Wright-type generalized coherent states

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Abstract: In this paper we construct a new generalization of coherent states based on the application of Wright functions. This explicit family of coherent states is based on the generalization of the classical coherent states by using a positive weight function. We analyze in detail by means of the Mandel parameter, the deviations from the conventional coherent states due to this generalization that leads to sub- or super-Poissonian behaviour. We also discuss the connection between generalized coherent states and weighted Poisson distributions. Finally, we briefly show the relation between the normalizing function here used and the solution of a fractional differential equation with variable coefficients.

Key–Words: Generalized coherent states, Wright functions, Weighted Poisson distributions

1 Introduction

Coherent states play a central role in quantum mechanics and quantum optics, to name but a few. A general definition of coherent states was proposed by Klauder by requiring, among others, the property of completeness. See Refs. [4, 5, 6, 9, 10] for details. For the sake of clarity, some properties on the quantum harmonic oscillator and conventional coherent states are reported below.

Coherent states are introduced as state kets $|z\rangle$ which belong to the Hilbert space $\mathcal{H}$ of a quantum harmonic oscillator, for every complex value of the variable z, and are characterized by minimum certainty [6].These states are represented in the Fock basis as below,

$$|z\rangle = \left(U\left(|z|^2\right)\right)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle \quad (1)$$

The state kets $|0\rangle$, $|1\rangle$, $\ldots$, represent the Fock states. These states form an orthonormal and complete set of eigenstates of the Hamiltonian operator over the Hilbert space $\mathcal{H}$. Every Fock state is related to the ground state $|0\rangle$ of the harmonic oscillator via the rising operator $a^\dagger$ in the following way, $|n\rangle = \left(a^\dagger\right)^n |0\rangle/n!$, for every $n = 0, 1, \ldots$. Refer to [10, 9] for details about the general theory of the quantum harmonic oscillator. The normalization of the state $|z\rangle$ to unity determines the function U,

$$U\left(|z|^2\right) = \exp\left(|z|^2\right) \quad (2)$$

In Ref. [12] the conventional coherent states are generalized by substituting the exponential function with the generalized Mittag-Leffler function,

$$|z;\alpha,\beta\rangle = \left(E_{\alpha,\beta}\left(|z|^2\right)\right)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_{\alpha,\beta}!}}|n\rangle \quad (3)$$

for every $\alpha, \beta > 0$. The generalization $[n]_{\alpha,\beta}!$ of the factorial is performed via the box function $[x]_{\alpha,\beta}$ which is defined as below,

$$[x]_{\alpha,\beta} \equiv \frac{\Gamma(\alpha x + \beta)}{\Gamma(\alpha(x-1)+\beta)}. \quad (4)$$

The probability $p_{\alpha,\beta}(n,z)$ that the state ket $|z;\alpha,\beta\rangle$ consists in n excitations results to be

$$p_{\alpha,\beta}(n,z) = \frac{|z|^{2n}\,\Gamma(\beta)}{\Gamma(\alpha n+\beta)\,E_{\alpha,\beta}\left(|z|^2\right)} \quad (5)$$

This approach suggests the following generalization of the destruction operator a and creation operator $a^\dagger$, respectively,

$$b_{\alpha,\beta}|n\rangle = \sqrt{[n]_{\alpha,\beta}}|n-1\rangle, \quad n = 1,2,3,\ldots, \quad (6)$$

$$b^\dagger_{\alpha,\beta}|n\rangle = \sqrt{[n+1]_{\alpha,\beta}}|n+1\rangle, \quad n = 0,1,2,. \quad (7)$$

The resulting commutation rule is deformed as below [13],

$$b_{\alpha,\beta}b^\dagger_{\alpha,\beta} - b^\dagger_{\alpha,\beta}b_{\alpha,\beta} = [n+1]_{\alpha,\beta} - [n]_{\alpha,\beta} \quad (8)$$

See Refs. [13, 12] for details.

In this paper, we consider a different generalization, based on the application of Wright functions, a class of special functions that play a key-role for example in the analysis of fractional diffusion equations, we refer for example to [7].

We discuss the meaning and possible utility of this new generalization by means of the Mandel parameter. We recall that there is a wide literature about generalized coherent states by using special functions such as Bessel functions (see e.g. [11]). These works represent an interesting bridge between the theory of special functions and the possible applications in quantum physics. In the last part of this note, we also provide a new connection between generalized coherent states and weighted Poisson distribution used in statistics in order to take into account deviations from purely Poissonian counting distributions. Finally, we show the relation between the normalizing function used in our model and a fractional equation with variable coefficients.

1.1 Wright coherent states

Instead of the Mittag-Leffler function, a generalization of the coherent states can be performed by adopting the Wright function [7],

$$W_{\lambda,\mu}\left(z\right)=\sum_{n=0}^{\infty}\frac{z^{n}}{n!\Gamma\left(\lambda n+\mu\right)} \tag{9}$$

in case $\lambda,\mu>0$ and for every $z\in C$. The W coherent state $|z;\lambda,\mu\rangle$ is defined by the expression below,

$$|z;\lambda,\mu\rangle=\left(W_{\lambda,\mu}\left(|z|^{2}\right)\right)^{-1/2}\sum_{n=0}^{\infty}\frac{z^{n}}{\sqrt{[n]_{\lambda,\mu}!}}|n\rangle \tag{10}$$

for every $\lambda,\mu>0$. The factorial is generalized by the term $[n]_{\lambda,\mu}!$ which is defined as follows,

$$[n]_{\lambda,\mu}!=[1]_{\lambda,\mu}\ldots[n]_{\lambda,\mu}=n!\Gamma\left(\lambda n+\mu\right).$$

In this case the box function $[x]_{\lambda,\mu}$ reads

$$[x]_{\lambda,\mu}\equiv x\frac{\Gamma\left(\lambda x+\mu\right)}{\Gamma\left(\lambda\left(x-1\right)+\mu\right)} \tag{11}$$

The probability $p_{\lambda,\mu}\left(n,z\right)$ that the state ket $|z;\lambda,\mu\rangle$ consists in n excitations results to be

$$p_{\lambda,\mu}\left(n,z\right)=\frac{|z|^{2n}}{n!\Gamma\left(\lambda n+\mu\right)W_{\lambda,\mu}\left(|z|^{2}\right)} \tag{12}$$

The destruction and creation operators are generalized as follows,

$$A_{\lambda,\mu}|n\rangle=\sqrt{[n]_{\lambda,\mu}}|n-1\rangle,\quad n=1,2,3,\ldots, \tag{13}$$

$$A^{\dagger}_{\lambda,\mu}|n\rangle=\sqrt{[n+1]_{\lambda,\mu}}|n+1\rangle,\quad n=0,1,2, \tag{14}$$

The deformed commutation rule reads

$$A_{\lambda,\mu}A^{\dagger}_{\lambda,\mu}-A^{\dagger}_{\lambda,\mu}A_{\lambda,\mu}=[n+1]_{\lambda,\mu}-[n]_{\lambda,\mu}\,. \tag{15}$$

The expectation values of the observable position and momentum of the harmonic oscillators are evaluated over the Wright coherent states via a list of expressions which are reported below. For the sake of shortness, let the quantities $\langle a\rangle_z$, $\langle a^2\rangle_z$, $\langle a^{\dagger}a\rangle_z$ represent the amplitudes which are evaluated over the Wright coherent state (10). These quantities read

$$\langle a\rangle_z=\frac{z}{W_{\lambda,\mu}\left(|z|^{2}\right)}\times \tag{16}$$

$$\times\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!\sqrt{\Gamma\left(\lambda n+\mu\right)\Gamma\left(\lambda(n+1)+\mu\right)}},$$

$$\langle a^2\rangle_z=\frac{z^2}{W_{\lambda,\mu}\left(|z|^{2}\right)}\times \tag{17}$$

$$\times\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!\sqrt{\Gamma\left(\lambda n+\mu\right)\Gamma\left(\lambda(n+2)+\mu\right)}},$$

$$\langle a^{\dagger}a\rangle_z=\frac{|z|^2}{W_{\lambda,\mu}\left(|z|^{2}\right)}\times \tag{18}$$

$$\times\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!\Gamma\left(\lambda(n+1)+\mu\right)},$$

$$\langle\left(a^{\dagger}\right)^{2}a^2\rangle_z=\frac{|z|^4}{W_{\lambda,\mu}\left(|z|^{2}\right)}\times \tag{19}$$

$$\times\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!\Gamma\left(\lambda(n+2)+\mu\right)}.$$

Along with the relation $\langle aa^{\dagger}\rangle_z=1+\langle a^{\dagger}a\rangle_z$, the above forms allow the evaluation of the variances of position, $\langle\Delta x^2(t)\rangle_z$, and momentum, $\langle\Delta p^2(t)\rangle_z$ of the harmonic oscillator in the Heisenberg picture,

$$\langle\Delta x^2(t)\rangle_z=\frac{\hbar}{2m\omega}\Bigg(2Re\Big\{\left(\langle a^2\rangle_z-(\langle a\rangle_z)^2\right)\times$$

$$\times\exp\left(-2\imath\omega t\right)\Big\}+1+2\left(\langle a^{\dagger}a\rangle_z-\langle a^{\dagger}\rangle_z\langle a\rangle_z\right)\Bigg),$$

$$\langle\Delta p^2(t)\rangle_z=\frac{-\hbar m\omega}{2}\Bigg(2Re\Big\{\left(\langle a^2\rangle_z-(\langle a\rangle_z)^2\right)\times$$

$$\times\exp\left(-2\imath\omega t\right)\Big\}-1-2\left(\langle a^{\dagger}a\rangle_z-\langle a^{\dagger}\rangle_z\langle a\rangle_z\right)\Bigg),$$

The parameters m and ω represent mass and frequency of the harmonic oscillator, respectively. The correlation between position and momentum reads

$$\langle\Delta x_z(t)\Delta p_z(t)+\Delta p_z(t)\Delta x_z(t)\rangle_z=$$

$$=2\hbar Im\Big\{\left(\langle a^2\rangle_z-(\langle a\rangle_z)^2\right)\exp\left(-2\imath\omega t\right)\Big\}$$

where

$$\Delta x_z(t) \equiv x(t) - \langle x(t)\rangle_z, \quad \Delta p_z(t) \equiv p(t) - \langle p(t)\rangle_z.$$

Notice that vanishing correlations are obtained in case the quantity $\left(\langle a^2\rangle_z - (\langle a\rangle_z)^2\right)$ vanishes.

1.2 Mandel Q parameter

The Mandel Q parameter has been introduced in Quantum Optics [8] to estimate the departure of the distribution of excitations from the Poisson statistic. This factor is defined as follows,

$$Q \equiv \frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle} - 1, \tag{20}$$

where N is the number operator. For the Wright coherent states (10), the corresponding Mandel Q_z parameter can be evaluated via the expectation values which are listed above and via the expression below,

$$Q_z = \frac{\langle \left(a^\dagger\right)^2 a^2\rangle_z - \left(\langle a^\dagger a\rangle_z\right)^2}{\langle a^\dagger a\rangle_z}. \tag{21}$$

Negative values of the Mandel factor, $-1 \leq Q < 0$, witness the non-classical nature of the states and indicate the appearance of a sub-Poissonian statistics of the photon number, which exhibits no classical analog. See Refs. [8, 9] for details.

2 Generalized coherent states as weighted Poisson distributions

The more general representation of modified coherent states is given by (see e.g. [12])

$$|z\rangle_c = \left(N_c\left(|z|^2\right)\right)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{c(n)}}|n\rangle, \tag{22}$$

where

$$N_c(x) = \sum_{n=0}^{\infty} \frac{x^n}{c(n)} \tag{23}$$

is the normalization. This formal representation should satisfy physical and mathematical constraints, in particular the convergence of the series (23) and the resolution of unity. Obviously we recover the classical coherent states when $c(n) = n!$ Here we would like to underline the link between this general scheme of derivation and the interpretation in terms of weighted Poisson distributions.

In the probabilistic literature, in order to take into account deviations from Poissonian counting distributions, it was introduced the general class of weighted Poisson distributions (see e.g. [1] and the references therein). Adopting the notation of [1], the probability mass function of a weighted Poisson process is

$$\Pr\{N^w(x) = n\} = \frac{w(n)p_n(x)}{E[w(N)]}, \quad n \geq 0 \tag{24}$$

where N is a random variable with a Poisson distribution $p_n(x)$, and $w(\cdot)$ a positive weight function with non-zero, finite expectation

$$0 < E[w(N)] = \sum_{n=0}^{\infty} w(n)p_n(x) < \infty. \tag{25}$$

We can give therefore an interpretation of the general coherent states (22) in terms of weighted Poisson distributions. Indeed, we recall that for classical coherent states, the probability $p\,(n, z)$ that the state ket $|z\rangle$ consists in n excitations results to be

$$p\,(n, z) = \frac{|z|^{2n}}{n!} e^{-|z|^2} \tag{26}$$

that is a Poissonian distribution. In the general case, we have

$$p\,(n, c, z) = \frac{|z|^{2n}}{c(n)} \frac{1}{N_c(|z|^2)} \tag{27}$$

that is a weighted Poisson distribution with $w(n) = n!/c(n)$. This interpretation can be obviously implemented to the particular case here considered of Wright-type coherent states. The link between weighted Poisson distributions and generalized coherent states can be useful in order to characterize for example photon counting distributions directly by using the properties of weighted distributions.

2.1 The connection with fractional ODEs

We finally point out that our generalization is somehow connected to the theory of fractional differential equations. Let us consider the normalizing function appearing in (12), that is

$$W_{\lambda,\mu}\left(|z|^2\right) = \sum_{k=0}^{\infty} \frac{|z|^{2k}}{k!\Gamma(\lambda k + \mu)}. \tag{28}$$

We here consider for simplicity $\mu = 1$ and take $x = |z|^2$. Then, it is quite simple to prove that this normalizing function satisfies the following fractional ODE

$$\frac{d}{dx} x^\lambda \frac{d^\lambda u}{dx^\lambda} - \frac{\lambda u}{x^{1-\lambda}} = 0, \quad x \geq 0, \lambda \in (0, 1) \tag{29}$$

involving the Caputo fractional derivative of order λ (see e.g. [3]). This relation can be directly proved by substitution, recalling that

$$\frac{d^\lambda x^\beta}{dx^\lambda} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\lambda)} x^{\beta-\lambda}. \tag{30}$$

Observe that, for $\lambda = 1$ we obtain the following equation

$$\frac{d}{dx} x \frac{du}{dx} = u \tag{31}$$

whose solution is the so-called Tricomi function (see [2])

$$C_0(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!^2}. \tag{32}$$

The operator $\frac{d}{dx} x \frac{d}{dx}$ appearing in (31) is also named Laguerre derivative in the literature.

3 Conclusion

In this paper we investigate the application of Wright-type special functions in the construction of a new family of generalized coherent states. It is well-known that standard coherent states play a key-role in many physical applications, in particular in photon counting problems and quantum optics. In the previous literature different modifications based on other relevant special functions such as Mittag-Leffler functions and Bessel functions have been considered. Here we analyze a new definition based on Wright functions and the consequences on modified destruction and creation operators. The corresponding statistics deviates from the classical Poissonian distribution. We also consider the connection between modified coherent states and weighted Poisson distributions, widely used in statistics in order to consider deviations from Poissonian distributions (i.e. sub- or super-Poissonian behaviour). We finally underline that this generalization is somehow connected to fractional differential equations with variable coefficients (including as a special case the ODE involving a Laguerre derivative). Possible implications of our results in the context of deformed bosons could be object of future studies. Moreover, one of the aims of our paper is to suggest the application of the theory of weighted Poisson statistics in modified coherent states models.

Acknowledgements: The work of all the authors has been carried out in the framework of the activities of the National Group of Mathematical Physics (GNFM, INdAM).

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