ωb – Topological Vector Spaces

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Abstract: - The purpose of the present paper is to introduce the new class of $\omega\mathrm{b}-$topological vector spaces. We study several basic and fundamental properties of $\omega\mathrm{b}-$topological and investigate their relationships with certain existing spaces. Along with other results, we prove that transformation of an open (resp. closed) set in a $\omega\mathrm{b}-$topological vector space is $\omega\mathrm{b}-$open (resp. closed). In addition, some important and useful characterizations of $\omega\mathrm{b}-$topological vector spaces are established. We also introduce the notion of almost $\omega\mathrm{b}-$topological vector spaces and present several general properties of almost $\omega\mathrm{b}-$topological vector spaces.

Key-Words: - $\omega\mathrm{b}-$open set, $\omega\mathrm{b}-$closed set, topological vector space $\omega\mathrm{b}-$topological vector space, almost $\omega\mathrm{b}-$topological vector space

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1 Introduction

Functional analysis in its traditional sense deals primarily with Banach spaces and, in particular, Hilbert spaces. However, many classical vector spaces have natural topologies not given by the norm. Such, for example, are many spaces of smooth, holomorphic and generalized function. The theory of topological vector spaces is the science of spaces of precisely this kind. The concept of topological vector spaces was first introduced and studied by Kolmogroff [19] in 1934. Since then, different researchers have explored many interesting and useful properties of topological vector spaces. Due to nice properties, topological vector spaces earn a great importance and remain a fundamental notion in fixed point theory, operator theory, and various advanced branches of Mathematics. Now a days the researchers not only make use of topological vector spaces in other fields for developing new notions but also strength and extend the concept of topological vector spaces with every possible way, making the field of study a more convenient and understandable. Recently, Khan et al [17] defined $\mathrm{s}-$topological vector spaces as a generalization of topological vector spaces. Khan and Iqbal [18], in 2016, put forth the concept of irresolute topological vector spaces which is independent of topological vector spaces but is included in $\mathrm{s}-$topological vector spaces. Ibrahim [15] introduced the study of $\alpha-$topological vector spaces. In 2018, Sharma et al [31] introduced and studied another class of spaces, namely almost pre topological vector spaces. In [28, 31, 32, 33} the Ram et al introduced the $\beta-$topological vector spaces and pre topological vector spaces, almost $\beta-$topological vector spaces and almost $\mathrm{s}-$topological vector spaces and studied properties and characterizations of these spaces. In 2018, Noiri et al [26] introduced and investigated a new class of sets called $\omega\mathrm{b}-$open sets which is weaker than $\omega-$open sets and $\mathrm{b}-$open sets. In the present paper, we introduce and investigate several properties and characterizations of $\omega\mathrm{b}-$topological vector spaces. The relationship of $\omega\mathrm{b}-$topological vector spaces with certain types of spaces is investigated as well. We will also introduce and study almost $\omega\mathrm{b}-$topological vector spaces.

2 Preliminaries

Throughout this paper, (X,τ) and (Y,σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated.

For a subset A of a space (X, τ), $Cl(A)$, $Int(A)$ and $X - A$ denote the closure of A, the interior of A and the complement of A in X, respectively. Recently, as generalization of closed sets, the notion of $\omega-$closed sets were introduced and studied by Hdeib [13]. Let (X, τ) be a topological space and let A be a subset of X. A point x of X is called a condensation point of A if for each open set U with $x \in U$, the set $U - A$ is uncountable. A subset A is said to be $\omega-$closed [7] if it contains all its condensation points. The complement of an $\omega-$closed set is said to be an $\omega-$open set. It is well known that a subset W of a space (X, τ) is $\omega-$open if and only if for each $x \in W$, there exists an open set U such that $x \in U$ and $U - W$ is countable. The family of all $\omega-$open subsets of a topological space (X, τ), is denoted by $\omega O(X, \tau)$, forms a topology on X which is finer than τ. The set of all $\omega-$open sets of (X, τ) containing a point $x \in X$ is denoted by $\omega O(X, x)$. The complement of an $\omega-$open set is said to be an $\omega-$closed set. The intersection of all $\omega-$closed sets of X containing A is called the $\omega-$closure of A and is denoted by $\omega Cl(A)$. The union of all $\omega-$open sets of X contained in A is called $\omega-$interior of A and is denoted by $\omega Int(A)$. A subset A of a topological space X is said to be $\mathrm{b}-$open if $A \subseteq Int[Cl(A)] \cup Cl[Int(A)]$. The complement of a $\mathrm{b}-$open set is called a $\mathrm{b}-$closed set. The intersection of all $\mathrm{b}-$closed sets containing A is called the $\mathrm{b}-$closure of A and is denoted by $bCl(A)$. The union of all $\mathrm{b}-$open sets of X contained in A is called the $\mathrm{b}-$interior A and is denoted by $bInt(A)$.

Definition 2.1. A subset A of a space X is said to be $\omega\mathrm{b}-$open if for every $x \in A$, there exists a $\mathrm{b}-$open subset $U_x \subseteq X$ containing x such that $U_x - A$ is countable. The complement of an $\omega\mathrm{b}-$open subset of X is called an $\omega\mathrm{b}-$closed subset of X.

The family of all $\omega\mathrm{b}-$open sets in a topological space (X, τ) is denoted by $\omega b-O(X, \tau)$ or $\omega b-O(X)$. The family of all $\omega\mathrm{b}-$closed sets in a topological space (X, τ) is denoted by $\omega b-C(X, \tau)$ or $\omega b-C(X)$.

For any $x \in X$, we present $\omega b-O(X, x) = \{U \subseteq X : x \in U \text{ and } U \text{ is } \omega b-open \text{ in } X\}$.

Lemma 2.2. For a subset of a topological space, both $\omega-$openness and $\mathrm{b}-$openness imply $\omega\mathrm{b}-$openness.

Proof. (1) Assume A is $\omega-$open, then for each $x \in A$, there is an open set U_x containing x such that $U_x - A$ is a countable set. Since every open set is $\mathrm{b}-$open, A is $\omega\mathrm{b}-$open.

(2) Let A be $\mathrm{b}-$open. For each $x \in A$, there exists a $\mathrm{b}-$open set open set $U_x = A$ such that $x \in U_x$ and $U_x - A = \phi$. Therefore, A is $\omega\mathrm{b}-$open.

The following diagram shows the implications for properties of subsets

$$\begin{array}{ccc} \text{open set} & \Rightarrow & \mathrm{b}-\text{open set} \\ \Downarrow & & \Downarrow \\ \omega-\text{open set} & \Rightarrow & \omega\mathrm{b}-\text{open set} \end{array}$$

The converses need not be true as shown by the examples by Noiri et al [26] in 2008.

Lemma 2.3. A subset A of a space X is $\omega\mathrm{b}-$open if and only if for every $x \in A$, there exists a $\mathrm{b}-$open subset U containing x and a countable subset C such that $U - C \subseteq A$.

Lemma 2.4. Let (X, τ) be a topological space.

(1) The intersection of an open set and a $\mathrm{b}-$open set is $\mathrm{b}-$open set.

(2) The union of any family of $\mathrm{b}-$open sets is a $\mathrm{b}-$open set.

Proposition 2.5. The intersection of an $\omega\mathrm{b}-$open set and a $\mathrm{b}-$open set is $\omega\mathrm{b}-$open.

Proposition 2.6. The intersection of an $\omega\mathrm{b}-$open set with an open set is $\omega\mathrm{b}-$open.

The intersection of two $\omega\mathrm{b}-$open sets is not always $\omega\mathrm{b}-$open.

Example 2.7. Let $X = \mathbb{R}$ with the usual topology τ. Let $A = Q$ be the set of rational numbers and $B = [0, 1)$. Then A and B are $\omega\mathrm{b}-$open, but $A \cap B$ is not $\omega\mathrm{b}-$open, since each b-open set containing 0 is uncountable set.

Proposition 2.8. The union of any family of $\omega\mathrm{b}-$open set is $\omega\mathrm{b}-$open.

Proof. Let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of $\omega\mathrm{b}-$open subsets of a space X. Then for any $x \in \bigcup_{\lambda \in \Lambda} A_\lambda$, there exists $\mu \in \Lambda$ such that $x \in A_\mu$. Hence there exists a $\mathrm{b}-$open subset U of X such that $U - A_\mu$ is countable. Now because $\left(U - \bigcup_{\lambda \in \Lambda} A_\lambda\right) \subseteq U - A_\mu$, and hence $\left(U - \bigcup_{\lambda \in \Lambda} A_\lambda\right)$ is countable. Therefore, $x \in \bigcup_{\lambda \in \Lambda} A_\lambda$ is $\omega\mathrm{b}-$open.

The intersection of all ωb − closed sets of X containing A is called the ωb − closure of A and is denoted by $\omega bCl(A)$. The union of all ωb − open sets of X contained in A is called the ωb − interior of A and is denoted by $\omega bInt(A)$.

Definition 2.9. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be ωb − continuous if for each open set W in Y, the inverse image $f^{-1}(W)\in \omega b-O(X)$.

Equivalently a mapping $f:(X,\tau)\to(Y,\sigma)$ is called ωb − continuous if for each $x\in X$ and each open set V in Y containing $f(x)$, there exists an ωb − open set U in X containing x such that $f(U)\subseteq V$.

Definition 2.10. Let X be a vector space over the field K, where $K=\mathbb{R}$ or $\mathbb{C}$ with standard topology. Let τ be a topology on X such that the following conditions are satisfied:

(1). For each $x, y\in X$ and each open set $W\subseteq X$ containing $x+y$, there exist open sets U and V in X containing x and y respectively, such that $U+V\subseteq W$,

(2). For each $\lambda\in$ K, $x\in X$ and each open set $W\subseteq X$ containing λx, there exist open sets U in K containing λ and V in X containing x such that $U.V\subseteq W$.

Then the pair $\left(X_{(K)},\tau\right)$ is called a topological vector space.

3 ωb − Topological Vector Spaces

Definition 3.1. Let X be a vector space over the field K, where $K=\mathbb{R}$ or $\mathbb{C}$ with standard topology. Let τ be a topology on X such that the following conditions are satisfied:

(1). For each $x, y\in X$ and each open set $W\subseteq X$ containing $x+y$, there exist ωb − open sets U and V in X containing x and y respectively, such that $U+V\subseteq W$,

(2). For each $\lambda\in$ K, $x\in X$ and each open set $W\subseteq X$ containing λx, there exist ωb − open sets U in K containing λ and V in X containing x such that $U.V\subseteq W$. Then the pair $\left(X_{(\mathrm{K})},\tau\right)$ is called an ωb − topogical vector space. (written in short, $\omega b-TVS$).

Example 3.2. Let $X=\mathbb{R}$ be the vector space of real numbers over the field $K=\mathbb{R}$, where $X=K=\mathbb{R}$ is endowed with standard topology. Then $\left(X_{(K)},\tau\right)$ is a ωb − topogical vector space.

Obviously. every topological vector space is an ωb − topogical vector space but the converse is not always true. The following are examples of ωb − topogical vector spaces which are not topological vector space.

Example 3.3. Consider the field $K=\mathbb{R}$ with standard topology. Let $X=\mathbb{R}$ be endowed with the topology $\tau=\{\phi,D,\mathbb{R}\}$, where D denotes the set of irrational numbers. We show that $\left(X_{(\mathbb{R})},\tau\right)$ is an $\omega b-TVS$. For this purpose, we have to prove the following:

(1) Let $x,y\in X$. If $x+y$ is rational, then the only open set in X containing $x+y$ is $\mathbb{R}$. So, there is nothing to prove.

If $x+y$ is irrational, then for open neighbourhood $W=D$ of $x+y$ in X, we have following cases:

Case (i). If both x and y are irrational, we can choose ωb − open sets $U=\{x\}$ and $V=\{y\}$ in X such that $U+V\subseteq W$.

Case (ii).If one of x or y is rational, say y. Then, for the selection of ωb − open sets $U=\{x\}$ and $V=\{p,y\}$ in X, where $p\in D$ such that $p+x\in D$, we have $U+V\subseteq W$.

This proves the first condition of the definition of $\omega\mathrm{b}-$topologoical vector space.

(2) Let $\lambda \in \mathbb{R}$ and $x \in X$. If λx is rational, then verification is straightforward.

Suppose λx is irrational. Let $W = D$ be an open neighbourhood of λx. Then the following cases arise:

Case (i). If λ and x are irrational, then, choose $\omega\mathrm{b}-$open sets $U = \left[(\lambda-\varepsilon, \lambda+\varepsilon) \cap Q\right] \cup \{\lambda\}$ in $\mathbb{R}$ and $V = \{x\}$ in X, we have $U.V \subseteq W$.

Case (ii). If λ is rational and x is irrational, then for the selection of $\omega\mathrm{b}-$open sets $U = (\lambda-\varepsilon, \lambda+\varepsilon) \cap Q$ in $\mathbb{R}$ and $V = \{x\}$ in X, we have $U.V \subseteq W$.

Case (iii). Finally, suppose λ is irrational and x is rational. Then for the choice of $\omega\mathrm{b}-$open sets, take $U = (\lambda-\varepsilon, \lambda+\varepsilon) \cap D$ in $\mathbb{R}$ and $V = \{x, p\}$ in X, where the selection of $p \in D$ is such that $p.q$ is irrational for each $q \in U$. Then we obtain $U.V \subseteq W$.

Hence $\left(X_{(\mathbb{R})}, \tau\right)$ is a $\omega\mathrm{b}-$topological vector space.

Example 3.4. Let $X = \mathbb{R}$ be a vector space of real numbers over the field $K = \mathbb{R}$ and let τ be a topology on X induced by open intervals (a, b) and $[1, c)$ where $a, b, c \in \mathbb{R}$. We observe that $\left(\mathbb{R}_{(\mathbb{R})}, \tau\right)$ is an $\omega\mathrm{b}-$topological vector space over the field $\mathbb{R}$ with the topology τ defined on $\mathbb{R}$. We note that for all $x, y \in \mathbb{R}$ and each open neighbourhood W of $x+y$ in X, there exist $\omega\mathrm{b}-$open neighbourhoods U and V of x and y respectively in X such that $U+V \subseteq W$.

Also for each $\lambda \in \mathbb{R}$, $x \in \mathbb{R}$ and for each open neighbourhood W of λx in X, there exist $\omega\mathrm{b}-$open neighbourhoods U of λ in $\mathbb{R}$ and V of x in X such that $U.V \subseteq W$. However, $\left(\mathbb{R}_{(\mathbb{R})}, \tau\right)$ is not a topological vector space because, for instance if, we choose $x = -3$, $y = 4$ and an open neighbourhood $W = [1, 2)$ of $x+y$ in X, we can not find open neighbourhoods U and V containing x and y respectively in X which satisfy the condition $U.V \subseteq W$.

Theorem 3.5. Let $\left(X_{(K)}, \tau\right)$ be an $\omega\mathrm{b}-$topological vector space. Suppose $T_x : X \to X$ is a translation mapping defined by $T_x(y) = y + x$ for each $y \in X$ $(\textit{where } x \textit{ is any fixed element of } X)$, and $M_\lambda : X \to X$ is a multiplication mapping defined by $M_\lambda(x) = \lambda x$, for each $x \in X$ $(\lambda \neq 0 \text{ is a fixed scalar})$. Then prove that T_x and M_x both are $\omega\mathrm{b}-$continuous.

Proof. Let y be an arbitrary element in X and let W be an open neighbourhood of $T_x(y) = y + x$. By definition of $\omega\mathrm{b}-$topological vector spaces, there exist $\omega\mathrm{b}-$open neighbourhoods U and V containing y and x respectively, such that $U+V \subseteq W$. In particular, we have $U + x \subseteq W$ which means $T_x(U) \subseteq W$. The inclusion shows that T_x is $\omega\mathrm{b}-$continuous at y. Hence T_x is $\omega\mathrm{b}-$continuous on X. Now we prove the statement for multiplication mapping. Let $\lambda \in K$ and $x \in X$. Let W be an open neighbourhood of $M_\lambda(x) = \lambda.x$. By definition of $\omega\mathrm{b}-$topological vector spaces, there exist $\omega\mathrm{b}-$open neighbourhoods U and V containing λ and x respectively, such that $U.V \subseteq W$. In particular, we have $\lambda.V \subseteq W$, which means $M_\lambda(V) \subseteq W$. The inclusion shows that M_x is $\omega\mathrm{b}-$continuous at x. Hence M_λ is $\omega\mathrm{b}-$continuous on X.

Theorem 3.6. Let A be any open subset of an $\omega\mathrm{b}-$topogical vector space $\left(X_{(\mathrm{K})}, \tau\right)$. Then the following statements are true:

(1). $x + A \in \omega b - O(X)$ for each $x \in X$,

(2). $\lambda.A \in \omega b - O(X)$ for each non-zero scalar λ.

Proof. (1). Let $y \in x + A$. Then there exists an $a \in A$ such that $y = x + a$. Hence, $a = (-x) + y \in A$. Since A is open set in X. Thus there exist $\omega\mathrm{b}-$open sets $U, V \in \omega b - O(X)$ containing $-x$ and y, respectively such that $U + V \subseteq A$. Therefore, $-x + V \subseteq A \Rightarrow$

$V \subseteq x+A \Rightarrow y \in \omega b-Int(x+A) \Rightarrow x+A \subseteq \omega b-Int(x+A)$. Hence it follows that $x+A = \omega b-Int(x+A)$. This proves that $x+A$ is an ωb−open set in X.

(2). Let $x \in \lambda A$. Then $x = \lambda a$ for some $a \in A$. Hence $a = \left(\frac{1}{\lambda}\right)x \in A$ and A is an open set in X. So by the definition of ωb−topological vector spaces, there exist ωb−open sets U in K containing $\frac{1}{\lambda}$ and V in X containing x such that $U.V \subseteq A$. Whence we find that $x \in V \subseteq \lambda.A$. This shows that $x \in \omega bInt(\lambda A)$. Since $x \in \lambda.A$ was arbitrary, it follows that $\lambda.A = \omega bInt(\lambda.A)$. Thus, $\lambda.A \in \omega b-O(X)$.

Corollary 3.7. Let A be any non-empty open subset of a ωb−topological vector space X. Then the following statements are true:

(i). $x+A \subseteq Int[Cl(Int(x+A))]$ for each $x \in X$.

(ii). $\lambda.A \subseteq Int[Cl(Int(\lambda.A))]$ for each non-zero scalar λ.

Corollary 3.3. If U is any non-empty open set and B is any set in an ωb−topological vector space X, then $B+U \in \omega b-O(X)$.

Theorem 3.4. Let (X,τ) be an ωb−topological vector space and let F be a closed subset of X. Then the following statements are true:

(1). $x+F \in \omega b-C(X)$ for each $x \in X$,

(2). $\lambda F \in \omega b-C(X)$ for each non-zero scalar λ.

Proof. (1). Suppose that $y \in \omega b-Cl(x+F)$. Consider $z = -x+y$ and let W be any open set in X containing z. Then there exists ωb−open sets U and V in X such that $-x \in U$, $y \in V$ and $U+V \subseteq W$. Since $y \in \omega b-Cl(x+F)$, $(x+F) \cap V \neq \phi$. So, there is an $a \in (x+F) \cap V$. Now, $-x+a \in F \cap (U+V) \subseteq F \cap W \Rightarrow F \cap W \neq \phi \Rightarrow z \in Cl(F) = F \Rightarrow y \in x+F$. It follows that $x+F = \omega b-Cl(x+F)$. This proves that $x+F$ is ωb−closed set in X.

(2). Assume that $x \in \omega b-Cl(\lambda.F)$ and let W be any open set in X containing $\frac{1}{\lambda}x$. Since X is ωb−topogical vector space, there exist ωb−open sets U in K containing $\frac{1}{\lambda}$ and V in X containing x such that $U.V \subseteq W$. Since $x \in \omega b-Cl(\lambda.F)$, there exists $a \in V \cap (\lambda.F)$. This implies that $\frac{1}{\lambda}a \in \left(\frac{1}{\lambda}\lambda F\right) \cap (U.V) \subseteq F \cap W \Rightarrow F \cap W \neq \phi \Rightarrow \frac{1}{\lambda}x \in Cl(F) = F \Rightarrow x \in \lambda F$. It proves that λF is ωb−closed in X.

Theorem 3.5. Let A and B be any subsets of a ωb−topogical vector space X. Then $\omega b-Cl(A)+\omega b-Cl(B) \subseteq Cl(A+B)$.

Proof. Let $z \in \omega b-Cl(A)+\omega b-Cl(B)$. Then $z = x+y$ for some $x \in \omega b-Cl(A)$ and $y \in \omega b-Cl(B)$. Let W be any open neighbourhood of z in X. By definition of ωb−topogical vector spaces, there exist ωb−open sets U and V in X containing x and y respectively, such that $U+V \subseteq W$. Since $x \in \omega b-Cl(A)$ and $y \in \omega b-Cl(B)$, there are $a \in A \cap U$ and $b \in B \cap V \Rightarrow a+b \in (A+B) \cap (U+V) \subseteq (A+B) \cap W \Rightarrow (A+B) \cap W \neq \phi$ and hence $z \in Cl(A+B)$. It follows that $\omega b-Cl(A)+\omega b-Cl(B) \subseteq Cl(A+B)$.

Theorem 3.6. Let A and B be any subsets of an ωb−topogical vector space X. Then $A+Int(B) \subseteq \omega b-Int(A+B)$.

Proof. Let $z \in A+Int(B)$ be arbitrary. Then $z = x+y$ for some $x \in A$ and $y \in Int(B)$. This results in $-x+z \in Int(B)$ and consequently by definition of

$\omega b-TVS$, there exist $\omega\text{b}-$open sets U and V in X containing $-x$ and z respectively, such that $U+V\subseteq Int(B)$. In particular, $-x+V\subseteq Int(B)$ $\Rightarrow$ $V\subseteq x+Int(B)\subseteq A+B.$ Since $V\in\omega b-O(X)$, $z\in V\subseteq\omega b-Int(A+B)$ $\Rightarrow z\in\omega b-Int(A+B).$ Thus $A+Int(B)\subseteq\omega b-Int(A+B).$

4 Characterizations of $\omega\text{b}-$TVS.

Theorem 4.1. Let A be any subset of an $\omega\text{b}-$topological vector space X. Then the following assertions are true:

(1). $\omega b-Cl(x+A)\subseteq x+Cl(A)$ for each $x\in X$.

(2). $x+\omega b-Cl(A)\subseteq Cl(x+A)$ for each $x\in X$.

(3). $x+Int(A)\subseteq\omega b-Int(x+A)$ for each $x\in X$.

(4). $Int(x+A)\subseteq x+\omega b-Int(A)$ for each $x\in X$.

Proof. (1). Let $y\in\omega b-Cl(x+A)$ and consider

$z=-x+y$ in X. Let W be any open set in X containing z. Since X is an $\omega\text{b}-$topological vector space, there exist $\omega\text{b}-$open sets U and V in X containing $-x$ and y respectively, such that $U+V\subseteq W$. Since $y\in\omega b-Cl(x+A)$, there is $a\in(x+A)\cap V.$ Then it follows that $-x+a\in(-x+x+A)\cap(U+V)=A\cap(U+V)\subseteq A\cap W$ showing that $A\cap W\neq\phi$ and hence $z\in Cl(A)$. That is, $y\in x+Cl(A).$ Therefore, $\omega b-Cl(x+A)\subseteq x+Cl(A).$

(2). Let $z\in x+\omega b-Cl(A)$. Then $z=x+y$ for some $y\in\omega b-Cl(A)$. Let W be an open neighbourhood of $x+y$ in X. Since X is $\omega\text{b}-$topological vector space, there exist $\omega\text{b}-$open sets U and V in X containing x and y respectively, such that $U+V\subseteq W$. Since $y\in\omega b-Cl(A)$, $A\cap V\neq\phi$. So, there is $a\in A\cap V$ and thus $x+a\in(x+A)\cap(U+V)\subseteq(x+A)\cap W$. Hence we have $(x+A)\cap W\neq\phi.$ Therefore it implies $z\in Cl(x+A)$. This proves the assertion.

(3). Let $y\in x+int(A)$. Then $y=x+a$ for some $a\in Int(A)$ and hence there exist $\omega\text{b}-$open sets U and V in X containing $-x$ and y respectively, such that $U+V\subseteq Int(A).$ Now, $-x+V\subseteq U+V\subseteq Int(A)\subseteq A$ implies that $V\subseteq x+A.$ Since V is $\omega\text{b}-$open set in X containing y, we must have $y\in\omega b-Int(x+A).$ Thus $x+Int(A)\subseteq\omega b-Int(x+A).$

(4). Let $z\in Int(x+A)$. Then $z=x+y$ for some $y\in A.$ Since X is $\omega\text{b}-$topological vector space, there exist $\omega\text{b}-$open sets U and V in X containing x and y respectively, such that $x+V\subseteq U+V\subseteq Int(x+A)\subseteq x+A$ implies that $x+V\subseteq x+\omega b-Int(A)$. Since $z=x+y\in x+V$, so $z\in x+\omega b-Int(A)$. Therefore it follows that $Int(x+A)\subseteq x+\omega b-Int(A)$.

Theorem 4.2. Let A be any subset of a $\omega\text{b}-$topological vector space X. Then the following assertions are true:

(a). $\omega b-Cl(\lambda.A)\subseteq\lambda.Cl(A)$ for each non-zero scalar λ,

(b). $\lambda.[\omega b-Cl(A)]\subseteq Cl(\lambda.A)$ for each non-zero scalar λ,

(c). $\lambda.Int(A)\subseteq\omega b-In(\lambda.A)$ for each non-zero scalar λ,

(d). $Int(\lambda.A)\subseteq\lambda.[\omega b-Int(A)]$ for each non-zero scalar λ,

Proof. (a). Assume that $x\in\omega b-Cl(\lambda A)$ and let W be any open set in X containing $\frac{1}{\lambda}x.$ Since X is a $\omega\text{b}-$topological vector space, there exist $\omega\text{b}-$open sets U in K containing $\frac{1}{\lambda}$ and V in X containing x such that $U.V\subseteq W$. Since $x\in\omega b-Cl(\lambda A)$, there exists $a \in V \cap (\lambda A)$. This implies that $\frac{1}{\lambda}a \in \left(\frac{1}{\lambda}\lambda A\right) \cap (U.V) \subseteq A \cap W \Rightarrow A \cap W \neq \phi \Rightarrow \frac{1}{\lambda}x \in Cl(A) \Rightarrow x \in \lambda.Cl(A)$. It proves that $\omega b - Cl(\lambda.A) \subseteq \lambda.Cl(A)$.

(b). Suppose that $B = \lambda A$ and $\lambda^* = \frac{1}{\lambda}$. Then, by (a). it follows that $\omega b - Cl(\lambda^* B) \subseteq \lambda^*.Cl(B) \Rightarrow \omega b - Cl\left(\frac{1}{\lambda}\lambda A\right) \subseteq \frac{1}{\lambda}.Cl(\lambda A) \Rightarrow \lambda.[\omega b - Cl(A)] \subseteq Cl(\lambda.A)$.

(c). Let $y = \lambda x$ for any $x \in Int(A)$. Then $x = \frac{1}{\lambda}y \in Int(A)$. Since X is $\omega\mathrm{b}$ – topogical vector space. So there exist $\omega\mathrm{b}$ – open sets U in K containing $\frac{1}{\lambda}$ and V in X containing y such that $U.V \subseteq Int(A)$. Then it implies that $x = \frac{1}{\lambda}y \in \frac{1}{\lambda}V \subseteq U.V \subseteq Int(A) \subseteq A \Rightarrow y \in V \subseteq \lambda.A.$ Since V is $\omega\mathrm{b}$ – open set. Hence $y \in \omega b - Int(\lambda A)$. Therefore it follows that $\lambda.Int(A) \subseteq \omega b - In(\lambda.A)$.

(d). Suppose that $B = \lambda A$ and $\lambda^* = \frac{1}{\lambda}$. Then by (c), it follows that $\lambda^*[Int(B)] \subseteq \omega b - Int(\lambda^*.B) \Rightarrow \frac{1}{\lambda}[Int(\lambda.A)] \subseteq \omega b - Int(A) \Rightarrow Int(\lambda.A) \subseteq \lambda.[\omega b - Int(A)]$.

Theorem 4.3. Let A be any subset of an $\omega\mathrm{b}$ – topogical vector space X. Then $\omega b - Cl[x + Cl(A)] \subseteq x + Cl(A)$ for each $x \in X$.

Proof. Assume $y \in \omega b - Cl[x + Cl(A)]$. Consider $z = -x + y$ and let W be an open neighbourhood of $z \in X$. Then there exist $\omega\mathrm{b}$ – open neighbourhoods U and V of $-x$ and y in X respectively, such that $U + V \subseteq W$. Then $-x + V \subseteq U + V \subseteq W \Rightarrow V \subseteq x + W$. Since $y \in \omega b - Cl[x + Cl(A)]$, we have $V \cap [x + Cl(A)] \neq \phi,$ which implies that $(x + W) \cap [x + Cl(A)] \neq \phi \Rightarrow W \cap Cl(A) \neq \phi$. Since W is open, $W \cap A \neq \phi$. Hence $z \in Cl(A)$, that is $-x + y \in Cl(A) \Rightarrow y \in x + Cl(A)$. Consequently, $\omega b - Cl[x + Cl(A)] \subseteq x + Cl(A)$.

Theorem 4.4. Let A be any subset of an $\omega\mathrm{b}$ – topogical vector space X. Then prove that $x + Int(A) \subseteq \omega b - Int[x + \omega b - Int(A)]$ for each $x \in X$.

Proof. Let $y \in x + Int(A)$. Then $y = x + a$ for some $a \in Int(A)$. As a result of this, we get $\omega\mathrm{b}$ – open sets U and V in X containing $-x$ and y respectively, such that $U + V \subseteq Int(A)$. Now $-x + V \subseteq U + V \subseteq Int(A) \subseteq \omega b - Int(A)$. Then it implies $V \subseteq x + \omega b - Int(A)$. Since V is $\omega\mathrm{b}$ – open set in X such that $y \in V$, we have that $y \in \omega b - Int[x + \omega b - Int(A)]$ proving that $x + Int(A) \subseteq \omega b - Int[x + \omega b - Int(A)]$.

Definition 4.5. Let B be a subset of a space X. A collection $\{U_i : i \in I\}$ of $\omega\mathrm{b}$ – open sets of X is called a $\omega\mathrm{b}$ – open cover of B if $B \subseteq \bigcup_{i \in I} U_i$. A topological space X is said to be $\omega\mathrm{b}$ – compact if every $\omega\mathrm{b}$ – open cover of X has a finite sub cover. A subset B of X is said to be $\omega\mathrm{b}$ – compact relative to X if every cover of X by $\omega\mathrm{b}$ – open sets of X has a finite sub cover.

Theorem 4.6. Let A be any $\omega\mathrm{b}$ – compact set in a $\omega\mathrm{b}$ – topogical vector space X. Then $x + A$ is compact, for each $x \in X$.

Proof. Let $\Psi = \{U_i : i \in I\}$ be an open cover of $x + A$. Then $A \subseteq \bigcup_{i \in I}(-x + U_i)$. By hypothesis and Theorem 3.6, $A \subseteq \bigcup_{i \in I_0}(-x + U_i)$ for some finite subset I_0 of I. Hence we find that $x + A \subseteq \bigcup_{i \in I_0} U_i$. Therefore, it follows that $x + A$ is compact.

Theorem 4.7. Let X be an ωb – topological vector space. Then scalar multiple of any ωb – compact subset of X is compact.

Proof. Suppose that A is a ωb – compact set in X. Let λ be any scalar. If $\lambda = 0$, we are done. Assume that $\lambda \neq 0$. We have to show that $\lambda.A$ is compact. Let $\Psi = \{U_i : i \in I\}$ be any open cover of $\lambda.A$. Then $\lambda.A \subseteq \cup\{U_i : i \in I\}$ $\Rightarrow A \subseteq \frac{1}{\lambda}.\cup\{U_i : i \in I\}$ $\Rightarrow$ $A \subseteq \cup\left\{\frac{1}{\lambda}U_i : i \in I\right\}$. Since U_i is open in X, so by Theorem 3.6, $\frac{1}{\lambda}U_i$ is ωb – open set in X. Consequently, $\left\{\frac{1}{\lambda}U_i : i \in I\right\}$ is an ωb – open cover of A. But A is ωb – compact, there exists a finite subset I_0 of I such that $A \subseteq \cup\left\{\frac{1}{\lambda}U_i : i \in I_0\right\}$. It implies that $\lambda.A \subseteq \cup\{U_i : i \in I_0\}$. Hence $\lambda.A$ is ωb – compact.

Theorem 4.8. Let X be an ωb – topological vector space and Y be a topological vector space over the same field K. Let $f : X \to Y$ be a linear map such that f is continuous at 0. Then f is ωb – continuous everywhere.

Proof. Let x be any non-zero element of X and V be any open set in Y containing $f(x)$. Since the translation of an open set in a topological vector space is open, $V - f(x)$ is open set in Y containing 0. Since f is continuous at 0, there exists an open set U in X containing 0 such that $f(U) \subseteq V - f(x)$.

Furthermore, linearity of f implies that $f(x+U) \subseteq V$. By theorem 3.6, $x+U$ is ωb – open and hence f is ωb – continuous at x. By hypothesis, f is ωb – continuous at 0. This reflects that f is ωb – continuous.

Corollary 4.9. Let X be an ωb – topological vector space over the field K. Let $f : X \to K$ be a linear functional which is continuous at 0. Then the set $F = \{x \in X : f(x) = 0\}$ is ωb – closed.

5 Almost ωb – Topological Vector Spaces

In this section, we define almost ωb – topological vector spaces and investigate their relationships with certain other types of spaces. Some general properties of almost ωb – topological vector spaces are also discussed.

Definition 5.1. Let X be a vector space over the field K, where $K = R$ or C with standard topology. Let τ be a topology on X such that the following conditions are satisfied:

(1). For each $x, y \in X$ and each regular open set $W \subseteq X$ containing $x + y$, there exist ωb – open sets U and V in X containing x and y respectively such that $U + V \subseteq W$, and

(2). For each $x \in X$ and every $\lambda \in K$ and each regular open set $W \subseteq X$ containing λx, there exist ωb – open sets U in K containing λ and V in X containing x such that $U.V \subseteq W$.

Then the pair $(X_{(K)}, \tau)$ is called an almost ωb – topological vector space

Theorem 5.2. Let A be any δ – open subset of an almost ωb – topological vector space X. Then the following statements are true:

(1). $x + A \in \omega b - O(X)$ for each $x \in X$

(2). $\lambda A \in \omega b - O(X)$ for each non-zero scalar λ.

Proof. (1). Let $y \in x + A$. Then $y = x + a$ for some $a \in A$. Since A is δ – open set in X, there exists a regular open set W in X such that $a \in W \subseteq A$. This implies that $-x + y \in W$. Since X is an almost ωb – topological vector space, there exist ωb – open sets U and V in X such that $-x \in U$, $y \in V$ and $U + V \subseteq W \Rightarrow$ $U + V \subseteq A$ $\Rightarrow$ $-x + V \subseteq A$ $\Rightarrow$ $V \subseteq x + A$. Since V is ωb – open. Therefore

$y \in \omega b - Int(x+A)$. This shows that $x+A = \omega b - Int(x+A)$. Hence $x+A \in \omega b - O(X)$.

(2). Let $x \in \lambda A$ be an arbitrary element. Since A is $\delta-$open set in X, there exists a regular open set W in X such that $\frac{1}{\lambda}x \in W \subseteq A$. By definition of almost ωb$-$topological vector spaces, there exist ωb$-$open sets U in the topological field K containing $\frac{1}{\lambda}$ and V in X containing x such that $U.V \subseteq W$. Then $\frac{1}{\lambda}V \subseteq U.V \subseteq W \subseteq A$ implies that $x \in V \subseteq \lambda A$. Since V is ωb$-$open. Therefore $x \in \omega b - Int(\lambda A)$ and hence $\lambda A = \omega b - Int(\lambda A)$. Thus λA is ωb$-$open in X; that is, $\lambda A \in \omega b - O(X)$.

Corollary 5.3. For any $\delta-$open set A in an almost ωb$-$topological vector space X, the following statements are valid:

(1). $x + A \subseteq Cl\left[Int(X+A)\right]$ for each $x \in X$,

(2). $\lambda A \subseteq Cl\left[Int(\lambda A)\right]$ for each non-zero scalar λ.

Theorem 5.4. Let A be any $\delta-$open set in an almost ωb$-$topological vector space X. Then $A + B \in \omega b - O(X)$ for any subset $B \subseteq X$.

Proof. Follows directly from Theorem 5.2.

Theorem 5.5. Let B be any $\delta-$closed closed set in an almost ωb$-$topological vector space X. Then the following statements are true:

(1). $x + B \in \omega b - C(X)$ for each $x \in X$,

(2). $\lambda B \in \omega b - C(X)$ for each non-zero scalar λ.

Proof. (1). We need to show that $x + B = \omega b - Cl(x+B)$. For, let $y \in \omega b - Cl(x+B)$ be an arbitrary element. Fix $z = -x + y$. Let W be any $\delta-$open set in X containing z. By definition of $\delta-$open sets, there is a regular open set G in X such that $z \in G \subseteq W$. Then there exist $U, V \in \omega b - O(X)$ such that $-x \in U$, $y \in V$ and $U + V \subseteq G$. Since $y \in \omega b - Cl(x+B)$, then by definition, $(x+B) \cap V \neq \phi$. Let $a \in (x+B) \cap V$. Then it implies $-x + a \in B \cap (-x+V) \subseteq B \cap (U+V) \subseteq B \cap G \subseteq B \cap W$. This shows that $B \cap W \neq \phi$. Thus shows that $z = -x + y \in \delta Cl(B)$. Since B is $\delta-$closed set, we have $z = -x + y \in B$. This implies $y \in x + B$. Thus $x + B = \omega b - Cl(x+B)$. Hence $x + B \in \omega b - C(X)$.

(2). We have to prove that $\lambda B = \omega b - Cl(\lambda B)$. For, let $x \in \omega b - Cl(\lambda B)$ be arbitrary and let W be any $\delta-$open set in X containing $\lambda^{-1}x$. By definition, there is a regular open set G in X such that $\lambda^{-1}x \in G \subseteq W$. Then there exist ωb$-$open sets U containing λ^{-1} in topological field K and V containing x in X such that $U.V \subseteq G$. Since $x \in \omega b - Cl(\lambda B)$, then there is an $a \in (\lambda B) \cap V$. Now $\lambda^{-1}a \in B \cap (\lambda^{-1}.V) \subseteq B \cap (U.V) \subseteq B \cap G \subseteq B \cap W$. This implies that $B \cap W \neq \phi$. Therefore $\lambda^{-1}x \in \delta Cl(B) = B$. This shows that $x \in \lambda B$. Hence $\lambda B = \omega b - Cl(\lambda B)$. Thus $\lambda B \in \omega b - C(X)$.

Theorem 5.6. Let A be a subset of an almost ωb$-$topological vector space X. Then the following assertions hold:

(1). $x + \omega b - Cl(A) \subseteq \delta Cl(x+A)$ for each $x \in X$,

(2). $\lambda\left[\omega b - Cl(A)\right] \subseteq \delta Cl(\lambda A)$ for each non-zero scalar λ.

Proof. (1). Let $z \in x + \omega b - Cl(A)$. Then $z = x + y$ for some $y \in \omega b - Cl(A)$. Let W be an open set in X containing z, then $z \in W \subseteq Int\left[Cl(W)\right]$. Since X is an almost ωb$-$topological vector space, then there exist $U, V \in \omega b - O(X)$ containing x and y respectively such that $U + V \subseteq Int\left[Cl(W)\right]$. Since $y \in \omega b - Cl(A)$, then there is some $a \in A \cap V$. Hence $x + a \in (x+A) \cap (U+V) \subseteq (x+A) \cap Int\left[Cl(W)\right]$.

This implies $(x+A)\cap Int[Cl(W)]\neq\phi$. Thus $z\in\delta Cl(x+A)$. Therefore it follows that $x+\omega b-Cl(A)\subseteq\delta Cl(x+A)$.

(2). Let $x\in\omega b-Cl(A)$ and let W be an open set in X containing λx. Then $\lambda x\in W\subseteq Int[Cl(W)]$,

so there exist ωb−open sets U containing λ in topological field K and V containing x in X such that $U.V\subseteq Int[Cl(W)]$. Since $x\in\omega b-Cl(A)$, then there is some $b\in A\cap V$. Therefore it follows immediately $\lambda b\in(\lambda A)\cap(\lambda V)\subseteq(\lambda A)\cap(U.V)\subseteq$ $(\lambda A)\cap Int[Cl(W)] \Rightarrow (\lambda A)\cap Int[Cl(W)]\neq\phi$. This implies that $\lambda x\in\delta Cl(\lambda A)$. Consequently, it shows that $\lambda[\omega b-Cl(A)]\subseteq\delta Cl(\lambda A)$.

Theorem 5.7. Let A be a subset of an almost ωb−topological vector space X. Then the following statements are true:

(1). $\omega b-Cl(x+A)\subseteq x+\delta Cl(A)$ for each $x\in X$,

(2). $\omega b-Cl(\lambda A)\subseteq\lambda[\delta Cl(A)]$ for each non-zero scalar λ.

Proof. (1). Let $y\in\omega b-Cl(x+A)$ and let W be an open set in X containing $-x+y$. Since X is an almost ωb−topological vector space and $W\subseteq Int[Cl(W)]$, there exist ωb−open sets U and V in X such that $-x\in U$, $y\in V$ and $U+V\subseteq Int[Cl(W)]$. Since $y\in\omega b-Cl(x+A)$, there is some $a\in(x+A)\cap V$ and hence $-x+a\in A\cap(U+V)\subseteq A\cap Int[Cl(W)] \Rightarrow A\cap Int[Cl(W)]\neq\phi \Rightarrow -x+y\in\delta Cl(A) \Rightarrow y\in x+\delta Cl(A)$. Thus $\omega b-Cl(x+A)\subseteq x+\delta Cl(A)$.

(2). Let $x\in\omega b-Cl(\lambda A)$ and W be an open set in X containing $\lambda^{-1}x$. So there exist ωb−open sets U in topological field K containing λ^{-1} and V in X containing x such that $U.V\subseteq Int[Cl(W)]$. As $x\in\omega b-Cl(\lambda.A)$, $(\lambda.A)\cap V\neq\phi$. Let $z\in(\lambda A)\cap V$. Then we have $\lambda^{-1}z\in A\cap(\lambda^{-1}V)\subseteq A\cap(U.V)\subseteq$ $A\cap Int[Cl(W)] \Rightarrow A\cap Int[Cl(W)]\neq\varphi$. Therefore $\lambda^{-1}x\in\delta Cl(A)$ and hence $x\in\lambda[\delta Cl(A)]$. Thus $\omega b-Cl(\lambda A)\subseteq\lambda[\delta Cl(A)]$.

Theorem 5.8. Let A be an open set in an almost ωb−topological vector space X. Then the following assertions hold:

(1). $\omega b-Cl(x+A)\subseteq x+Cl(A)$ for each $x\in X$,

(2). $\omega b-Cl(\lambda A)\subseteq\lambda[Cl(A)]$ for each non-zero scalar λ.

Proof. (1). Let $y\in\omega b-Cl(x+A)$ and let W be an open set in X containing $-x+y$. Since X is an almost ωb−topological vector space and $W\subseteq Int[Cl(W)]$, there exist ωb−open sets U and V in X such that $-x\in U$, $y\in V$ and $U+V\subseteq Int[Cl(W)]$. Since $y\in\omega b-Cl(x+A)$, there is an $a\in(x+A)\cap V$. Now $-x+a\in A\cap(-x+V)\subseteq A\cap(U+V)\subseteq A\cap Int[Cl(W)]$ $\Rightarrow A\cap Int[Cl(W)]\neq\phi \Rightarrow A\cap Cl(W)\neq\phi$. Since A is open, $A\cap W\neq\phi$. Therefore it follows that $-x+y\in Cl(A)$; that is, $y\in x+Cl(A)$. Thus it shows that $\omega b-Cl(x+A)\subseteq x+Cl(A)$.

(2). Let $y\in\omega b-Cl(\lambda A)$ and W be an open set in X containing $\lambda^{-1}y$. Then there exist ωb−open sets U in topological field K containing λ^{-1} and V in X containing y such that $U.V\subseteq Int[Cl(W)]$. As $y\in\omega b-Cl(\lambda A)$, there is some $b\in(\lambda A)\cap V$. Thus $\lambda^{-1}b\in A\cap(\lambda^{-1}V)\subseteq A\cap(U.V)\subseteq A\cap Int[Cl(W)] \Rightarrow$ $A\cap Int[Cl(W)]\neq\varphi \Rightarrow A\cap Cl(W)\neq\varphi$. Since A is open, so $A\cap W\neq\phi$. Therefore it follows that $\lambda^{-1}y\in Cl(A)\Rightarrow y\in\lambda[Cl(A)]$. Consequently, we obtain $\omega b-Cl(\lambda A)\subseteq\lambda[Cl(A)]$.

Theorem 5.9. Let A and B be subsets of an almost ωb−topological vector space X. Prove that

$\omega b-Cl(A)+\omega b-Cl(B)\subseteq \delta Cl(A+B)$.

Proof. Let $x\in \omega b-Cl(A)$ and $y\in \omega b-Cl(B)$. Let W be an open neighbourhood of $x+y$ in X. Since $W\subseteq Int[Cl(W)]$ and $Int[Cl(W)]$ is regular open, there exist $U,V\in \omega b-O(X)$ such that $x\in U$, $y\in V$ and $U+V\subseteq Int[Cl(W)]$. Since $x\in \omega b-Cl(A)$ and $y\in \omega b-Cl(B)$, there are $a\in A\cap U$ and $b\in B\cap V$. So $a+b\in (A+B)\cap (U+V)\subseteq (A+B)\cap Int[Cl(W)] \Rightarrow (A+B)\cap Int[Cl(W)]\neq \phi$. Thus $x+y\in \delta Cl(A+B)$; That is, $\omega b-Cl(A)+\omega b-Cl(B)\subseteq \delta Cl(A+B)$.

Theorem 5.10. Let A be any subset of an almost ωb-topological vector space X. Then the following assertions are true:

(1). $\delta Int(x+A)\subseteq x+\omega b-Int(A)$ for each $x\in X$,

(2). $x+\delta Int(A)\subseteq \omega b-Int(x+A)$ for each $x\in X$,

Proof. (1). Let $y\in \delta Int(x+A)$. We know that $\delta Int(x+A)$ is δ-open. Then for each $y\in \delta Int(x+A)$, there exist a regular open set W in X such that $y\in W\subseteq \delta Int(x+A)$. Since $y\in \delta Int(x+A)\subseteq (x+A)$, $y=x+a$ for some $a\in A$. Since X is an almost ωb-topological vector space, then there exist ωb-open sets U and V in X containing x and a respectively such that $U+V\subseteq W$. Thus it follows that $x+V\subseteq W\subseteq \delta Int(x+A)\subseteq x+A$. Hence we have $V\subseteq (-x)+x+A=A$. Since V is ωb-open, then $V\subseteq \omega b-Int(A)$ and therefore $a\in \omega b-Int(A) \Rightarrow -x+y\in \omega b-Int(A) \Rightarrow y\in x+\omega b-Int(A)$. It shows that $\delta Int(x+A)\subseteq x+\omega b-Int(A)$.

(2). Let $y\in x+\delta Int(A)$. Then there exists a regular open set W in X such that $-x+y\in W\subseteq \delta Int(A)\subseteq A$. By definition of almost ωb-topological vector spaces, there exist ωb-open sets U and V in X containing $-x$ and y respectively, such that $U+V\subseteq W$. Thus $-x+V\subseteq U+V\subseteq W \Rightarrow V\subseteq x+W\subseteq x+A$. Sine V is ωb-open set. Thus, $y\in \omega b-Int(x+A)$. Hence it follows that $x+\delta Int(A)\subseteq \omega b-Int(x+A)$.

Theorem 5.11. Let A be a subset of an almost ωb-topological vector space X. Then the following assertions are true:

(1). $\delta Int(\lambda A)\subseteq \lambda[\omega b-Int(A)]$, for any nonzero scalar λ.

(2). $\lambda[\delta Int(A)]\subseteq \omega b-Int(\lambda A)$, for each nonzero scalar λ.

Proof. (1). Let $y\in \delta Int(\lambda A)$. Then there exists a regular open set W in X such that $y\in W\subseteq \delta Int(\lambda .A)$. Then $y=\lambda a$ for some $a\in A$. By definition of almost ωb-topological vector spaces, there exist ωb-open sets U in the topological field K containing λ and V in X containing a respectively, such that $U.V\subseteq W$. Now $y=\lambda a\in \lambda V\subseteq U.V\subseteq W\subseteq \delta Int(\lambda A)\subseteq \lambda A \Rightarrow a\in V\subseteq A$. Since V is ωb-open set in X. Hence $a\in \omega b-Int(A) \Rightarrow y=\lambda a\in \lambda[\omega b-Int(A)]$. Therefore it shows that $\delta Int(\lambda A)\subseteq \lambda[\omega b-Int(A)]$,

(2). Let $y\in \lambda[\delta Int(A)]$. Then $y=\lambda a$ for some $a\in \delta Int(A)$ and $\delta Int(A)$ is δ-open in X. Thus there exists a regular open set W in X such that $a=\lambda^{-1}y\in W\subseteq \delta Int(A)$. By definition of almost ωb-topological vector spaces, there exist ωb-open sets U in topological field K containing λ^{-1} and V in X containing y respectively such that $U.V\subseteq W$. Now $a=\lambda^{-1}y\in \lambda^{-1}V\subseteq U.V\subseteq W \subseteq \delta Int(A)\subseteq A \Rightarrow y\in V\subseteq \lambda A$. Since V is ωb-open set. It follows that $y\in \omega b-Int(\lambda A)$. Therefore it shows that $\lambda[\delta Int(A)]\subseteq \omega b-Int(\lambda A)$,

Theorem 5.12. Let X be an almost ωb-topological vector space. Then the following assertions are true:

(1). The translation mapping $T_x : X\to X$ defined by $T_x(y)=x+y$, $\forall x,y\in X$, is almost ωb-continuous.

(2). The multiplication mapping $M_\lambda : X \to X$ defined by $M_\lambda(y) = \lambda x,\ \forall x \in X$, is almost $\alpha-$continuous, where λ is non-zero fixed scalar in K.

Proof. (1). Let $y \in X$ be an arbitrary element. Let W be any regular open set in X containing $T_x(y)$. Then, by definition of almost ωb$-$topological vector spaces, there exist ωb$-$open sets U in X containing x and V in X containing y such that $U+V \subseteq W$. This results in $x+V \subseteq W \Rightarrow T_x(V) \subseteq W$. This indicates that T_x is almost ωb$-$continuous at y and hence T_x is almost ωb$-$continuous.

(2). Let $x \in X$ and W be any regular open set in X containing x. Then there exist ωb$-$open sets U in the topological field K containing λ and V in X containing x such that $U.V \subseteq W$. This gives that $\lambda V \subseteq W$. This means that $M_\lambda(V) \subseteq W$ showing that M_λ is almost ωb$-$continuous. at x. Since $x \in X$ was an arbitrary element, it follows that M_λ is almost ωb$-$continuous.

6 Conclusion

Topological vector spaces are a fundamental notion and play an important role in various advanced branches of mathematics like fixed point theory, operator theory, etc. This paper expounds ωb$-$topological vector spaces and the almost ωb$-$topological vector spaces which are basically a generalization of topological vector spaces. This paper makes us familiar with ωb$-$topological vector spaces as well as with the origin of almost ωb$-$topological vector spaces and the elementary concepts that are used to develop the theory of almost ωb$-$topological vector spaces. We investigate several new properties and characterizations of ωb$-$topological vector spaces and almost ωb$-$topological vector spaces. We explore some more basic features of ωb$-$topological vector spaces and almost ωb$-$topological vector spaces and interprets their relationships with some well-known existing spaces.

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