Some Notes on the Wright Functions in Probability Theory

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Abstract: We start with a short survey of the basic properties of the Wright functions and distinguish between the functions of the first and the second kind. Then we focus on the key role of the Wright functions of the second kind for the probability theory.

Key–Words: Probability theory, Wright functions, Mittag-Leffler functions, Laplace and Mellin transforms, Completely monotone functions.

1 Introduction

The Wright function in general and its particular cases known as the Mainardi functions play a very important role in the theory of the time-fractional partial differential equations. The purpose of this paper is to outline the relevant properties of the Wright functions from the viewpoint of the theory of probability.
This work is organized as follows. In Section 2, we recall some of the basic properties of the Wright functions that are entire functions in the complex plane. In doing so we distinguish between the functions of the first and the second kind.
In Section 3, we devote our attention to two special cases of the Wright function of the second kind introduced by Mainardi in the 1990's in virtue of their importance for the time-fractional diffusion equations. Nowadays in the FC literature they are referred to as the Mainardi functions. In contrast to the general case of the Wright function, they depend just on one parameter $\nu \in [0,1)$. One of the Mainardi functions, known as the M-Wright function, generalizes the Gaussian function and degenerates to the delta function in the limiting case $\nu = 1$. We present some plots of the M-Wright function on $\mathbb{R}$ for several parameter values $\nu \in [0,1/2]$ and $\nu \in [1/2,1]$. The plots demonstrate the non-negativity of the M-Wright function for $\nu \in [1/2,1]$.
For the asymptotic behavior of the Wright functions in the complex domain, we refer the interested readers to the papers by Wong and Zhao [19, 20] and to the surveys by Luchko and by Paris in the Handbook of Fractional Calculus and Applications, see respectively [7], [16], and references therein. The results presented in the above mentioned papers implicate in particular a very different behavior of the M-Wright function on $\mathbb{R}^+$ and on $\mathbb{R}^-$.
Then we recall how the Mainardi functions are related to an important class of the probability density functions (pdf's) known as the extremal Lévy stable densities. This emphasizes the relevance of the Mainardi functions in the probability theory independently on the framework of the fractional diffusion equations.
In Section 4, we outline the (unilateral) Laplace transforms of the Wright functions of the first and second kinds that turn out to be expressed differently in terms of the Mittag-Leffler function in two parameters.
In Section 5, we prove the non-negativity of the Wright function of the second find on $\mathbb{R}$ and discuss its interpretation as a pdf. The derivations are based on the Laplace transform formulas and on the properties of the completely monotone functions
Finally, in Section 6, we provide some concluding remarks and pose a problem related to the Wright functions worth to be considered in the next future.

2 The classical Wright function

The classical *Wright function*, that we denote by $W_{\lambda,\mu}(z)$, is defined by the series representation convergent on the whole complex plane $\mathbb{C}$,

$$W_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\lambda n+\mu)},\ \lambda>-1,\ \mu\in\mathbb{C}. \quad (1)$$

One of its *integral representations* reads as:

$$W_{\lambda,\mu}(z)=\frac{1}{2\pi i}\int_{Ha}\mathrm{e}^{\sigma+z\sigma^{-\lambda}}\frac{d\sigma}{\sigma^{\mu}},\ \lambda>-1,\ \mu\in\mathbb{C}, \quad (2)$$

where Ha denotes the Hankel path. It is a loop, which starts from $-\infty$ along the lower side of negative real axis, encircles the axes origin and ends at $-\infty$ along the upper side of the negative real axis.

$W_{\lambda,\mu}(z)$ is then an *entire function* for all $\lambda \in (-1, +\infty)$. Originally, in 1930's Wright assumed $\lambda \geq 0$ in connection with his investigations on the asymptotic theory of partitions [21, 22], and only in 1940 [23] he considered $-1 < \lambda < 0$.

We note that in the Vol 3, Chapter 18 of the handbook of the Bateman Project [2], presumably for a misprint, the parameter λ is restricted to be non-negative, whereas the Wright functions remained practically ignored in other handbooks. In 1990's Mainardi, being aware only of the Bateman handbook, proved that the Wright function is entire also for $-1 < \lambda < 0$ in his approaches to the time fractional diffusion equation, see [9, 10, 11].

In view of the asymptotic representation in the complex domain and of the Laplace transform for positive argument $z = r > 0$ (r can denote the time variable t or the positive space variable x) the Wright functions are distinguished in *first kind* ($\lambda \geq 0$) and *second kind* ($-1 < \lambda < 0$) as outlined in the Appendix F of the book by Mainardi [12].

3 The Mainardi auxiliary functions

We note that two particular Wright functions of the second kind, were introduced by Mainardi in 1990's [9, 10, 11] named $F_\nu(z)$ and $M_\nu(z)$ ($0 < \nu < 1$), called *auxiliary functions* in virtue of their role in the time fractional diffusion equations. These functions are indeed special cases of the Wright function of the second kind $W_{\lambda,\mu}(z)$ by setting, respectively, $\lambda = -\nu$ and $\mu = 0$ or $\mu = 1 - \nu$. Hence we have:

$$F_\nu(z) := W_{-\nu,0}(-z), \quad 0 < \nu < 1, \tag{3}$$

and

$$M_\nu(z) := W_{-\nu,1-\nu}(-z), \quad 0 < \nu < 1, \tag{4}$$

These functions are interrelated through the following relation:

$$F_\nu(z) = \nu z M_\nu(z). \tag{5}$$

The series and integral representations of the auxiliary functions are derived from those of the general Wright functions. Then for $z \in \mathbb{C}$ and $0 < \nu < 1$ we have:

$$\begin{aligned} F_\nu(z) &= \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\Gamma(-\nu n)} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{n!} \Gamma(\nu n + 1) \sin(\pi \nu n), \end{aligned} \tag{6}$$

$$\begin{aligned} M_\nu(z) &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[-\nu n + (1-\nu)]} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\nu n) \sin(\pi \nu n), \end{aligned} \tag{7}$$

and

$$F_\nu(z) := \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma - z\sigma^\nu} d\sigma, \tag{8}$$

$$M_\nu(z) := \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma - z\sigma^\nu} \frac{d\sigma}{\sigma^{1-\nu}}. \tag{9}$$

Explicit expressions of $F_\nu(z)$ and $M_\nu(z)$ in terms of known functions are expected for some particular values of ν as shown and recalled in [9, 10, 11], that is

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-z^2/4}, \tag{10}$$

$$M_{1/3}(z) = 3^{2/3} \mathrm{Ai}(z/3^{1/3}), . \tag{11}$$

Liemert and Klenie [5] have added the following expression for $\nu = 2/3$

$$\begin{aligned} M_{2/3}(z) &= 3^{-2/3}\, \mathrm{e}^{-2z^3/27} \\ &\left[3^{1/3}\, z\, \mathrm{Ai}\left(z^2/3^{4/3}\right) - 3\mathrm{Ai}'\left(z^2/3^{4/3}\right)\right] \end{aligned} \tag{12}$$

where Ai and Ai′ denote the *Airy function* and its first derivative. Furthermore they have suggested in the positive real field $\mathbb{R}^+$ the following remarkably integral representation

$$\begin{aligned} M_\nu(x) &= \frac{1}{\pi} \frac{x^{\nu/(1-\nu)}}{1-\nu} \\ &\cdot \int_0^\pi C_\nu(\phi) \exp\left(-C_\nu(\phi)\right)\, x^{1/(1-\nu)}\, d\phi, \end{aligned} \tag{13}$$

where

$$C_\nu(\phi) = \frac{\sin(1-\nu)}{\sin\phi} \left(\frac{\sin\nu\phi}{\sin\phi}\right)^{\nu/(1-\nu)}, \tag{14}$$

corresponding to equation (7) of the article written by Saa and Venegeroles [17] .

We find it convenient to show the plots of the M-Wright functions on a space symmetric interval of $\mathbb{R}$ in Figs 1, 2, corresponding to the cases $0 \leq \nu \leq 1/2$ and $1/2 \leq \nu \leq 1$, respectively. We recognize the non-negativity of the M-Wright function on $\mathbb{R}$ for $1/2 \leq \nu \leq 1$ consistently with the analysis on distribution of zeros and asymptotics of Wright functions carried out by Luchko, see [6], [7].

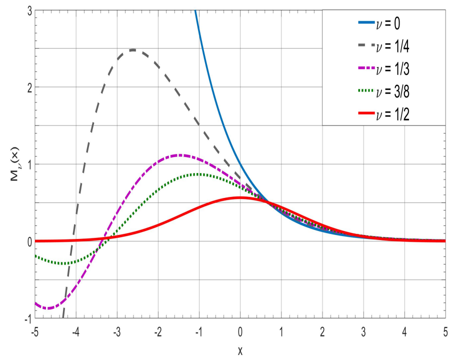

Figure 1: Plots of the M-Wright function as a function of the x variable, for $0 \leq \nu \leq 1/2$.

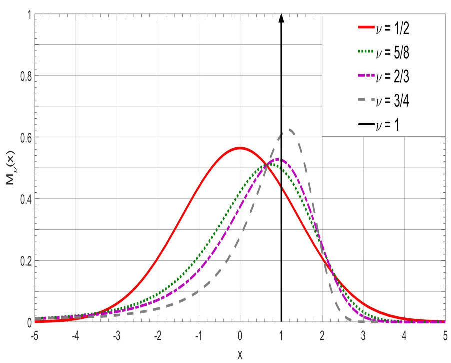

Figure 2: Plots of the M-Wright function as a function of the x variable, for $1/2 \leq \nu \leq 1$.

3.1 The auxiliary functions as extremal stable densities

We find it worthwhile to recall the relations between the Mainardi auxilary functions and the extremal Lévy stable densities as proven in the 1997 paper by Mainardi and Tomirotti [15]. For an essential account of the general Lévy stable distributions in probability we refer to the 2001 paper by Mainardi et al. [14] recalled in the Appendix F in [12]. Indeed, from a comparison between the series expansions of stable densities according to the Fekller-Takayasu canonic form with index of stability $\alpha \in (0, 2]$ and skewness θ ($|\theta| \leq \min\{\alpha, 2 - \alpha\}$), and those of the auxiliary functions in Eqs. (6) - (7), we recognize, see also [13], that the auxiliary functions are related to the extremal stable densities as follows

$$L_\alpha^{-\alpha}(x) = \frac{1}{x} F_\alpha(x^{-\alpha}) = \frac{\alpha}{x^{\alpha+1}} M_\alpha(x^{-\alpha}), \quad 0 < \alpha < 1, \quad x \geq 0. \tag{15}$$

$$L_\alpha^{\alpha-2}(x) = \frac{1}{x} F_{1/\alpha}(x) = \frac{1}{\alpha} M_{1/\alpha}(x)\,, \quad 1 < \alpha \leq 2\,, \quad -\infty < x < +\infty. \tag{16}$$

In the above equations, for $\alpha = 1$, the skewness parameter turns out to be $\theta = -1$, so we get the singular limit

$$L_1^{-1}(x) = M_1(x) = \delta(x - 1)\,. \tag{17}$$

Hereafter we show the plots the extremal stable densities according to their expressions in terms of the M-Wright functions, see Eq. (15), Eq. (16) for $\alpha = 1/2$ and $\alpha = 3/2$, respectively.

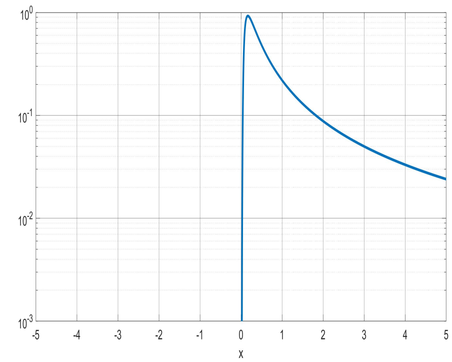

Figure 3: Plot of the unilateral extremal stable pdf for $\alpha = 1/2$

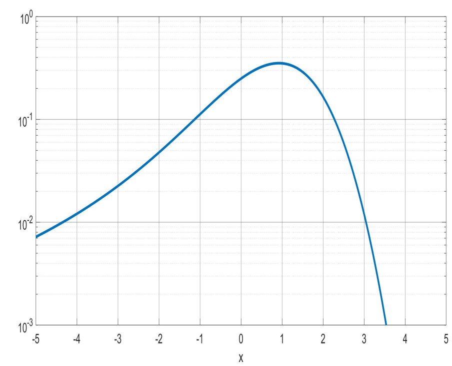

Figure 4: Plot of the bilateral extremal stable pdf for $\alpha = 3/2$

We recognize that the above plots are consistent with the corresponding ones shown by Mainardi et al. [14] for the stable pdf's derived as fundamental solutions of a suitable space-fractional diffusion equation.

4 Laplace transform pairs related to the Wright function

We note that the Wright functions are entire of order $1/(1 + \lambda)$ hence only the functions of the first kind ($\lambda \geq 0$) are of exponential order whereas those of the second kind ($-1 < \lambda < 0$) are not of exponential order. The case $\lambda = 0$ is trivial since

$W_{0,\mu}(z) = \exp(z)/\Gamma(\mu)$.
As a consequence of the different orders we must point out the different Laplace transforms proved e.g. in [4],[12], see also the recent survey on Wright functions by Luchko [7].
In the case $\lambda > 0$ the Wright function is an entire function of order less than 1 and consequently its Laplace transform can be obtained by transforming term-by-term its Taylor expansion (1) in the origin. As a result we get for the first kind:

$$W_{\lambda,\mu}(\pm r) \div \frac{1}{s} E_{\lambda,\mu}\left(\pm\frac{1}{s}\right) ; \qquad (18)$$

where the sign $\div$ is used for the juxtaposition of a function with its Laplace transform.
For the second kind, when $-1 < \lambda < 0$ and putting for convenience $\nu = -\lambda$ so $0 < \nu < 1$, we get following a method by Mainardi see eg [10]

$$W_{-\nu,\mu}(-r) \div E_{\nu,\mu+\nu}(-s) . \qquad (19)$$

Above we have introduced the Mittag-Leffler function in two parameters $\alpha > 0$, $\beta \in \mathbb{C}$ defined as its convergent series for all $z \in \mathbb{C}$

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}. \qquad (20)$$

For more details on the special functions of the Mittag-Leffler type we refer the interested readers to the treatise by Gorenflo et al. [3], where in the forthcoming 2-nd edition also the Wright functions are treated in some detail.

5 Wright functions as pdf's

Using the known completely monotone functions, the technique of the Laplace transform, and the Bernstein theorem, one can prove non-negativity of some Wright functions. Say, the function

$$p_{\nu,\mu}(r) = \Gamma(\mu)\, W_{-\nu,\mu-\nu}(-r) \qquad (21)$$

can be interpreted as a one-sided pdf for $0 < \nu \le 1$, $\nu \le \mu$ (see [8]). To show this, we use the Laplace transform pair (19) that we rewrite in the form

$$W_{-\nu,\mu-\nu}(-r) \div E_{\nu,\mu}(-s),\ 0 < \nu \le 1$$

and the fact that the Mittag-Leffler function $E_{\nu,\mu}(-s)$ is completely monotone for $0 < \nu \le 1$, $\nu \le \mu$. According to the Bernstein theorem, the function $p_{\nu,\mu}(r)$ is non-negative.
To calculate the integral of $p_{\nu,\mu}(r)$ over $\mathbb{R}^+$ let us mention that it can be interpreted as the Laplace transform of $p_{\nu,\mu}$ at the point $s = 0$ or the Mellin transform at $s = 1$. Using the Mellin integral transform of the Wright function as in [7] leads now to the following chain of equalities:

$$\begin{aligned}\int_0^\infty p_{\nu,\mu}(r)\, dr &= \int_0^\infty \Gamma(\mu)\, W_{-\nu,\mu-\nu}(-r)\, dr \\ &= \frac{\Gamma(\mu)\Gamma(s)}{\Gamma(\mu-\nu+\nu s)}\Big|_{s=1} = \frac{\Gamma(\mu)}{\Gamma(\mu)} = 1.\end{aligned}$$

The Mellin transform technique allows us to calculate also all moments of order $s > 0$ of the pdf $p_{\nu,\mu}(r)$ on $\mathbb{R}^+$:

$$\begin{aligned}\int_0^\infty p_{\nu,\mu}(r)\, r^s\, dr &= \int_0^\infty \Gamma(\mu) W_{-\nu,\mu-\nu}(-r)\, r^{s+1-1}\, dr \\ &= \frac{\Gamma(\mu)\Gamma(s+1)}{\Gamma(\mu+\nu s)}.\end{aligned} \qquad (22)$$

For $\mu = 1$, the pdf $p_{\nu,\mu}(r)$ can be expressed in terms of the M-Wright function $M_\nu(r)$, $0 < \nu < 1$ defined by Eq. (7). As it is well known (see, e.g., [12]), $M_\nu(r)$ can be interpreted as a one-sided pdf on $\mathbb{R}^+$ with the moments given by the formula

$$\int_0^\infty M_\nu(r)\, r^s\, dr = \frac{\Gamma(s+1)}{\Gamma(1+\nu s)},\ s > 0.$$

6 Conclusions

After a short survey of the basic properties of the Wright functions distinguished in the functions of the first and the second kind, we have shown the key role of the functions of the second kind in the probability theory. For readers' convenience we presented some illuminating plots of a special case of the Wright function of the second kind known as the M-Wright function. In our opinion, it would be worth to consider the relevance of the Wright function of the first kind for the probability theory. This problem will be discussed elsewhere.

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