$\beta*-$Compactness and $\beta*-$Connectedness

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Abstract: - In 2014 Mubarki, Al-Rshudi, and Al-Juhani introduced and studied the notion of a set in general topology called $\beta*-$open sets and investigated its fundamental properties and studied the relationships between $\beta*-$open set and other topological sets including $\beta*-$continuity in topological spaces. The objective of this paper is to introduce the new concepts called $\beta*-$compact space, countably $\beta*-$compact space, $\beta*-$Lindelof space, almost $\beta*-$compact space, mildly $\beta*-$compact space and $\beta*-$connected space in general topology and investigate several properties and characterizations of these new concepts in topological spaces.

Key-Words: - Topological space, generalized open set, $\beta*-$open set, $\beta*-$compact space, countably $\beta*-$compact space, $\beta*-$Lindelof space, almost $\beta*-$compact space, mildly $\beta*-$compact space, $\beta*-$connected space.

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1 Introduction

The concept of supra topology was introduced by A. S. Mashhour et al [13] in the year 1983. They studied about $s-$continuous functions and $s*-$continuous functions. In 2008, R. Devi et al [5] introduced the concept of supra $\alpha-$open sets and supra $\alpha-$continuous maps. Jamal. M. Mustafa [16] studied about supra $b-$compact and supra $b-$Lindelof spaces. Vidyarani et al in [30] introduced the concept of supra $N-$compact, countably supra $N-$compact, supra $N-$Lindelof and supra $N-$connectedness and investigated about their relationships using the concept of continuity. In 2013, Missier and Rodrigo [14] introduced new class of set in general topology called an $\alpha-$open (supra $\alpha-$open) set. In 2014 Mubarki, Al-Rshudi, and Al-Juhani [15] introduced and studied the notion of set in general topology called $\beta*-$open sets and investigated its fundamental properties and studied the relationship between $\beta*-$open set and other topological sets including $\beta*-$continuity in topological spaces. The objective of this paper is to introduce the new concepts called $\beta*-$compact space, countably $\beta*-$compact space, $\beta*-$Lindelof space, almost $\beta*-$compact space, mildly $\beta*-$compact space and $\beta*-$connected space in general topology and investigate several properties and characterizations of these new concepts in topological spaces.

Throughout this paper (X,τ) or simply by X we denote topological space on which no separation axioms are assumed unless explicitly stated and $f:(X,\tau)\rightarrow(Y,\sigma)$ means a mapping f from a topological space X to a topological space Y. If U is a set and x is a point in X, then $N(x)$, $Int(U)$, $Cl(U)$ and U^c denote respectively, the neighbourhood system of x, the interior of U, the closure of U and complement of U.

2 Preliminaries

Definition 2.1. A subset A of a topological space X is called semi$-$open set if $A\subseteq Cl[Int(A)]$.

Definition 2.2. A subset A of a topological space X is called $\alpha-$open set if $A\subseteq Int[Cl(Int(A))]$.

Definition 2.3. A subset A of a topological space X is called $\beta-$open set if $A\subseteq Cl\left[Int\left(Cl\left(A\right)\right)\right]$.

Definition 2.4. A subset A of a topological space X is called pre$-$open set if $A\subseteq Int\left[Cl\left(A\right)\right]$.

Definition 2.5. A subset A of a topological space X is said to be b$-$open set if $A\subseteq Cl\left[Int\left(A\right)\right]\cup Int\left[Cl\left(A\right)\right]$.

Definition 2.6. Let $\left(X,\tau\right)$ be a topological space. Then a point $x\in X$ is called the $\delta-$cluster point of $A\subseteq X$ if $A\cap Int\left[Cl\left(U\right)\right]\neq\phi$ for every open set U of X containing x. The set of all cluster points of A is called the $\delta-$cluster points of A, denoted by $Cl_{\delta}\left(A\right)$. A subset $A\subseteq X$ is called $\delta-$closed if $A=Cl_{\delta}\left(A\right)$.

Definition 2.7. Let $\left(X,\tau\right)$ be a topological space and $A\subseteq X$. Then A is called $\delta-$open set if its compliment $X-A$ is $\delta-$closed in X. The collection of all $\delta-$open sets in a topological space $\left(X,\tau\right)$ forms a topology τ_{δ} on X, weaker than τ and the class of all regular open sets in τ forms an open basis for τ_{δ}.

Definition 2.8. A subset A of a topological space X is called e$*-$open set if $A\subseteq Cl\left[Int\left(Cl_{\delta}\left(A\right)\right)\right]$.

Definition 2.9. Let $\left(X,\tau\right)$ be a topological space. Then a subset A of X is said to be $\beta*-$open if $A\subseteq Cl\left[Int\left(Cl\left(A\right)\right)\right]\cup Int\left[Cl_{\delta}\left(A\right)\right]$. The family of all $\beta*-$open subsets of a topological space $\left(X,\tau\right)$ will be as always denoted by $\beta*O\left(X\right)$.

Definition 2.10. A subset A of a topological space $\left(X,\tau\right)$ is said to be a $\beta*-$closed set if $Int\left[Cl\left(Int\left(A\right)\right)\right]\cap Cl\left[Int_{\delta}\left(A\right)\right]\subseteq A$.

The family of all $\beta*-$closed subsets of a topological space $\left(X,\tau\right)$ will be as denoted by $\beta^{*}C\left(X\right)$.

Remark 2.11. The following diagram holds for each a subset A of X.

open set $\rightarrow\alpha-$open set $\rightarrow$ preopen set $\rightarrow$ b$-$open set $\rightarrow\beta-$open set $\rightarrow\beta*-$open set $\rightarrow$ e$*-$open set

Theorem 2.12. Let $\left(X,\tau\right)$ be a topological space. Then the following assertions hold:

(1) The arbitrary union of $\beta*-$open sets is $\beta*-$open.

(2) The arbitrary intersections of $\beta*-$closed is $\beta*-$closed.

Proof. (1) Let $\left\{A_i:i\in I\right\}$ be a family of $\beta*-$open sets. Then $A_i\subseteq Cl\left[Int\left(Cl\left(A_i\right)\right)\right]\cup Int\left[Cl_{\delta}\left(A_i\right)\right]$ and therefore immediately it follows that $\bigcup_{i\in I}A_i\subseteq\bigcup_{i\in I}\left(Cl\left[Int\left(Cl\left(A_i\right)\right)\right]\cup Int\left[Cl_{\delta}\left(A_i\right)\right]\right)\subseteq$ $Cl\left[Int\left(Cl\left(\bigcup_{i\in I}A_i\right)\right)\right]\cup Int\left[Cl_{\delta}\left(\bigcup_{i\in I}A_i\right)\right]$, for all $i\in I$. Thus $\bigcup_{i\in I}A_i$ is $\beta*-$open.

(2) It follows from (1).

Remark 2.13. The next example shows that the intersection of any two $\beta*-$open sets is not $\beta*-$open.

Example 2.14. Let $X=\left\{1,2,3\right\}$ with topology $\tau=\left\{\phi,\left\{1\right\},\left\{2\right\},\left\{1,2\right\},X\right\}$. Then $A=\left\{1,3\right\}$ and $B=\left\{2,3\right\}$ are $\beta^{*}-$open sets. But $A\cap B=\left\{3\right\}$ is not $\beta*-$open.

Definition 2.15. Let $\left(X,\tau\right)$ be a topological space. Then:

(1) The union of all $\beta*-$open sets of X contained in A is called the $\beta*-$interior of A and is denoted by $\beta*-Int\left(A\right)$.

(2) The intersection of all $\beta*-$closed sets of X containing A is called the $\beta*-$closure of A and is denoted by $\beta*-Cl\left(A\right)$.

Theorem 2.16. Let A, B be two subsets of a topological space $\left(X,\tau\right)$. Then the following assertions are true:

(1) $\beta^*\text{-}Cl(X)=X$ and $\beta^*\text{-}Cl(\phi)=\phi.$

(2) $A\subseteq \beta^*\text{-}Cl(A).$

(3) If $A\subseteq B,$ then $\beta^*\text{-}Cl(A)\subseteq \beta^*\text{-}Cl(B).$

(4) $x\in \beta^*\text{-}Cl(A)$ if and only if for each a β^*-open set U containing $x,$ $U\cap A\neq \phi.$

(5) A is β^*-closed set if and only if $A=\beta^*\text{-}Cl(A).$

(6) $\beta^*\text{-}Cl[\beta^*\text{-}Cl(A)]=\beta^*\text{-}Cl(A).$

(7) $\beta^*\text{-}Cl(A)\cup \beta^*\text{-}Cl(B)\subseteq \beta^*\text{-}Cl(A\cup B).$

(8) $\beta^*\text{-}Cl(A\cap B)\subseteq \beta^*\text{-}Cl(A)\cap \beta^*\text{-}Cl(B).$

Theorem 2.17. Let $A,$ B be two subsets of a topological space $(X,\tau).$ Then the following assertions are true:

(1) $\beta^*\text{-}Int(X)=X$ and $\beta^*\text{-}Int(\phi)=\phi.$

(2) $\beta^*\text{-}Int(A)\subseteq A.$

(3) If $A\subseteq B,$ then $\beta^*\text{-}Int(A)\subseteq \beta^*\text{-}Int(B).$

(4) $x\in \beta^*\text{-}Int(A)$ if and only if there exists β^*-open set W such that $x\in W\subseteq A.$

(5) A is β^*-open set if and only if $A=\beta^*\text{-}Int(A).$

(6) $\beta^*\text{-}Int[\beta^*\text{-}Int(A)]=\beta^*\text{-}Int(A).$

(7) $\beta^*\text{-}Int(A\cap B)\subseteq \beta^*\text{-}Int(A)\cap \beta^*\text{-}Int(B).$

(8) $\beta^*\text{-}Int(A)\cup \beta^*\text{-}Int(B)\subseteq \beta^*\text{-}Int(A\cup B).$

Definition 2.18. Let X be a non-empty set. The subfamily $\mu\subseteq P(X)$ is said to be a supra topology on X if $\phi, X\in\mu$ and μ is closed under arbitrary unions. The pair (X,μ) is called a supra topological space. The elements of μ are said to be supra open in $(X,\mu).$ Complement of supra open sets are called supra closed sets.

Definition 2.19. A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be a β^*-continuous if $f^{-1}(V)$ is a β^*-open $(\beta^*\text{-}closed)$ set in X for each open (closed) set V in Y.

Definition 2.20. A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be a β^*-irresolute if $f^{-1}(V)$ is a β^*-open $(\beta^*\text{-closed})$ set in X for X each β^*-open $(\beta^*\text{-closed})$ set V in Y.

Definition 2.21. A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be a β^*-open $(\beta^*\text{-closed})$ if $f(U)$ is a β^*-open $(\beta^*\text{-}closed)$ set in Y for each open (closed) set U in X.

Definition 2.22. A set $A\subseteq X$ is said to be β^*-connected if A cannot be written as the union of two β^*-separated sets.

Definition 2.23. Let X be any nonempty set and $\tau\subseteq P(X)$. We say that τ is a supra topology on X if $\phi, X\in\tau$ and τ is closed under arbitrary union. The pair (X,τ) is called supra topological space. The elements of τ are called supra open sets in (X,τ) and complement of a supra open set is called a supra closed set.

Definition 2.24. A supra topological space is called supra compact $(\text{S}-\text{compact})$ if and only if every supra open cover of X has a finite sub cover.

Definition 2.25. A function $f:(X,\tau)\to(Y,\sigma)$ is called perfectly β^*-continuous if the inverse image $f^{-1}(V)$ of every β^*-open set V of Y is both open and closed in X.

Definition 2.26. A function $f:(X,\tau)\to(Y,\sigma)$ is called strongly β^*-continuous if the inverse image $f^{-1}(V)$ of every β^*-open V in Y is open in X.

Definition 2.27. A function $f:(X,\tau)\to(Y,\sigma)$ is called β^*-irresolute if the inverse image $f^{-1}(V)$ of every β^*-open V in Y is β^*-open in X.

3 β*–Compact Spaces

Definition 3.1. A collection $\{A_i : i \in I\}$ of β^*–open sets in a topological space (X, τ) is called a β^*–open cover of a subset B of X if $B \subseteq \cup\{A_i : i \in I\}$ holds.

Definition 3.2. A topological space (X, τ) is called β^*–compact if every β^*–open cover of X has a finite sub cover.

Definition 3.3. A subset B of a topological space (X, τ) is said to be β^*–compact relative to (X, τ) if, for every collection $\{A_i : i \in I\}$ of β^*–open subsets of X such that $B \subseteq \cup\{A_i : i \in I\}$ there exists a finite subset I_0 of I such that $B \subseteq \cup\{A_i : i \in I_0\}$.

Definition 3.4. A subset B of a topological space (X, τ) is said to be β^*–compact if B is β^*–compact as a subspace of X.

Theorem 3.5. Every β^*–compact space is compact.

Proof. Let $\{A_i : i \in I\}$ be an open cover of (X, τ). Since every open set in X is β^*–open in X. So $\{A_i : i \in I\}$ is a β^*–open cover of (X, τ). Since (X, τ) is β^*–compact, β^*–open cover $\{A_i : i \in I\}$ of (X, τ) has a finite sub cover say $\{A_i : i = 1,2,3,...,n\}$ for X. Hence (X, τ) is a compact space.

Theorem 3.6. Every β^*–closed subset of a β^*–compact space (X, τ) is β^*–compact, relative to X.

Proof. Let A be a β^*–closed closed subset of a topological space (X, τ). Then A^c is β^*–open in (X, τ). Let $\Gamma = \{A_i : i \in I\}$ be a β^*–open cover of A by β^*–open subsets of (X, τ). Then $\Gamma^* = \Gamma \cup \{A^c\}$ is a β^*–open cover of (X, τ). That is $X = \left(\bigcup_{i \in I} A_i\right) \cup A^c$. By hypothesis (X, τ) is a β^*–compact space and hence Γ^* is reducible to a finite sub cover of (X, τ) say $X = \left(\bigcup_{i \in I_0} A_i\right) \cup A^c$ for some finite subset I_0 of I. But A and A^c are disjoint. Hence $A \subseteq \cup\{A_i : i \in I_0\}$. Thus β^*–open cover $\Gamma = \{A_i : i \in I\}$ of A contains a finite sub cover. Hence A is β^*–compact relative to (X, τ).

Theorem 3.7. A β^*–continuous image of a β^*–compact space is compact.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a β^*–continuous map from a β^*–compact (X, τ) onto a topological space (Y, σ). Let $\Gamma = \{A_i : i \in I\}$ be an open cover of Y. Therefore $f^{-1}(\Gamma) = \{f^{-1}(A_i) : i \in I\}$ is a β^*–open cover of X, as f is β^*–continuous. Since X is β^*–compact, the β^*–open cover $f^{-1}(\Gamma) = \{f^{-1}(A_i) : i \in I\}$ of X, has a finite sub cover say $\{f^{-1}(A_i) : i = 1,2,3,...,n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$, which implies $Y = f(X) = \bigcup_{i=1}^{n} A_i$. That is $\{A_i : i = 1,2,3,...,n\}$ is a finite sub cover of $\Gamma = \{A_i : i \in I\}$. Hence (Y, σ) is compact.

Theorem 3.8. Suppose that a function $f : (X, \tau) \to (Y, \sigma)$ is β^*–irresolute and a subset S of X is β^*–compact relative to (X, τ), then the image $f(S)$ is β^*–compact relative to (Y, σ).

Proof. Let $\Gamma = \{A_i : i \in I\}$ be a collection of β^*–open cover of (Y, σ), such that $f(S) \subseteq \cup\{A_i : i \in I\}$. Since f is β^*–irresolute. So $S \subseteq \cup\{f^{-1}(A_i) : i \in I\}$, where $\{f^{-1}(A_i) : i \in I\} \subseteq \beta^*\text{–}O(X, \tau)$. Since S is β^*–compact relative to (X, τ), there exists a finite sub collection $\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}$ such that $S \subseteq \cup\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}$. That is $f(S) \subseteq \cup\{A_1, A_2, ..., A_n\}$. Hence $f(S)$ is β^*–compact relative to (Y, σ).

Theorem 3.9. Suppose that a map $f : (X, \tau) \to (Y, \sigma)$ is strongly β^*–continuous map from a compact space (X, τ) onto a topological space (Y, σ), then (Y, σ) is β^*–compact.

Proof. Let $\{A_i : i \in I\}$ be a β^*–open cover of (Y, σ). Since f is strongly β^*–continuous, $\{f^{-1}(A_i) : i \in I\}$ is an open cover of (X, τ). Again, since (X, τ) is compact, the open cover

$\{f^{-1}(A_i) : i \in I\}$ of (X,τ) has a finite sub cover say $\{f^{-1}(A_i) : i = 1,2,3,...,n\}$. Therefore $X = \cup\{f^{-1}(A_i) : i = 1,2,3,...,n\}$, which implies $f(X) = \cup\{A_i : i = 1,2,3,...,n\}$, so that $Y = \cup\{A_i : i = i = 1,2,3,...,n\}$. That is $\{A_1, A_2, ..., A_n\}$ is a finite sub cover of $\{A_i : i \in I\}$ for (Y,σ). Hence (Y,σ) is β^*–compact.

Theorem 3.10. Suppose that a map $f:(X,\tau)\to(Y,\sigma)$ is perfectly β^*–continuous map from a compact space (X,τ) onto a topological space (Y,σ). Then (Y,σ) is β^*–compact.

Proof. Let $\{A_i : i \in I\}$ be a β^*–open cover of (Y,σ). Since f is perfectly β^*–continuous, $\{f^{-1}(A_i) : i \in I\}$ is an open cover of (X,τ). Again, since (X,τ) is compact, the open cover $\{f^{-1}(A_i) : i \in I\}$ of (X,τ) has a finite sub cover say $\{f^{-1}(A_i) : i = 1,2,3,...,n\}$. Therefore $X = \cup\{f^{-1}(A_i) : i = 1,2,3,...,n\}$, which implies $f(X) = \cup\{A_i : i = 1,2,3,...,n\}$, so that $Y = \cup\{A_i : i = 1,2,3,...,n\}$. That is $\{A_1, A_2, \ldots, A_n\}$ is a finite sub cover of $\{A_i : i \in I\}$ for (Y,σ). Hence (Y,σ) is β^*–compact.

Theorem 3.11. Suppose that a function $f:(X,\tau)\to(Y,\sigma)$ is β^*–irresolute map from a β^*–compact space (X,τ) onto a topological space (Y,σ). Then (Y,σ) is β^*–compact.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a β^*–irresolute map from a β^*–compact space (X,τ) onto a topological space (Y,σ). Let $\{A_i : i \in I\}$ be a β^*–open cover of (Y,σ). Then $\{f^{-1}(A_i) : i \in I\}$ is a β^*–open cover of (X,τ), since $\boldsymbol{f}$ is β^*–irresolute. As (X,τ) is β^*–compact, the β^*–open cover $\{f^{-1}(A_i) : i \in I\}$ of (X,τ) has a finite sub cover say $\{f^{-1}(A_i) : i = 1,2,3,...,n\}$. Therefore $X = \cup\{f^{-1}(A_i) : i = 1,2,3,...,n\}$, which implies $f(X) = \cup\{A_i : i = 1,2,3,...,n\}$, so that $Y = \cup\{A_i : i = 1,2,3,...,n\}$. That is $\{A_1, A_2, \ldots, A_n\}$ is a finite sub cover of $\{A_i : i \in I\}$ for (Y,σ). Hence (Y,σ) is β^*–compact.

Theorem 3.12. If (X,τ) is compact and every β^*–closed set in X is also closed in X, then (X,τ) is β^*–compact.

Proof. Let $\{A_i : i \in I\}$ be a β^*–open cover of X. Since every β^*–closed set in X is also closed in X. Thus $\{X - A_i : i \in I\}$ is a closed cover of X and hence $\{A_i : i \in I\}$ is an open cover of X. Since (X,τ) is compact. So there exists a finite sub cover $\{A_i : i = 1,2,3,...,n\}$ of $\{A_i : i \in I\}$ such that $X = \cup\{A_i : i = 1,2,3,...,n\}$. Hence (X,τ) is β^*–compact.

Theorem 3.13. A topological space (X,τ) is β^*–compact if and only if every family of β^*–closed sets of (X,τ) having finite intersection property has a non empty intersection.

Proof. Suppose (X,τ) is β^*–compact. Let $\{A_i : i \in I\}$ be a family of β^*–closed sets with finite intersection property. Suppose $\bigcap_{i\in I} A_i = \phi$, then $X - \bigcap(\{A_i : i \in I\}) = X$. This implies $\bigcup\{(X - A_i) : i \in I\} = X$. Thus the cover $\{(X - A_i) : i \in I\}$ is a β^*–open cover of (X,τ). Then as (X,τ) is β^*–compact, the β^*–open cover $\{(X - A_i) : i \in I\}$ has a finite sub cover say $\{(X - A_i) : i = 1,2,3,...,n\}$. This implies that $X = \cup\{(X - A_i) : i = 1,2,3,...,n\}$ which implies $X = X - \cap\{A_i : i = 1,2,3,...,n\}$, which implies $X - X = X - [X - \cap\{A_i : i = 1,2,3,...,n\}]$, which implies $\phi = \cap\{A_i : i = 1,2,3,...,n\}$. This disproves the assumption. Hence $\cap\{A_i : i \in I\} \neq \phi$.

Conversely, suppose (X,τ) is not β^*–compact. Then there exits a β^*–open cover of (X,τ) say $\{G_i : i \in I\}$ having no finite sub cover. This implies that for any finite sub family $\{G_i : i = 1,2,3,...,n\}$ of $\{G_i : i \in I\}$, we have $\bigcup\{G_i : i = 1,2,3,...,n\} \neq X$, which implies $X - \left(\bigcup\{G_i : i = 1,2,3,...,n\}\right) \neq X - X$, therefore $\bigcap\{X - G_i : i = 1,2,3,...,n\} \neq \phi$. Then the family $\{X - G_i : i \in I\}$ of $\beta*-$closed sets has a finite intersection property. Also by assumption $\bigcap\{X - G_i : i \in I\} \neq \phi$ which implies $X - \left(\bigcup\{G_i : i \in I\}\right) \neq \phi$, so that $\bigcup\{G_i : i \in I\} \neq X$. This implies $\{G_i : i \in I\}$ is not a cover of (X, τ). This disproves the fact that $\{G_i : i \in I\}$ is a cover for (X, τ). Therefore a $\beta*-$open cover $\{G_i : i \in I\}$ of (X, τ) has a finite sub cover $\{G_i : i = 1,2,3,...,n\}$. Hence (X, τ) is $\beta*-$compact.

Theorem 3.14. Let A be a $\beta*-$compact set relative to a topological space X and B be a $\beta*-$closed subset of X. Then $A \cap B$ is $\beta*-$compact relative to X.

Proof. Let A be $\beta*-$compact relative to X. Let $\{A_i : i \in I\}$ be a cover of $A \cap B$ by $\beta*-$open sets in X. Then $\{A_i : i \in I\} \cup \{B^C\}$ is a cover of A by $\beta*-$open sets in X, but A is $\beta*-$compact relative to X, so there exists a finite subset $I_0 = \{i_1, i_2, i_3, \ldots, i_n\} \subseteq I$ such that $A \subseteq \left(\bigcup\{A_{i_k} : k = 1, 2, 3, \ldots, n\}\right) \cup B^C$. Then it follows that $A \cap B \subseteq \bigcup\{A_{i_k} \cap B : k = 1, 2, 3, \ldots, n\} \subseteq \bigcup\{A_{i_k} : k = 1,2,3,...,n\}$. Hence $A \cap B$ is $\beta*-$compact.

Theorem 3.15. Suppose that a function $f : (X, \tau) \to (Y, \sigma)$ is $\beta*-$irresolute and a subset B of X is $\beta*-$compact relative to X. Then $f(B)$ is $\beta*-$compact relative to Y.

Proof. Let $\{A_i : i \in I\}$ be a cover of $f(B)$ by $\beta*-$open subsets of Y. Since f is $\beta*-$irresolute. Then $\{f^{-1}(A_i) : i \in I\}$ is a cover of B by $\beta*-$open subsets of X. Since B is $\beta*-$compact relative to X, $\{f^{-1}(A_i) : i \in I\}$ has a finite sub cover say $\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}$ for B. Then it implies that $\{A_i : i = 1,2,3,...,n\}$ is a finite sub cover of $\{A_i : i \in I\}$ for $f(B)$. So $f(B)$ is $\beta*-$compact relative to Y.

Definition 3.16. Let (X, τ) be a topological space and let E be a subset of X. Let $\tau_E^{i\alpha} = \{A \cap E : A \in \beta*-O(X, \tau)\}$. Then $\left(E, \tau_E^{i\alpha}\right)$ is a supra topological space.

Theorem 3.17. Let (X, τ) be a topological space and let E be a subset of X. Then $\left(E, \tau_E^{i\alpha}\right)$ is supra compact if and only if for any $\beta*-$open cover Γ of E has a finite sub cover of E.

Proof. Suppose E is supra compact. Let $\Gamma \subseteq \beta*-O(X, \tau)$ such that $E \subseteq \bigcup\Gamma$. Let $\Gamma_E = \{A \cap E : A \in \Gamma\}$. Then $E = \bigcup\Gamma_E$ and $\Gamma_E \subseteq \tau_E^{i\alpha}$. By hypothesis there exists a finite subset $\Gamma_E^* = \{A_i \cap E : i = 1,2,3,...,n\} \subseteq \Gamma_E$ such that $E = \bigcup\Gamma_E^*$. Then $\Gamma^* = \{A_i : i = 1,2,3,...,n\} \subseteq \Gamma$ and $E \subseteq \bigcup\Gamma^*$.

Conversely, let $\Upsilon = \{A_i \cap E : i \in I\} \subseteq \tau_E^{i\alpha}$ such that $E = \bigcup\Upsilon$. Then $\Upsilon^* = \{A_i : i \in A\}$ is a $\beta*-$open covering of E. By hypothesis there exists $\Upsilon^{**} = \{A_i : i = 1,2,3,...,n\}$ a finite subset of Υ^* such that $E \subseteq \bigcup\Upsilon^{**}$. Then $\Upsilon^{\#} = \{A_i \cap E : i = 1,2,3,...,n\}$ is a finite subset of Υ such that $E = \bigcup\Upsilon^{\#}$. This proves that $\left(E, \tau_E^{i\alpha}\right)$ is supra compact.

4 Countably $\beta*-$Compact Spaces

In this section, we present the concept of countably $\beta*-$compactness and its properties.

Definition 4.1. A topological space (X, τ) is said to be countably $\beta*-$compact if every countable $\beta*-$open cover of X has a finite sub cover.

Theorem 4.2. If (X, τ) is a countably $\beta*-$compact space, then (X, τ) is countably compact.

Proof. Let (X, τ) be a countably $\beta*-$compact space. Let $\{A_i : i \in I\}$ be a countable open cover of (X, τ). Since $\tau \subseteq \beta*-O(X, \tau)$. So $\{A_i : i \in I\}$ is a countable $\beta*-$open cover of (X, τ). Since (X, τ) is countably $\beta*-$compact, therefore countable $\beta*-$open cover

$\{A_i : i \in I\}$ of (X, τ) has a finite sub cover say $\{A_i : i = 1, 2, 3, \ldots, n\}$ for X. Hence (X, τ) is a countably compact space.

Theorem 4.3. If (X, τ) is countably compact and every β^*–closed subset of X is closed in X, then (X, τ) is countably β^*–compact.

Proof. Let (X, τ) be a countably compact space. Let $\{A_i : i \in I\}$ be a countable β^*–open cover of (X, τ). Since every β^*–closed subset of X is closed in X. Thus every β^*–open set in X is open in X. Therefore $\{A_i : i \in I\}$ is a countable open cover of (X, τ). Since (X, τ) is countably compact, so countable open cover $\{A_i : i \in I\}$ of (X, τ) has a finite sub cover say $\{A_i : i = 1,2,3,\ldots,n\}$ for X. Hence (X, τ) is a countably β^*–compact space.

Theorem 4.4. Every β^*–compact space is countably β^*–compact.

Proof. Let (X, τ) be a β^*–compact space. Let $\{A_i : i \in I\}$ be a countable β^*–open cover of (X, τ). Since (X, τ) is β^*–compact, so β^*–open cover $\{A_i : i \in I\}$ of (X, τ) has a finite sub cover say $\{A_i : i = 1,2,3,\ldots,n\}$ for (X, τ). Hence (X, τ) is countably β^*–compact space.

Theorem 4.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a β^*–continuous onjective mapping. If X is countably β^*–compact space, then (Y, σ) is countably compact.

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be a β^*–continuous map from a countably β^*–compact space (X, τ) onto a topological space (Y, σ). Let $\{A_i : i \in I\}$ be a countable open cover of Y. Then $\{f^{-1}(A_i) : i \in I\}$ is a countable β^*–open cover of X, as f is β^*–continuous. Since X is countably β^*–compact, the countable β^*–open cover $\{f^{-1}(A_i) : i \in I\}$ of X has a finite sub cover say $\{f^{-1}(A_i) : i = 1,2,3,\ldots,n\}$. Therefore $X = \cup\{f^{-1}(A_i) : i = 1,2,3,\ldots,n\}$, which implies $Y = f(X) = \cup\{A_i : i = 1,2,3,\ldots,n\}$. That is $\{A_i : i = 1,2,3,\ldots,n\}$ is a finite sub cover of $\{A_i : i \in I\}$ for Y. Hence Y is countably compact.

Theorem 4.6. Suppose that a map $f : (X, \tau) \to (Y, \sigma)$ is perfectly β^*–continuous map from a countably compact space (X, τ) onto a topological space (Y, σ). Then (Y, σ) is countably β^*–compact.

Proof. Let $\{A_i : i \in I\}$ be a countable β^*–open cover of (Y, σ). Since f is perfectly β^*–continuous, $\{f^{-1}(A_i) : i \in I\}$ is a countable open cover of (Y, σ). Again, since (X, τ) is countably β^*–compact, the countable open cover $\{f^{-1}(A_i) : i \in I\}$ of (X, τ) has a finite sub cover say $\{f^{-1}(A_i) : i = 1,2,3,\ldots,n\}$. Therefore $X = \cup\{f^{-1}(A_i) : i = 1,2,3,\ldots,n\}$, which implies $f(X) = \cup\{A_i : i = 1,2,3,\ldots,n\}$, so that $Y = \cup\{A_i : i = 1,2,3,\ldots,n\}$. That is $\{A_1, A_2, \ldots, A_n\}$ is a finite sub cover of $\{A_i : i \in I\}$ for (Y, σ). Hence (Y, σ) is countably β^*–compact.

Theorem 4.7. Suppose that a map $f : (X, \tau) \to (Y, \sigma)$ is strongly β^*–continuous map from a countably compact space (X, τ) onto a topological space (Y, σ). Then (Y, σ) is countably β^*–compact.

Proof. Let $\{A_i : i \in I\}$ be a countable β^*–open cover of (Y, σ). Since f is strongly β^*–continuous, $\{f^{-1}(A_i) : i \in I\}$ is a countable open cover of (X, τ). Again, since (X, τ) is countably compact, the countable supra open cover $\{f^{-1}(A_i) : i \in I\}$ of (X, τ) has a finite sub cover say $\{f^{-1}(A_i) : i = 1,2,3,\ldots,n\}$. Therefore $X = \cup\{f^{-1}(A_i) : i = 1,2,3,\ldots,n\}$, which implies $f(X) = \cup\{A_i : i = 1,2,3,\ldots,n\}$, so that $Y = \cup\{A_i : i = 1,2,3,\ldots,n\}$. That is $\{A_1, A_2, \ldots, A_n\}$ is a finite sub cover of $\{A_i : i \in I\}$ for (Y, σ) Hence (Y, σ) is countably β^*–compact.

Theorem 4.8. The image of a countably β^*–compact space under a β^*–irresolute map is countably β^*–compact.

Proof. Suppose that a map $f:(X,\tau)\to(Y,\sigma)$ is β^*-irresolute from a countably β^*-compact space (X,τ) onto a topological space (Y,σ). Let $\{A_i : i\in I\}$ be a countable β^*-open cover of (Y,σ). Then $\{f^{-1}(A_i) : i\in I\}$ is a countable β^*-open cover of (X,τ), since f is β^*-irresolute. As (X,τ) is countably β^*-compact, the countable β^*-open cover $\{f^{-1}(A_i) : i\in I\}$ of (X,τ) has a finite sub cover say $\{f^{-1}(A_i) : i=1,2,3,\ldots,n\}$. Then it follows that $X=\cup\{f^{-1}(A_i) : i=1,2,3,\ldots,n\}$, which implies $f(X)=\cup\{A_i : i=1,2,3,\ldots,n\}$, so that $Y=\cup\{A_i : i=1,2,3,\ldots,n\}$. That is $\{A_1,A_2,\ldots,A_n\}$ is a finite sub cover of $\{A_i : i\in I\}$ for (Y,σ). Hence (Y,σ) is countably β^*-compact.

Definition 4.9. Let (X,τ) be a topological space and $x\in X$. A point x is said to be β^*-limit point of $A\subseteq X$ provided that every β^*-neighbourhood of x contains at least one point of A different from x.

Theorem 4.10. Every infinite subset of a β^*-compact space has a β^*-limit point.

Proof. Let A be an infinite subset of a β^*-compact space (X,τ). Assume A does not have a β^*-limit point. Then for each $x\in X$, there exists a β^*-open set G_x containing at most one point of A. Now, the collection $\Lambda=\{G_x : x\in X\}$ forms a β^*-open cover of X. Since X is β^*-compact, therefore there exist $x_1,x_2,\ldots,x_n\in X$ such that $X=\bigcup_{i=1}^{i=n}G_{x_i}$. Therefore X has at most n points of A. This implies that A is finite. But this contradicts that A is infinite. Thus A has a β^*-limit point.

5 β*−Lindelof Spaces

In this section, we concentrate on the concept of β^*-Lindelof space and its properties.

Definition 5.1. A topological space (X,τ) is said to be β^*-Lindelof space if every β^*-open cover of X has a countable sub cover.

Theorem 5.2. Every β^*-Lindelof space (X,τ) is Lindeloff space.

Proof. Let (X,τ) be a β^*-Lindelof space. Let $\{A_i : i\in I\}$ be an open cover of (X,τ). Since $\tau\subseteq\beta^*\text{-}O(X,\tau)$. Therefore $\{A_i : i\in I\}$ is a β^*-open cover of (X,τ). Since (X,τ) is β^*-Lindelof space. So there exists a countable subset I_0 of I such that $\{A_i : i\in I_0\}$ is a β^*-open sub cover of (X,τ). Hence (X,τ) is a Lindelof space.

Theorem 5.3. Every β^*-compact space is β^*-Lindelof.

Proof. Let (X,τ) be a β^*-compact space. Let $\{A_i : i\in I\}$ be a β^*-open cover of (X,τ). Since (X,τ) is β^*-compact space. Then $\{A_i : i\in I\}$ has a finite sub cover say $\{A_i : i=1,2,3,\ldots,n\}$. Since every finite sub cover is always countable sub cover and therefore $\{A_i : i=1,2,3,\ldots,n\}$. is countable sub cover of $\{A_i : i\in I\}$. Hence (X,τ) is β^*-Lindelof space.

Theorem 5.4. Every β^*-closed subset of a β^*-Lindelof space is β^*-Lindelof.

Proof. Let F be a β^*-closed subset of X and $\{G_i : i\in I\}$ be β^*-open cover of F. Then F^c is β^*-open and $F\subseteq\cup\{G_i : i\in I\}$. Hence $X=(\cup\{G_i : i\in I\})\cup F^c$. Since X is β^*-Lindelof, then $X=(\cup\{G_i : i\in I_0\})\cup F^c$ for some countable subset I_0 of I. Therefore $F\subseteq\cup\{G_i : i\in I_0\}$. Thus F is β^*-Lindelof.

Theorem 5.5. Let A be a β^*-Lindelof subset of X and B be a β^*-closed subset of X. Then $A\cap B$ is β^*-Lindelof.

Proof. Let $\{G_i : i\in I\}$ be a β^*-open cover of $A\cap B$. Then $A\subseteq\left(\bigcup_{i\in I}G_i\right)\cup B^c$. Since A is β^*-Lindelof, then there exists a countable subset I_0 of I such that $A\subseteq\left(\bigcup_{i\in I_0}G_i\right)\cup B^c$. Therefore $A\cap B\subseteq\bigcup_{i\in I_0}G_i$. Thus $A\cap B$ is β^*-Lindelof.

Theorem 5.6. A topological space (X,τ) is β^*-Lindelof if and only if every collection of β^*-closed subsets of X satisfying the countable intersection property, has, itself, a non-empty intersection.

Necessity: Let $\Lambda=\{F_i : i\in I\}$ be a collection of β^*-closed subsets of X which has the countable intersection property. Assume that $\bigcap_{i\in I} F_i=\phi$. Then $X=\bigcup_{i\in I} F_i^c$. Since X is β^*-Lindelof, then there exists a countable subset I_0 of I such that $X=\bigcup_{i\in I_0} F_i^c$. Therefore, $\bigcap_{i\in I_0} F_i=\phi$ contradicts that Λ has the countable intersection property. Thus Λ has, itself, a non-empty intersection.

Sufficiency: Let $\{G_i : i\in I\}$ be a β^*-open cover of X. Suppose $\{G_i : i\in I\}$ has no countable sub cover. Then $X-\bigcup_{i\in J} G_i\neq\phi$, for any countable subset J of I. Now, $\bigcap_{i\in J} G_i^c\neq\phi$ implies that $\{G_i^c : i\in I\}$ is a collection of β^*-closed subsets of X which has the countable intersection property. Therefore $\bigcap_{i\in I} G_i^c\neq\phi$. Thus $X\neq\bigcup_{i\in I} G_i$ contradicts that $\{G_i : i\in I\}$ is a β^*-open cover of X. Hence X is β^*-Lindelof.

Theorem 5.7. A β^*-continuous image of a β^*-Lindelof space is a Lindeloff space.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a β^*-continuous map from a β^*-Lindelof space X onto a topological space Y. Let $\{A_i : i\in I\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i\in I\}$ is a β^*-open cover of X, as f is β^*-continuous. Since X is β^*-Lindelof space, the β^*-open cover $\{f^{-1}(A_i) : i\in I\}$ of X has a countable sub cover say $\{f^{-1}(A_i) : i\in I_0\}$ for some countable set $I_0\subseteq I$. Therefore $X=\bigcup\{f^{-1}(A_i) : i\in I_0\}$, which implies $f(X)=\bigcup\{A_i : i\in I_0\}$, then $Y=\bigcup\{A_i : i\in I_0\}$. That is $\{A_i : i\in I_0\}$ is a countable sub cover of $\{A_i : i\in I\}$ for Y. Hence (Y,σ) is a Lindeloff space.

Theorem 5.8. The image of a β^*-Lindelof space under a β^*-irresolute map is β^*-Lindelof space.

Proof. Suppose that a map $f:(X,\tau)\to(Y,\sigma)$ is a β^*-irresolute map from a β^*-Lindelof space (X,τ) onto a topological space (Y,σ). Let $\{B_i : i\in I\}$ be a β^*-open cover of (Y,σ). Since f is β^*-irresolute. Therefore $\{f^{-1}(B_i) : i\in I\}$ is a β^*-open cover of (X,τ). As (X,τ) is β^*-Lindelof space. the β^*-open cover $\{f^{-1}(B_i) : i\in I\}$ of (X,τ) has a countable sub cover say $\{f^{-1}(B_i) : i\in I_0\}$ for some countable set $I_0\subseteq I$. Therefore $X=\bigcup\{f^{-1}(B_i) : i\in I_0\}$, which implies $f(X)=\bigcup\{B_i : i\in I_0\}$, so that $Y=\bigcup\{B_i : i\in I_0\}$. That is $\{B_i : i\in I_0\}$ a countable sub cover of $\{B_i : i\in I\}$ for Y. Hence (Y,σ) is a β^*-Lindelof space.

Theorem 5.9. If (X,τ) is β^*-Lindelof space and countably β^*-compact space, then (X,τ) is β^*-compact space.

Proof. Suppose (X,τ) is β^*-Lindelof space and countably β^*-compact space. Let $\{A_i : i\in I\}$ be a β^*-open cover of (X,τ). Since (X,τ) is β^*-Lindelof space, $\{A_i : i\in I\}$ has a countable sub cover say $\{A_i : i\in I_0\}$ for some countable set $I_0\subseteq I$. Therefore $\{A_i : i\in I_0\}$ is a countable β^*-open cover of (X,τ). Again, since (X,τ) is countably β^*-compact space, $\{A_i : i\in I_0\}$ has a finite sub cover and say $\{A_i : i=1,2,3,\dots,n\}$. Therefore $\{A_i : i=1,2,3,\dots,n\}$ is a finite sub cover of $\{A_i : i\in I\}$ for (X,τ). Hence (X,τ) is a β^*-compact space.

Theorem 5.10. If a function $f:(X,\tau)\to(Y,\sigma)$ is β^*-irresolute and a subset A of X is β^*-Lindelof relative to X, then $f(A)$ is β^*-Lindelof relative to Y.

Proof. Let $\{B_i : i\in I\}$ be a cover of $f(A)$ by β^*-open subsets of Y. By hypothesis f is β^*-irresolute and so $\{f^{-1}(B_i) : i\in I\}$ is a cover of A by β^*-open subsets of X. Since A is β^*-Lindelof relative to X, $\{f^{-1}(B_i) : i\in I\}$ has a countable sub cover say $\{f^{-1}(B_i) : i\in I_0\}$ for A, where I_0 is a countable subset of I. Now $\{B_i : i\in I_0\}$ is a countable sub cover of $\{B_i : i \in I\}$ for $f(A)$. So $f(A)$ is $\beta*$–Lindelof relative to Y.

6 Almost $\beta*$–Compact Spaces

Definition 6.1. A topological space (X,τ) is called almost $\beta*$–compact $(\beta*$–Lindelof$)$ provided that every $\beta*$–open cover of X has a finite (countable) sub collection, the $\beta*$–closure of whose members cover X.

The proofs of the following four propositions are straightforward and therefore will be omitted.

Proposion 6.2. Every almost $\beta*$–compact space is almost $\beta*$–Lindelof space.

Proposion 6.3. Every $\beta*$–compact space $(\beta*$–Lindelof space$)$ is almost $\beta*$–compact (almost $\beta*$–Lindelof).

Proposion 6.4. Any finite (countable) topological space (X,τ) is almost $\beta*$–compact (almost $\beta*$–Lindelof).

Proposition 6.5. A finite (countable) union of almost $\beta*$–compact (almost $\beta*$–Lindelof) subsets of (X,τ) is almost $\beta*$–compact (almost $\beta*$–Lindelof).

Definition 6.6. A subset E of (X,τ) is called $\beta*$–clopen provided that it is $\beta*$–open and $\beta*$–closed.

Theorem 6.7. Let F be a $\beta*$–clopen subset of an almost $\beta*$–compact (almost $\beta*$–Lindelof) space (X,τ). Then F is almost $\beta*$–compact (almost $\beta*$–Lindelof).

Proof. Let F be a $\beta*$–clopen subset of an almost $\beta*$–compact space X and $\{G_i : i \in I\}$ be a $\beta*$–open cover of F. Then F^c is $\beta*$–open and $X \subseteq \left(\bigcup\{G_i : i \in I\}\right) \cup F^c$. Since X is almost $\beta*$–compact, then there exists a finite subset I_0 of I such that $X = \left(\bigcup\{\beta*\text{–}Cl(G_i) : i \in I_0\}\right) \cup F^c$. Thus it follows that $F \subseteq \bigcup\{\beta*\text{–}Cl(G_i) : i \in I_0\}$. Hence F is almost $\beta*$–compact.

The proof is similar in case of almost $\beta*$–Lindelof.

Theorem 6.8. If A is an almost $\beta*$–compact (almost $\beta*$–Lindelof) subset of (X,τ) and B is a $\beta*$–open subset of X, then $A \cap B$ is almost $\beta*$–compact (almost $\beta*$–Lindelof).

Proof. Let $\Lambda = \{G_i : i \in I\}$ be a $\beta*$–open cover of $A \cap B$. Then $A \subseteq \left(\bigcup\{G_i : i \in I\}\right) \cup B^c$. Since A is almost $\beta*$–compact, then there exists a finite subset I_0 of I such that $A \subseteq \left(\bigcup\{\beta*\text{–}Cl(G_i) : i \in I_0\}\right) \cup B^c$. Therefore $A \cap B \subseteq \bigcup\{\beta*\text{–}Cl(G_i) : i \in I_0\}$. Thus $A \cap B$ is almost $\beta*$–compact.

The proof is similar in case of almost $\beta*$–Lindelof.

Theorem 6.9. Let a map $f : (X,\tau) \to (Y,\sigma)$ be $\beta*$–irresolute. Suppose that A is almost $\beta*$–compact (almost $\beta*$–Lindelof) subset of X. Then $f(A)$ is almost $\beta*$–compact (almost $\beta*$–Lindelof).

Proof. Suppose that $\{G_i : i \in I\}$ is $\beta*$–open cover of $f(A)$. Then $f(A) \subseteq \bigcup\{G_i : i \in I\}$. Now, $A \subseteq \bigcup\{f^{-1}(G_i) : i \in I\}$. Since f is $\beta*$–irresolute, then $\{f^{-1}(G_i) : i \in I\}$ is a $\beta*$–open cover of A. By hypothesis, A is almost $\beta*$–compact, then there exists a finite subset I_0 of I such that $A \subseteq \bigcup\{\beta*\text{–}Cl[f^{-1}(G_i)] : i \in I_0\}$. Since f is $\beta*$–irresolute, then $\beta*\text{–}Cl(f^{-1}(G_i)) \subseteq f^{-1}[\beta*\text{–}Cl(G_i)]$, for all $i \in I_0$. Hence it follows that $f(A) \subseteq \bigcup_{i \in I_0} f\left[f^{-1}(\beta*\text{–}Cl(G_i))\right] \subseteq \bigcup_{i \in I_0} \beta*\text{–}Cl(G_i)$, which implies that $f(A) \subseteq \bigcup_{i \in I_0} \beta*\text{–}Cl(G_i)$. Thus $f(A)$ is almost $\beta*$–compact.

The proof is similar in case of almost $\beta*$–Lindelof.

Theorem 6.10. Let $f : (X,\tau) \to (Y,\sigma)$ be a $\beta*$–open bijective map and (Y,σ) is almost $\beta*$–compact. Then (X,τ) is almost compact.

Proof. Let $\{G_i : i \in I\}$ be an open cover of X. Then $f(X) = f\left(\bigcup_{i \in I} G_i\right)$. Therefore $Y = \bigcup_{i \in I} f(G_i)$. Now,

Y is almost $\beta*-$compact, then there exists a finite subset I_0 of I such that $Y = \bigcup_{i \in I_0} \beta*-Cl\left[f(G_i)\right]$. Since f is $\beta*-$open bijective map, then f is $\beta*-$closed map. Therefore, we have $\beta*-Cl\left[f(G_i)\right] \subseteq f\left[Cl(G_i)\right]$, for all $i \in I_0$. Thus $Y \subseteq \bigcup_{i \in I_0} f\left[Cl(G_i)\right] \subseteq f\left[\bigcup_{i \in I_0} Cl(G_i)\right]$, which implies that $X = f^{-1}(Y) \subseteq \bigcup_{i \in I_0} Cl(G_i)$. Thus $X = \bigcup_{i \in I_0} Cl(G_i)$. Hence X is almost compact.

Theorem 6.11. If every collection of $\beta*-$closed subsets of (X, τ), satisfying the finite (countable) intersection property, has, itself, a non-empty intersection, then X is almost $\beta*-$compact $(\text{almost } \beta*-\text{Lindelof})$.

Proof. Let $\{G_i : i \in I\}$ be a $\beta*-$open cover of X. Suppose $\{G_i : i \in I\}$ has no finite sub-collection such that the $\beta*-$closure of whose members cover X. Then $X - \bigcup_{i=1}^{i=n} I\alpha - Cl(G_i) \neq \phi$, for any $n \in N$. Therefore $X - \bigcup_{i=1}^{i=n} G_i \neq \phi$. Now, $\bigcap_{i=1}^{n} G_i^c \neq \phi$ implies $\{G_i^c : i \in I\}$ is a collection of $\beta*-$closed subsets of X which has the finite intersection property. Thus $\bigcap_{i \in I} G_i^c \neq \phi$ implies $X \neq \bigcup_{i \in I} G_i$. But this is a contradiction. Hence X is almost $\beta*-$compact.

A similar proof is given in a case of almost $\beta*-$Lindelof.

7 Mildly $\beta*-$Compact Spaces

Definition 7.1. A topological space (X, τ) is called mildly $\beta*-$compact $(\text{mildly } \beta*-\text{Lindelof})$ provided that every $\beta*-$clopen cover of X has a finite (countable) sub cover.

Theorem 7.2. Every mildly $\beta*-$compact space is mildly $\beta*-$Lindelof.

Proof. It is straight forward.

Theorem 7.3. Every almost $\beta*-$compact $(\text{almost } \beta*-\text{Lindelof})$ space (X, τ) is mildly $\beta*-$compact $(\text{mildly } \beta*-\text{Lindelof})$.

Proof. Let $\Lambda = \{H_i : i \in I\}$ be a $\beta*-$clopen cover of (X, τ). Since (X, τ) is almost $\beta*-$compact, then there exists a finite subset I_0 of I such that $X = \bigcup_{i \in I_0} \beta*-Cl(H_i)$. Now, $\beta*-Cl(H_i) = H_i$. Thus (X, τ) is mildly $\beta*-$compact.

A similar proof is given when (X, τ) is almost $\beta*-$Lindelof.

Corollary 7.4. Every $\beta*-$compact $(\beta*-\text{Lindelof})$ space is mildly $\beta*-$compact $(\text{mildly } \beta*-\text{Lindelof})$.

Theorem 7.5. If F is a $\beta*-$clopen subset of a mildly $\beta*-$compact $(\text{mildly } \beta*-\text{Lindelof})$ space X, then F is mildly $\beta*-$compact $(\text{mildly } \beta*-\text{Lindelof})$

Proof. Let F be a $\beta*-$clopen subset of X and $\{G_i : i \in I\}$ be a $\beta*-$clopen cover of F. Then F^c is a $\beta*-$clopen and $F \subseteq \bigcup_{i \in I} G_i$. Therefore $X = \left(\bigcup_{i \in I} G_i\right) \cup F^c$. Since X is mildly $\beta*-$compact, then there exists a finite subset I_0 of I such that $X = \left(\bigcup_{i \in I_0} G_i\right) \cup F^c$. So $F \subseteq \left(\bigcup_{i \in I_0} G_i\right)$. Hence F is mildly $\beta*-$compact.

The proof is similar in a case of mildly $\beta*-$Lindelof.

Theorem 7.6. If A is a mildly $\beta*-$compact $(\text{mildly } \beta*-\text{Lindelof})$ subset of X and B is a $\beta*-$clopen subset of X, then $A \cap B$ is mildly $\beta*-$compact $(\text{mildly } \beta*-\text{Lindelof})$.

Proof. Let $\Lambda = \{G_i : i \in I\}$ be a $\beta*-$clopen cover of $A \cap B$. Then $A \subseteq \left(\bigcup_{i \in I} G_i\right) \cup B^c$. Since A is mildly $\beta*-$compact, then there exists a finite subset I_0 of I such that $A \subseteq \left(\bigcup_{i \in I_0} G_i\right) \cup B^c$. Therefore $A \cap B \subseteq \bigcup_{i \in I_0} G_i$. Thus $A \cap B$ is mildly $\beta*-$compact.

The proof is similar in case of mildly $\beta*-$Lindelof.

Theorem 7.7. If $f : (X, \tau) \to (Y, \sigma)$ is a $\beta*-$open bijective map and (Y, σ) is mildly $\beta*-$compact, then (X, τ) is mildly compact.

Proof. Let $\{G_i : i \in I\}$ be a clopen cover for X. Then $f(X) = f\left(\bigcup_{i \in I} G_i\right)$. Hence $Y = \bigcup_{i \in I} f(G_i)$. Since f is $\beta*-$open bijective map, then f is $\beta*-$closed. Therefore $\{f(G_i) : i \in I\}$ is a $\beta*-$open cover of X. Since Y is mildly $\beta*-$compact, then there exists a finite subset I_0 of I such that $Y = \bigcup_{i \in I_0} f(G_i)$. Therefore $X = \bigcup_{i \in I_0} G_i$. Thus X is mildly compact.

Proposion 7.8. A subset A of (X,τ) is mildly compact (mildly Lindelof) if and only if (X,τ_A) is mildly compact (mildly Lindelof).

8 $\beta*$–Connected Spaces

Definition 8.1. A topological space (X,τ) is said to be connected if X cannot be written as a disjoint union of two non empty open sets. A subset of (X,τ) is connected if it is connected as a subspace.

Definition 8.2. A topological space (X,τ) is said to be $\beta*-$connected if X cannot be written as a disjoint union of two non empty $\beta*-$open sets. A subset of (X,τ) is $\beta*-$connected if it is $\beta*-$connected as a subspace.

Theorem 8.3. Every $\beta*-$connected space (X,τ) is connected.

Proof. Let A and B be two non empty disjoint proper open sets in X. Since every open set is $\beta*-$open set. Therefore A and B are non empty disjoint proper $\beta*-$open sets in X and X is $\beta*-$connected space. Hence $X \neq A \cup B$. Therefore X is $\beta*-$connected.

Theorem 8.4. Let (X,τ) be a topological space. Then the following statements are equivalent

(i) (X,τ) is $\beta*-$connected.

(ii) The only subsets of (X,τ) which are both $\beta*-$open and $\beta*-$closed are the empty set ϕ and X

(iii) Each $\beta*-$continuous map of (X,τ) into a discrete space (Y,σ) with at least two points is a constant map.

Proof. $(i)\Rightarrow(ii)$**:** Let G be a non empty proper $\beta*-$open and $\beta*-$closed subset of (X,τ). Then $X-G$ is also both $\beta*-$open and $\beta*-$closed. Then $X = G \cup (X-G)$ is a disjoint union of two non empty $\beta*-$open sets, which contradicts the fact that (X,τ) is $\beta*-$connected. Hence $G=\phi$ or $G=X$.

$(ii)\Rightarrow(i)$**:** Suppose that $X = A \cup B$ where A and B are disjoint non empty $\beta*-$open subsets of (X,τ). Since $A = X-B$, then A is both $\beta*-$open and $\beta*-$connected. By assumption $A=\phi$ or $A=X$, which is a contradiction. Hence (X,τ) is $\beta*-$connected.

$(ii)\Rightarrow(iii)$**:** Let $f:(X,\tau)\to(Y,\sigma)$ be a $\beta*-$continuous map, where (Y,σ) is discrete space with at least two points. Then $f^{-1}(y)$ is $\beta*-$closed and $\beta*-$open for each $y\in Y$. Thus (X,τ) is covered by $\beta*-$closed and $\beta*-$open covering $\{f^{-1}(y): y\in Y\}$. By assumption, $f^{-1}(y)=\phi$ or $f^{-1}(y)=X$ for each $y\in Y$. If $f^{-1}(y)=\phi$ for each $y\in Y$, then f fails to be a map. Therefore their exists at least one point say $y^*\in Y$ such that $f^{-1}(\{y^*\})\neq\phi$. Since $f^{-1}(\{y^*\})$ is also both $\beta*-$open and $\beta*-$closed set. Therefore by hypothesis $f^{-1}(\{y^*\})=X$. This shows that f is a constant map.

$(iii)\Rightarrow(ii)$**:** Let G be both $\beta*-$open and $\beta*-$closed set in (X,τ). Suppose $G\neq\phi$. Let $f:(X,\tau)\to(Y,\sigma)$ be a $\beta*-$continuous map defined by $f(G)=\{a\}$ and $f(X-G)=\{b\}$ where $a\neq b$ and $a,b\in Y$. By assumption, f is constant so $G=X$.

Theorem 8.5. Suppose $f:(X,\tau)\to(Y,\sigma)$ is a $\beta*-$continuous surjection and (X,τ) is $\beta*-$connected. Then (Y,σ) is connected.

Proof. Suppose (Y,σ) is not connected. Let $Y = A\cup B$, where A and B are disjoint non empty open subsets of (Y,σ). Since f is $\beta*-$continuous, $X = f^{-1}(A)\cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non empty $\beta*-$open subsets of X. This disproves the fact that (X,τ) is $\beta*-$connected. Hence (Y,σ) is connected.

Theorem 8.6. Suppose $f:(X,\tau)\to(Y,\sigma)$ is a $\beta*-$irresolute surjection and (X,τ) is $\beta*-$connected. Then Y is $\beta*-$connected.

Proof. Suppose that Y is not $\beta*-$connected. Let $Y = A\cup B$, where A and B are disjoint non empty $\beta*-$open sets in Y. Since f is $\beta*-$irresolute map and onto, $X = f^{-1}(A)\cup f^{-1}(B)$, where $f^{-1}(A)$ and

$f^{-1}(B)$ are disjoint non empty β^*-open sets in (X,τ). This contradicts the fact that (X,τ) is β^*-connected. Hence (Y,σ) is β^*-connected.

Theorem 8.7. If every β^*-closed set in X is closed in X and X is connected, then X is β^*-connected.

Proof. Suppose that X is connected. Then X cannot be expressed as a disjoint union of two nonempty proper open subset of X. Let X be not β^*-connected space. Let A and B be any two non empty β^*-open subsets of X such that $X = A \cup B$, where $A \cap B = \phi$. Since every β^*-closed set in X is closed in X. Therefore every β^*-open set in X is open in X. Hence A and B are open subsets of X, which contradicts that X is connected. Therefore X is β^*-connected.

Theorem 8.8. Every β^*-connected space (X,τ) is mildly β^*-compact.

Proof. Since (X,τ) is β^*-connected then the only β^*-clopen subsets of (X,τ) are X and ϕ. Therefore (X,τ) is mildly β^*-compact.

Theorem 8.9. If two β^*-open sets C and D form a separation of X and if Y is β^*-connected subspace of X, then Y lies entirely within C or D.

Proof. By hypothesis C and D are both β^*-open sets in X. The sets $C \cap Y$ and $D \cap Y$ are β^*-open in Y, these two sets are disjoint and their union is Y. If they were both non empty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely in C or D.

Theorem 8.10. Let A be a β^*-connected subspace of X. If $A \subseteq B \subseteq \beta^*-Cl(A)$, then B is also β^*-connected.

Proof. Let A be β^*-connected. Let $A \subseteq B \subseteq \beta^*-Cl(A)$. Suppose that $B = C \cup D$ is a separation of B by β^*-open sets. Thus by previous theorem A must lie entirely in C or D. Suppose that $A \subseteq C$, then it implies that $\beta^*-Cl(A) \subseteq \beta^*-Cl(C)$. Since $\beta^*-Cl(C)$ and D are disjoint, B cannot intersect D. This disproves the fact that D is non empty subset of B. So $D = \phi$ which implies B is β^*-connected.

9 Conclusion

We have used β^*-open sets to introduce the new concepts of notions in topological spaces namely β^*-compact space, countably β^*-compact space, β^*-Lindelof space, almost β^*-compact space, mildly β^*-compact space and β^*-connected space and have investigated several properties and characterizations of these new concepts.

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