Third Hankel Determinant for A Class of Functions with Respect to Symmetric Points Associated With Exponential Function

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Abstract: The purpose of the present work is to determine the possible upper bound of third order Hankel determinantforthefunctionsstarlikeandconvexwithrespecttosymmetricpointsassociatedwithexponentialfunctions.

Key–Words: Analytic function, Univalent function, Subordination, Fekete-Szegö inequality, Hankel determinant, Symmetricpoints.

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1 Introduction

The class of all analytic functions f with $f(0) = 0$ and $f'(0) = 1$ in the unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathcal{A}$ and has the Taylor's series expansion of the form given by

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

The class of functions that are univalent and analytic in E is denoted by $\mathbb{S}$. The family of functions that are analytic in E and maps E onto the right half plane is denoted by P and functions of this class are of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2}$$

For two analytic functions g and h in E, g is subordinate to h, denoted as $g \prec h$, if there is an analytic function w in E with $w(0) = 0$ and $|w(z)| < 1$ such that $g(z) = h(w(z))$, for all $z \in E$. If h is univalent in E, the subordination is same as $g(0) = h(0)$ and $g(E) \subseteq h(E)$. Pommerenke ([35], [36]) defined the Hankel determinant $H_k(n)$, for positive integers k, n for the functions in $\mathbb{S}$ of the form (1), as below:

$$H_k(n) = \begin{vmatrix} a_n & a_{n+1} & a_{n+2}\cdots\cdots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & a_{n+3}\cdots\cdots & a_{n+k} \\ a_{n+2} & a_{n+3} & a_{n+4}\cdots\cdots & a_{n+k+1} \\ -- & -- & ------- & --- \\ -- & -- & ------- & --- \\ a_{n+k-1} & a_{n+k} & a_{n+k+1}\cdots\cdots & a_{n+2k-2} \end{vmatrix} \tag{3}$$

For fixed positive integers k and n the growth of $H_k(n)$ as $n \longrightarrow \infty$ has been determined by Noor [30], with bounded boundary. The Hankel determinant for exponential polynomials was studied by Ehrenborg [9].The Hankel determinant of different orders is obtained for different values of k, n. For instance, when $k = 2$ and $n = 1$, the determinant

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_3 - a_2^2|, (a_1 = 1) \tag{4}$$

This determinant is the particular case of estimating the greatest value of the functional $|a_3 - \mu a_2^2|$ for functions in $\mathbb{S}$, where μ is real or complex. This is known as the Fekete-Szegö [10] problem. Many researchers like Ali et al. ([1],[2]), Cho and Owa ([6],[7]),Koegh and Merkes [16], Lewandowski et al.[21], Ma and Minda [24], Magesh et al. [25], Murugusundaramurthy et al. [28], Ram Reddy and Sharma [39], Ravichandran et al. [40],Shanmugam et al. [41], Ram Reddy et al.[42], Haripriya and Sharma [12], Srivastava H.M. et al.[43], Tuneski and Darus [44] have studied Fekete- Szegö inequality for various subclasses of univalent analytic functions.

Now for $k = 2, n = 2$, it can be obtained that

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2| \tag{5}$$

The maximum value of $H_2(2)$has been investigated by several authors. For instance the reader can see the work initiated by Hayman [13], Noonan and Thomas [29], Janteng et al. ([14],[15]), Bansal[5], Lee et al. [20], Liu et al. [23], Raina et al. [38], Ohran et al. [31], Laxmi and Sharma [18], Răducanu and Zaprawa [37].Very recently, Zaprawa[47] shown a new direction in estimating the upper bound of the Hankel determinant for $k = 2, n = 3$, i.e., $H_2(3)$ for various subclasses of $\mathbb{S}$. This determinant is given by

$$H_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = |a_3a_5 - a_4^2| \quad (6)$$

For $k = 3, n = 1$ the Hankel determinant $H_3(1)$ is called as third order Hankel determinant and is given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) \quad (7)$$

Babalola [4] was the first person to study the upper bound of $H_3(1)$ for subclasses of $\mathbb{S}$. Interested readers can see the work carried by several researchers like Vamshee Krishna et al. ([45],[46]), Prajapat et al. ([32],[33]),Altinkaya and Yalcin [3],Cho et al. [8], lecko et al. [19], Kowalczyk et al.[17],Mohd Narzan et al.[27]. Mendiratta et al. [26] introduced and studied the class of starlike functions $\mathbb{S}_e^* = \mathbb{S}^*(e^z)$ defined by

$$\frac{zf'(z)}{f(z)} \prec e^z, (z \in E) \quad (8)$$

Very recently, in 2018, Hai-Yan Zhang et al. [11] investigated the upper bound of the Third Hankel determinant for the function class $\mathbb{S}_e^*$ related with exponential function. The work of above researchers motivated us to determine the non-sharp upper bounds of third Hankel determinant for the class of starlike and convex functions with respect to symmetric points subordinate to exponential function. Now we define the following subclasses.

Definition 1 *A function $f \in \mathbb{S}$, is in the class $\mathbb{S}_s^*(e^z)$if and only if*

$$\frac{2[zf'(z)]}{f(z) - f(-z)} \prec e^z, \text{for all} z \in E. \quad (9)$$

Definition 2 *A function $f \in \mathbb{S}$, is in the class $C_s(e^z)$ if and only if*

$$\frac{2[zf'(z)]'}{(f(z) - f(-z))'} \prec e^z, \text{for all} z \in E. \quad (10)$$

1.1 Preliminaries

The lemmas listed below are needed to prove the desired results.

Lemma 3 *[34] If $p \in P$, then $|p_n| \leq 2, \forall n \in \mathbb{N}$.*

Lemma 4 *[22] If $p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots$, is such that $Re(p(z)) > 0$ in E, then for some x, z with $|x| \leq 1, |z| \leq 1$, we have*

$$2p_2 = p_1^2 + x(4 - p_1^2), for\ some\ x,\ |x| \leq 1 \quad (11)$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z \quad (12)$$

Lemma 5 *[40] If $p \in P$, then $|p_2 - \nu p_1^2| \leq max|\{1, |2\nu - 1|\}$ for any $\nu \in \mathbb{C}$.*

2 Mains Results

Theorem 6 *If $f \in \mathbb{S}_s^*(e^z)$ then $|a_2| \leq \frac{1}{2}$, $|a_3| \leq \frac{1}{2}, |a_4| \leq \frac{19}{48}, |a_5| \leq \frac{13}{24}$.*

Proof: As $f \in \mathbb{S}_s^*(e^z)$ from (9) and using the principle of subordination, we have

$$\frac{2[zf'(z)]}{f(z) - f(-z)} = e^{w(z)}, \quad (13)$$

Let us define the function $p(z) = \frac{1+w(z)}{1-w(z)} = 1+p_1z+p_2z^2+\cdots$, analytic in E with $p(0) = 1$ and maps E onto the right half of the $w-$plane. Computing $w(z)$ in terms of $p(z)$, we get,

$$w(z) = \frac{p_1z}{2} + \Big(\frac{p_2}{2} - \frac{p_1^2}{4}\Big)z^2 + \Big(\frac{p_3}{2} - \frac{p_1p_2}{2} + \frac{p_1^3}{8}\Big)z^3 + \Big(\frac{p_4}{2} - \frac{p_1p_4}{2} - \frac{p_2^2}{4} + \frac{3p_1^2p_2}{8} - \frac{p_1^4}{16}\Big)z^4 + \cdots \quad (14)$$

$$\text{Again} e^{w(z)} = 1 + w(z) + \frac{(w(z))^2}{2!} + \frac{(w(z))^3}{3!} + \frac{(w(z))^4}{4!} + \cdots \quad (15)$$

Substituting (14) in (15), we get

$$e^{w(z)} = 1 + \frac{p_1z}{2} + \Big(\frac{p_2}{2} - \frac{p_1^2}{8}\Big)z^2 + \Big(\frac{p_3}{2} - \frac{p_1p_2}{4} + \frac{p_1^3}{48}\Big)z^3 + \Big(\frac{p_4}{2} - \frac{p_1p_3}{4} - \frac{p_2^2}{8} + \frac{p_1^2p_2}{16} + \frac{p_1^4}{384}\Big)z^4 + \cdots \quad (16)$$

Substituting $f(z), f'(z)$ on L.H.S. of (13) and replacing $e^{w(z)}$ on R.H.S, upon equating like powers of z, we have

$$a_2 = \frac{p_1}{4}, a_3 = \frac{p_2}{4} - \frac{p_1^2}{16}, a_4 = \frac{p_4}{8} - \frac{p_1p_2}{32} - \frac{p_1^3}{384},$$
$$a_5 = \frac{p_4}{8} - \frac{p_1p_3}{16} + \frac{p_1^4}{384}. \quad (17)$$

Taking modulus on either side of each expression in (17) and applying Lemma 3 and Lemma 5, we obtain

$$|a_2| \le \frac{1}{2}, |a_3| \le \frac{1}{2}, |a_4| \le \frac{19}{48}, |a_5| \le \frac{13}{24}. \quad (18)$$

Theorem 7 *If* $f \in \mathbb{S}_s^*(e^z)$ *then* $|a_3 - a_2^2| \le \frac{1}{2}$.

Proof: Proceeding as in Theorem 6 and using (17), we have

$$a_2 = \frac{p_1}{4}, a_3 = \frac{p_2}{4} - \frac{p_1^2}{16}. \quad (19)$$

Consider

$$|a_3 - a_2^2| = \Big|\frac{p_2}{4} - \frac{p_1^2}{16} - \frac{p_1^2}{16}\Big| = \Big|\frac{p_2}{4} - \frac{p_1^2}{8}\Big| \quad (20)$$

Using Lemma 5, we obtain

$$|a_3 - a_2^2| \le \frac{1}{2} \quad (21)$$

Theorem 8 *If* $f \in \mathbb{S}_s^*(e^z)$ *then* $|a_2a_3 - a_4| \le \frac{765+59\sqrt{118}}{3468}$.

Proof: From equation (17) of Theorem 6, we have

$$|a_2a_3 - a_4| = \Big|\frac{3p_1p_2}{32} - \frac{p_3}{8} - \frac{5p_1^3}{384}\Big| \quad (22)$$

Now applying Lemma 4, we have

$$|a_2a_3 - a_4| = \Big|\frac{(4-p_1^2)p_1x^2}{32} - \frac{(4-p_1^2)p_1x}{64} - \frac{(4-p_1^2)(1-|x|^2)z}{16} + \frac{p_1^3}{384}\Big| \quad (23)$$

Denote $|x| = t \in [0,1], p_1 = c \in [0,2]$. Then, using triangle inequality, equation (23) gives

$$|a_2a_3 - a_4| \le \frac{(4-c^2)ct^2}{32} + \frac{(4-c^2)ct}{64} + \frac{(4-c^2)}{16} + \frac{c^3}{384}$$

Suppose that

$$F(c,1) \equiv \frac{(4-c^2)ct^2}{32} + \frac{(4-c^2)ct}{64} + \frac{(4-c^2)}{16} + \frac{c^3}{384}$$

Thus we get $\frac{\partial F}{\partial t} = \frac{(4-c^2)c}{64} + \frac{(4-c^2)ct}{16} \ge 0$, the function $F(c,t)$ is non-decreasing for any t in $[0,1]$. This shows that $F(c,t)$ has maximum value at $t = 1$.

$$\text{Max}F(c,t) = F(c,1) = \frac{(4-c^2)c}{32} + \frac{(4-c^2)c}{64} + \frac{(4-c^2)}{16} + \frac{c^3}{384}$$

Let us define $M(c) := \frac{(4-c^2)c}{32} + \frac{(4-c^2)c}{64} + \frac{(4-c^2)}{16} + \frac{c^3}{384}$, then $M'(c) := \frac{3(4-c^2)c}{64} - \frac{11c^2}{128} - \frac{c}{8}$.
$M'(c)$ vanishes at $c = r^* = \frac{-8+2\sqrt{118}}{17}$. A simple computation yields that $M''(c) < 0$, which means that the function $M(c)$ can take maximum value at $r^* = \frac{-8+2\sqrt{118}}{17}$. Hence, we have

$$|a_2a_3 - a_4| \le M(r^*) = \frac{765 + 59\sqrt{118}}{3468}. \quad (24)$$

Thus the theorem is proved.

Theorem 9 *If* $f \in \mathbb{S}_s^*(e^z)$ *then* $|a_2a_4 - a_3^2| \le \frac{3}{8}$.

Proof: From equation (17) of Theorem 6, we have

$$\begin{aligned} &|a_2a_4 - a_3^2| \\ &= \Big|\frac{p_2}{4}\Big(\frac{p_3}{8} - \frac{p_1p_2}{32} - \frac{p_1^3}{384}\Big) - \Big(\frac{p_2}{4} - \frac{p_1^2}{16}\Big)^2\Big| \\ &= \Big|\frac{p_1p_3}{32} - \frac{p_1^2p_2}{128} - \frac{p_1^4}{1536} - \frac{p_2^2}{16} + \frac{2p_1^2p_2}{64} - \frac{p_1^4}{256}\Big| \end{aligned} \quad (25)$$

According to Lemma 4, we obtain

$$|a_2a_4 - a_3^2| = \Big|\frac{(4-p_1^2)p_1(1-|x|^2)z}{64} - \frac{(4-p_1^2)p_1^2x^2}{128} - \frac{(4-p_1^2)x}{256} - \frac{(4-p_1^2)^2x^2}{64} - \frac{p_1^4}{1536}\Big|.$$

Denote $|x| = t \in [0,1], p_1 = c \in [0,2]$. Then, using triangle inequality, we get

$$|a_2a_4 - a_3^2| \le \frac{(4-c^2)}{32} + \frac{(4-c^2)c^2t^2}{128} + \frac{(4-c^2)c^2t}{256} + \frac{(4-c^2)c^2t^2}{64} + \frac{c^4}{1536}.$$

Let us consider

$$F(c,t) = \frac{(4-c^2)}{32} + \frac{(4-c^2)c^2t^2}{128} + \frac{(4-c^2)c^2t}{256} + \frac{(4-c^2)^2t^2}{64} + \frac{c^4}{1536}.$$

Thus we get $\frac{\partial F}{\partial t} = \frac{(4-c^2)c^2}{256} + \frac{(4-c^2)c^2t}{64} + \frac{(4-c^2)t}{32} \ge 0$, which gives that $F(c,t)$ is increasing for any then t in $[0,1]$. This shows that $F(c,t)$ has maximum value at $t = 1$.

$$\text{Max}F(c,t) = F(c,1) = \frac{(4-c^2)c^2}{256} + \frac{(4-c^2)c^2}{258} + \frac{(4-c^2)^2}{64} + \frac{(4-c^2)}{32} + \frac{c^4}{1536}$$

Let us define $M(c) := \frac{3(4-c^2)c^2}{256} + \frac{(4-c^2)^2}{64}$
$+ \frac{(4-c^2)}{32} + \frac{c^4}{1536}$, then
$M'(c) := \frac{3(4-c^2)c}{128} - \frac{(4-c^2)c}{32} - \frac{c^3}{8} - \frac{c}{16}$.
If $M'(c)$ vanishes at $c = 0$. A simple computation yields that $M''(c) < 0$, which means that the function $M(c)$ has maximum value at $c = 0$. Hence, we have

$$|a_2a_4 - a_3^2| \leq M(0) = \frac{3}{8}. \tag{26}$$

Theorem 10 *If* $f \in \mathbb{S}_s^*(e^z)$ *then*
$|H_3(1)| \leq \frac{90831+1121\sqrt{118}}{166464} = 0.618$.

Proof: Since $H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$, by applying triangle inequality, we get

$$|H_3(1)| \leq$$
$$|a_3|\big|a_2a_4 - a_3^2\big| + |a_4|\big|a_4 - a_2a_3\big| + |a_5|\big|a_3 - a_2^2\big|. \tag{27}$$

Now, substituting the equations (21),(24),(25) in (27) we obtain $|H_3(1)| \leq 0.618$.
Now, we state some results related to the class defined in Definition 2.

Theorem 11 *If* $f \in C_s(e^z)$ *then* $|a_2| \leq \frac{1}{4}$,
$|a_3| \leq \frac{1}{6}, |a_4| \leq \frac{19}{192}, |a_5| \leq \frac{13}{120}$.

Theorem 12 *If* $f \in C_s(e^z)$ *then* $|a_3 - a_2^2| \leq \frac{1}{6}$.

Theorem 13 *If* $f \in C_s(e^z)$ *then*
$|a_2a_3 - a_4| \leq \frac{157446-5575\sqrt{104}}{1168032} = 0.08612$.

Theorem 14 *If* $f \in C_s(e^z)$ *then* $|a_2a_4 - a_3^2| \leq \frac{25}{576}$.

Theorem 15 *If* $f \in C_s(e^z)$ *then* $|H_3(1)| \leq 0.0338$.

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