{-1}, \alpha(Q, T)) \\ &= \beta(R, (PQ^{-1}, QTQ^{-1})) \\ &= (PQ^{-1}R^{-1}, \alpha(R, QTQ^{-1})) \\ &= (PQ^{-1}R^{-1}, RQTQ^{-1}R^{-1}) \\ &= (P(RQ)^{-1}, \alpha(RQ, T)) \\ &= \beta(RQ, (P, T)), \end{aligned}$$

$$\mathcal{O}(P, T) = \{(\overline{P}, \overline{T}) = \beta(Q, (P, T)), \forall Q \in \mathfrak{G}\}$$

$$\begin{aligned} \varphi : \mathfrak{G} &\longrightarrow \mathcal{O}(P, T) \\ Q &\longrightarrow (\overline{P}, \overline{T}) = \beta(Q, (P, T)) \end{aligned}$$

φ is clearly a diffeomorphism:

If $\varphi(Q) = \varphi(\overline{Q})$, then $PQ = P\overline{Q}$ consequently $Q = \overline{Q}$

And, for $(\overline{P}, \overline{T}) \in \mathcal{O}(P, T)$, there exists $Q \in \mathfrak{G}$ with $(\overline{P}, \overline{T}) = (PQ^{-1}, QTQ^{-1})$, so $\varphi(Q) = (\overline{P}, \overline{T})$. □

Proposition 10. *The set $\mathfrak{M}$ is identified as the set of orbits class $\mathfrak{G} \times \mathfrak{M}/\beta$.*

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Proof. We define f as

$$\begin{aligned} \mathfrak{G} \times \mathfrak{M}/\beta &\longrightarrow \mathfrak{M} \\ (P, T) \circ \mathfrak{G} &\longrightarrow T' \end{aligned}$$

where T' is in such a way that there exist $Q \in \mathfrak{G}$ such that $\beta(Q, (P, T)) = (I, T')$

1) It suffices to take $Q = P$ to obtain $T' = P^{-1}TP$

2) f is well-defined because of unicity of T':
Let $(I, T') \sim (I, T'')$, then, there exist Q such that

$$\beta(Q, (I, T')) = (IQ^{-1}, \alpha(Q^{-1}, T')) = (I, T'')$$

So, $IQ^{-1} = I$ and $Q^{-1} = I = Q$ and $IQ^{-1}\alpha(Q^{-1}, T') = \alpha(I, T') = T'$.

3) f is bijective:

If $f((I,T')\circ\mathfrak{G} = f((I,T'')\circ\mathfrak{G}$, then $T' = T''$ and $f((I,T')\circ\mathfrak{G} = f((I,T'')\circ\mathfrak{G}$, so f is injective

And, clearly, for all $T\in\mathfrak{M}$, $f((I,T)\circ\mathfrak{G}) = T$ and f is surjective.

□

Proposition 11. *The $\mathfrak{G}$-action preserves the fibers $F_T = \alpha^{-1}(T)$ of $\alpha : \mathfrak{G}\times\mathfrak{M}\longrightarrow\mathfrak{M}$.*

Proof. Let $(P,\overline{T})\in\alpha^{-1}(T)$, then $\alpha(Q,(P,\overline{T})) = (PQ^{-1}, QTQ^{-1}) = PQ^{-1}Q\overline{T}Q^{-1}QP^{-1} = P\overline{T}P^{-1} = T$, then $(PQ^{-1}, QTQ^{-1})\in\alpha^{-1}(T)$. □

From propositions 9 and 11 we can deduce the following result.

Proposition 12. *$(\mathfrak{G}\times\mathfrak{M}, \mathfrak{M}, \alpha, \mathfrak{G})$ is a principal fiber bundle.*

Clearly, we observe that F_T is diffeomorphic to $\mathfrak{G}$:

$$\begin{aligned}\psi : F_T &\longrightarrow \mathfrak{G}\\ (Q,\overline{T}) &\longrightarrow Q\end{aligned}$$

If $\psi(Q,\overline{T}) = (\overline{Q},\overline{\overline{Q}})$, then $Q = \overline{Q}$ and $Q\overline{T}Q^{-1} = \overline{Q}\overline{\overline{T}}\overline{Q}^{-1} = Q\overline{\overline{T}}Q^{-1}$, so $\overline{T} = \overline{\overline{T}}$ the ψ is injective.

On the other hand, for all $Q\in\mathfrak{G}$, there exists $(Q, Q^{-1}TQ)\in F_T$ such that $\psi(Q,Q^{-1}TQ) = T$, the ψ is surjective.

3.1 Orbit space of a free, proper G-action principal bundle

We ask if $(\mathfrak{M},\pi,\mathfrak{M}/\mathfrak{G})$ determines a principal $\mathfrak{G}$-bundle.

For ensure that it is sufficient that the action of $\mathfrak{G}$ on $\mathfrak{M}$ to be free (this is obviously necessary) and proper

Remember that a group action $\alpha : \mathfrak{G}\times\mathfrak{M}\longrightarrow\mathfrak{M}$ is called free if, there exist $T\in\mathfrak{M}$ such that $\alpha(P,T) = T$ implies $P = I$.

In our particular setup the condition is written as $PX_iP^{-1} = X_i$, for $i = 1,\dots,m$, for some $T = (X_1,\dots,X_m)$

A function $f : X\longrightarrow Y$ between two topological spaces is proper if the preimage of every compact set in Y is compact in X.

3.2 Subbundle of n-tuples of simultaneously diagonalizable matrices

Let $T\in\mathfrak{M}$ be an n-tuple of simultaneous diagonalizable matrices, following definition 2 there exist Q such that $\alpha(Q,T) = D = (D_1,\dots,D_n)$ with D_i diagonal.

Proposition 13. *Let $T\in\mathfrak{M}$ be an n-tuple of simultaneous diagonalizable matrices. Then, all $\overline{T}\in\mathcal{O}(T)$ is simultaneous diagonalizable.*

Proof. $\overline{T} = Q_1TQ_1^{-1} = Q_1Q^{-1}DQQ_1^{-1} = Q_2DQ_2^{-1}$ □

Corollary 14. $F_D\subset\mathfrak{G}\times\mathfrak{D}$

So, defining

$$\begin{aligned}\alpha_1 : \mathfrak{G}\times\mathfrak{D} &\longrightarrow \mathfrak{D}\\ (Q,D) &\longrightarrow \alpha(Q,D)\end{aligned}$$

we have that

Proposition 15. *$(\mathfrak{G}\times\mathfrak{D}, \mathfrak{M}, \alpha_1, \mathfrak{G})$ is a principal subbundle of $(\mathfrak{G}\times\mathfrak{M}, \mathfrak{M}, \alpha, \mathfrak{G})$.*

4 Restriction of group structure

Let $\mathfrak{C}$ be the manifold of m-tuples of distinct points of $\mathbf{R}^n$:

$$\begin{aligned}\mathfrak{C} = \{(x_1,\dots,x_m)\in\mathbb{R}^n \quad &\times\dots^m\times\mathbb{R}^n \mid\\ &x_i\neq x_j,\ \forall i\neq j\}.\end{aligned}$$

The group of permutations $\mathfrak{S}_n$ acts over $\mathfrak{C}$ reordering in the same manner the elements of each x_i.

$$\begin{aligned}\gamma : \mathfrak{S}_n\times\mathfrak{C} &\longrightarrow \mathfrak{C}\\ (\sigma,(x_1,\dots x_m)) &\longrightarrow (\sigma(x_1),\dots,\sigma(x_m))\end{aligned}$$

Example 1 Suppose $n = 3$ and $m = 2$ and $\sigma = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$. Then

$$\begin{aligned}&\gamma(\sigma,((\lambda_1^1,\lambda_1^2,\lambda_1^3),(\lambda_2^1,\lambda_2^2,\lambda_2^3)) =\\ &\qquad((\lambda_1^3,\lambda_1^1,\lambda_1^2),(\lambda_2^3,\lambda_2^1,\lambda_2^2))\end{aligned}$$

The set $\mathfrak{C}$ can be identified with the set of m-tuples of diagonal matrices with distinct n-tuples of eigenvectors in the following manner:

$$\begin{aligned}\psi : \mathfrak{C} &\longrightarrow \mathfrak{D}\subset\mathfrak{M}\\ (x_1,\dots,x_m) &\longrightarrow (\operatorname{diag} x_1,\dots,\operatorname{diag} x_m)\end{aligned}$$

where $\operatorname{diag} x_i = \begin{pmatrix} x_i^1 & & \\ & \ddots & \\ & & x_i^n \end{pmatrix}$, for $1 \leq i \leq m$.
We will denote by $\mathfrak{C}_{\mathfrak{D}} = \psi(\mathfrak{C})$ this set of n-tuples of diagonal matrices.

The natural representation of a permutation in a matrix form permit us to conclude that the map ψ preserves the equivalence relation. So, we have the following commutative diagram.

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\psi"] \arrow[d, "\pi_S"'] & \mathfrak{D} \arrow[d, "\pi_{\mathfrak{G}}"] \\ \mathfrak{C}/\mathfrak{S}_n \arrow[r, "\widetilde{\varphi}"] & \mathfrak{D}/\mathfrak{G} \end{tikzcd}$$

Let $\mathfrak{X} = \{((x_1, \ldots, x_m), (P, T)) \in \mathfrak{C} \times (\mathfrak{G} \times \mathfrak{D}) \mid \psi(x_1, \ldots, x_m) = \alpha_1(P, T)\}$

The subgroup $\mathfrak{S}_n$ acts over $\mathfrak{X}$ in the following manner

$$\begin{array}{rl} \bar{\alpha} : \mathfrak{S}_n \times \mathfrak{X} & \longrightarrow \mathfrak{X} \\ (\sigma, \mathcal{X}) & \longrightarrow \mathcal{Y} \end{array}$$

where

$$\begin{array}{ll} \mathcal{X} & = (((x_1, \ldots, x_m), (P, T))) \\ \mathcal{Y} & = ((\sigma(x_1), \ldots, \sigma(x_m)), (PP_\sigma^{-1}, P_\sigma T P_\sigma^{-1}) \end{array}$$

and P_σ is the permutation matrix associated to σ.

From this action we can induce a $\mathfrak{S}_n$-bundle over $\mathfrak{C}$

$$\begin{tikzcd} & \mathfrak{X} \arrow[r] \arrow[dl] \arrow[d, "\pi_{\mathfrak{C}}"] & \mathfrak{G} \times \mathfrak{D} \arrow[d, "\alpha_1"] \\ \mathfrak{X}/\mathfrak{S}_n \arrow[r, "g"] & \mathfrak{C} \arrow[r, "\psi"] & \mathfrak{D} \end{tikzcd}$$

where $g(((x_1, \ldots, x_m), (P, T))\mathfrak{S}_n) = (x_1, \ldots, x_m)$ and $\pi_{\mathfrak{C}}((x_1, \ldots, x_m), (P, T)) = (x_1, \ldots, x_m)$

and

$$\begin{tikzcd} & \mathfrak{X} \arrow[r] \arrow[dl] \arrow[d, "\pi_{\mathfrak{C}}"] & \mathfrak{G} \times \mathfrak{D} \arrow[d, "\alpha_1"] \\ \mathfrak{X}/\mathfrak{S}_n \arrow[r, "g"] \arrow[dr] & \mathfrak{C} \arrow[r, "\psi"] \arrow[d, "\pi_{\mathfrak{S}_n}"] & \mathfrak{D} \arrow[d, "\pi_{\mathfrak{G}}"] \\ & \mathfrak{C}/\mathfrak{S}_n \arrow[r, "\widetilde{\varphi}"] & \mathfrak{D}/\mathfrak{G} \end{tikzcd}$$

5 Characteristic classes

Characteristic classes are global invariants that measure the deviation of the local product structure from a global product structure.

The theory of characteristic classes generalizes the idea of obstructions to construct cross sections of fiber bundles.

Definition 16. *Let $\mathfrak{G}$ be a topological group and $p : E \longrightarrow X$ be a $\mathfrak{G}$-principal bundle. Let h^* be a cohomology theory on topological spaces. A characteristic class is an element*

$$x(p) := h^*(f_p)(x) \in h^*(X)$$

where $x \in h^*(BG)$.

5.1 Invariant polynomials

Consider the m-tuple of matrices $T = (X_1, \ldots, X_m) = ((X_{ij}^1, \ldots, (X_{ij}^m) \in \mathfrak{M}$, we will denote by $\mathcal{P}(X)$ a polynomial of $m \cdot n^2$ variables, these will be considered as homogeneous polynomials, that is, it is formed by monomials of the same degree, and that the degree of this monomial would be the degree of the polynomial $\mathcal{P}(X)$.

Definition 17. *Let $T \in \mathfrak{M}$, a polynomial $\mathcal{P}(T)$, it is called invariant if and only if $\mathcal{P}(T) = \mathcal{P}(PAP^{-1})$ for all $P \in GL(n; \mathbb{R})$.*

Every polynomial determines a function $\mathcal{P} : \mathfrak{M} \longrightarrow \mathbb{R}$, this function is only determined by said polynomial.

Given the m-tuple of matrices T, we can associate the following polynomial:

$$\sigma(t) = \prod_{j=1}^{m} \det(I + tX_j)$$

This polynomial is invariant:

Proof.

$$\begin{array}{l} \sigma(PTP^{-1}) = \prod_{j=1}^m \det(I + tPX_jP^{-1}) = \\ (\det P)^m (\det P^{-1})^m \prod_{j=1}^m \det(I + tX_j) = \sigma(T) \end{array}$$

□

Polynomial $\sigma(T)$ can be written in the following manner

$$\sigma(T) = \prod_{j=1}^{m} \sigma_i(T) = \prod_{j=1}^{m} \left(\Sigma_{i=0}^{n} \sigma_i^j(X_j) t^i \right)$$

with $\sigma_0(T) = \prod_{j=1}^m \det X_j$.

Proposition 18. *Each polynomial $\sigma_i(T)$ is an invariant polynomial.*

Proof. It suffices to note that $\sigma_i^j(X_j)$ is invariant. □

Let be now $T \in \mathfrak{D}$, taking into account the invariance of the characteristic polynomial we have that

$$\det(I + tX_j) = \prod_{k=1}^{n}(1 + t\lambda_k^j) = \sum_{i=0}^{nm} \sigma_i(\lambda_1^1, \dots, \lambda_n^1, \dots, \lambda_1^m, \dots, \lambda_n^m)t^i$$

Polynomials $\sigma_i(\lambda_1^1, \dots, \lambda_n^1, \dots, \lambda_1^m, \dots, \lambda_n^m$ are called elementary symmetric polynomials in the variables $\lambda_1^1, \dots, \lambda_n^m)$. These polynomials are invariant for product of permutations $(s_1(\lambda_1^1, \dots, \lambda_n^1), \dots, s_m(\lambda_1^m, \dots, \lambda_n^m))$.

Now let us consider $\mathcal{P}(T)$ be an invariant polynomial where X_j are n-matrices with elements x_{ik}^j in $\Omega^2(\mathfrak{M})$ the space of 2-differential forms in $\mathfrak{M}$, and since the outer product of forming of even order is commutative, then the polynomial $\mathcal{P}(T)$ in the variables x_{ik}^j, belongs to in $\Omega^{2nm}(\mathfrak{M})$ defining $\mathcal{P}(F^\triangledown)$ a characteristic classe where $F^\triangledown$ is curvature shape of the variety.

6 Conclusion

In this paper, a review of the simultaneous diagonalization of n-tuples of matrices for its applications in sciences has been presented. This kind of family of linear operators has been analyzed using geometric constructions such as the principal bundles and associating them with a cohomology class defined from invariant polynomials that permit measure the deviation of the local product structure from the global product structure.

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