Products of Volterra-type operators and composition operators from analytic Morrey spaces into Zygmund spaces

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Abstract: In recent years the composition operator $C_\varphi$ has been received much attention and appear in various settings in the literature. It is interesting to provide a function theoretic characterization when $\varphi$ induces a bounded or compact composition operator on various function spaces. In this paper we consider the products of Volterra-type operators and composition operators. We characterize the boundedness and compactness of the products of Volterra-type operators and composition operators $T_\varphi C_\psi$ and $I_\varphi C_\psi$ from the analytic Morrey spaces $L^{2,\lambda}$ to the Zygmund space $Z\lambda$, and the little analytic Morrey spaces $L^{2,\lambda}_0$ to the little Zygmund space $Z\lambda_0$ over the unit disk, respectively.

Key–Words: Analytic Morrey space, Zygmund space, Volterra-type operators, Composition operators, Boundedness, Compactness

1 Introduction

Let $D = \{ z : |z| < 1 \}$ be the open unit disk in the complex plane and $H(D)$ denote the set of all analytic functions on $D$. Let $\varphi$ be an analytic self-map of the unit disk $D$. Associated with $\varphi$ is the composition operator $C_\varphi$ defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H(D).$$

It is interesting to provide a function theoretic characterization when $\varphi$ induces a bounded or compact composition operator on various function spaces. Boundedness and compactness of composition operators on various function spaces have been studied by numerous authors, for example, see [3, 4, 10, 11, 13, 14, 15, 17, 20, 25].

For an arc $I \subset \partial D$, let $|I| = \frac{1}{2\pi} \int_I |d\zeta|$ be the normalized arc length of $I$,

$$f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{|d\zeta|}{2\pi}, f \in H(D),$$

and $S(I)$ be the Carleson box based on $I$ with

$$S(I) = \{ z \in D : 1 - |I| \leq |z| < 1, \, \frac{z}{|z|} \in I \}.$$

Clearly, if $I = \partial D$, then $S(I) = D$.

Let $L^{2,\lambda}(D)$ represent the analytic Morrey spaces of all analytic functions $f \in H^2$ on $D$ such that

$$\sup_{I \subset \partial D} \left( \frac{1}{|I|} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} \right)^{1/2} < \infty,$$

where $0 < \lambda \leq 1$ and the Hardy space $H^2$ consists of analytic functions $f$ in $D$ satisfying

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

From Theorem 3.1 of [21] or Theorem 3.21 of [23], we can define the norm of function $f \in L^{2,\lambda}(D)$ and its equivalent formula as follows

$$\|f\|_{L^{2,\lambda}} = |f(0)| + \sup_{I \subset \partial D} \left( \frac{1}{|I|} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) dm(z) \right)^{1/2} \approx |f(0)| + \sup_{a \in D} ((1 - |a|^2)^{1-\lambda} \int_D |f'(z)|^2 (1 - |\varphi_a(z)|^2) dm(z))^{1/2}.$$

Similarly to the relation between $BMOA$ space and $VMOA$ space, we have that $f \in L^{2,\lambda}_0(D)$, the little analytic Morrey spaces, if $f \in L^{2,\lambda}(D)$ and

$$\lim_{|I| \to 0} \left( \frac{1}{|I|} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} \right)^{1/2} = 0.$$

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Clearly, $\mathcal{L}_0^{2,1}(D) = VMOA$. The following lemma gives equivalent conditions of $\mathcal{L}_0^{2,\lambda}$. The proof is similar to that of Theorem 6.3 in [6], we omit the details.

**Lemma 1** Suppose that $0 < \lambda < 1$ and $f \in H(D)$. Let $a \in D$, $\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}$, Then the following statements are equivalent.

(i) $f \in \mathcal{L}_0^{2,\lambda}(D)$;

(ii) $\lim_{|a|\to 1} (1 - |a|^2)^{1-\lambda} \times \int_D |f'(z)|^2 (1 - |\varphi_a(z)|^2) dm(z) = 0$; \hspace{1cm} (1)

(iii) $\lim_{|a|\to 1} (1 - |a|^2)^{1-\lambda} \times \int_D |f'(z)|^2 \log \frac{1}{|\varphi_a(z)|} dm(z) = 0$. \hspace{1cm} (2)

It is known that $\mathcal{L}_0^{2,1}(D) = BMOA$ and if $0 < \lambda < 1$, $BMOA \subseteq \mathcal{L}_0^{2,\lambda}(D)$. For more information on $BMOA$ and $VMOA$, see [6].

The Zygmund space $Z$ consists of all analytic functions $f$ defined on $D$ such that

$$z(f) = \sup \{1 - |z|^2 | f''(z): z \in D \} < +\infty.$$  

From a theorem of Zygmund (see [30, vol. I, p. 263] or [5, Theorem 5.3]), we see that $f \in Z$ if and only if $f$ is continuous in the close unit disk $D = \{ z : |z| \leq 1 \}$ and the boundary function $f(e^{i\theta})$ such that

$$\sup_{h>0,\theta} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$ 

An analytic function $f \in H(D)$ is said to belong to the little Zygmund space $Z_0$ consists of all $f \in Z$ satisfying $\lim_{|z|\to 1} (1 - |z|^2) |f''(z)| = 0$. It can easily proved that $Z$ is a Banach space under the norm

$$\|f\|_Z = |f(0)| + |f'(0)| + z(f)$$

and the polynomials are norm-dense in closed subspace $Z_0$ of $Z$. For some other information on this space and some operators on it, see, for example, [7, 8, 26, 27].

Suppose that $g : D \rightarrow C$ is a analytic map. Let $T_g$ and $I_g$ denote the Volterra-type operators with the analytic symbol $g$ on $D$ respectively:

$$T_g f(z) = \int_0^z f(w)g'(w) dw$$

and

$$I_g f(z) = \int_0^z f'(w)g(w) dw, \quad z \in D.$$  

In [12] Pommerenke introduced the Volterra-type operator $T_g$ and showed that $T_g$ is a bounded operator on the Hardy space $H^2$ if and only if $g \in BMOA$. In [26] the author studied the boundedness and compactness of $T_g$ between the $\alpha$-Bloch spaces $\beta_\alpha$ and the logarithmic Bloch space $LB^1$. Boundedness and compactness of this operators $T_g$ acting on various function spaces have been studied in many literature. See [1, 2, 16, 18, 19, 20, 22] for more information.

Here, we consider the products of Volterra-type operators and composition operators, which are defined by

$$(T_g C_\varphi f)(z) = \int_0^z (f \circ \varphi)(\zeta)g'(\zeta) d\zeta, \quad f \in H(D)$$

and

$$(I_g C_\varphi f)(z) = \int_0^z (f \circ \varphi)'(\zeta)g(\zeta) d\zeta, \quad f \in H(D).$$

In [8], Li and Stević studied those operators from $H^\infty$ and Bloch spaces to Zygmund Spaces. The author in [28] characterized the boundedness and compactness of those operators on the logarithmic Bloch space $LB^1$. Xiao and Xu [24] studied the composition operators on the analytic Morrey spaces $L^{2,\lambda}$ spaces. Li, Liu and Lou[9] studied the Volterra-type operators on $L^{2,\lambda}$ spaces. Zhou and Ye [29] considered this operators from $L^{2,\lambda}$ spaces to the classical Bloch space. In 2006, the boundedness of composition operators on the Zygmund space $Z$ was first studied by Choe, Koo, and Smith in [3]. Later, many researchers have studied composition operators and weighted composition operators acting on the Zygmund space $Z$. Li and Stević in [7] studied the boundedness and compactness of the generalized composition operators on Zygmund spaces and Bloch type spaces. Ye and Hu in [27] characterized boundedness and compactness of weighted composition operators on the Zygmund space $Z$. In this paper the boundedness and compactness of those operators from analytic Morrey spaces $L^{2,\lambda}$ into Zygmund spaces $Z$ are discussed. As some corollaries we obtain the boundedness and compactness for $T_g$ and $I_g$ from $L^{2,\lambda}$ into $Z$ spaces.

**Notations:** For two functions $F$ and $G$, if there is a constant $C > 0$ dependent only on indexes $p, \lambda, \ldots$ such that $F \leq CG$, then we say that $F \lesssim G$. Furthermore, denote that $F \approx G$ ($F$ is comparable with $G$) whenever $F \lesssim G \lesssim F$.  

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2 Auxiliary results

In order to prove the main results of this paper, we need some auxiliary results.

Lemma 2 Let $0 < \lambda < 1$. If $f \in \mathcal{L}^{2,\lambda}$, then

(i) $|f(z)| \lesssim \frac{\|f\|_{L^{2,\lambda}}}{(1 - |z|^2)^{\frac{\lambda}{2}}}$ for every $z \in D$;

(ii) $|f'(z)| \lesssim \frac{\|f\|_{L^{2,\lambda}}}{(1 - |z|^2)^{\frac{\lambda-1}{2}}}$ for every $z \in D$;

(iii) $|f''(z)| \lesssim \frac{\|f\|_{L^{2,\lambda}}}{(1 - |z|^2)^{\frac{\lambda-3}{2}}}$ for every $z \in D$.

Proof: (i) and (ii) are from Lemma 2.5 in [9].

For any $f \in \mathcal{L}^{2,\lambda}$. Fix $z \in D$ and let $\rho = \frac{1 + |z|}{2}$, by the Cauchy integral formula, we obtain that

\[ |f''(z)| = \frac{1}{2\pi i} \int_{|\xi|=\rho} \frac{f'(|\xi|)}{(|\xi| - z)^2} d\xi \]

\[ \leq \frac{\|f\|_{L^{2,\lambda}}}{(1 - \rho^2)^{\frac{\lambda-3}{2}}} \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho d\theta}{|\rho e^{i\theta} - z|^2} \]

\[ = \frac{\|f\|_{L^{2,\lambda}}}{(1 - \rho^2)^{\frac{\lambda-3}{2}} \rho^2 - |z|^2} \lesssim \frac{\|f\|_{L^{2,\lambda}}}{(1 - |z|^2)^{\frac{\lambda-3}{2}}}. \]

Hence (iii) holds.

Lemma 3 Let $0 < \lambda < 1$. If $f \in \mathcal{L}^{2,\lambda}_0$, then

(i) $\lim_{|z| \to 1} (1 - |z|^2)^{\frac{3-\lambda}{2}} |f'(z)| = 0$;

(ii) $\lim_{|z| \to 1} (1 - |z|^2)^{\frac{1-\lambda}{2}} |f(z)| = 0$;

(iii) $\lim_{|z| \to 1} (1 - |z|^2)^{\frac{5-\lambda}{2}} |f''(z)| = 0$.

The proof of (i) is similar to that of Lemma 2.5 in [9], and we easily obtain (ii) and (iii) by (i). These details are omitted here.

Lemma 4 Suppose $T_y C_\varphi$ (or $I_y C_\varphi$) : $\mathcal{L}^{2,\lambda}_0 \to \mathcal{Z}_0$ is a bounded operator, then $T_y C_\varphi$ (or $I_y C_\varphi$) : $\mathcal{L}^{2,\lambda} \to \mathcal{Z}$ is a bounded operator.

The proof is similar to that of Lemma 2.3 in [26]. The details are omitted.

3 Boundedness of $T_y C_\varphi$

In this section we characterize the boundedness of the operator $T_y C_\varphi$ from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$, and the little analytic Morrey spaces $\mathcal{L}^{2,\lambda}_0$ to the little Zygmund space $\mathcal{Z}_0$, respectively.

Theorem 5 Let $g$ be an analytic function on the unit disc $D$ and $\varphi$ an analytic self-map of $D$. Then $T_y C_\varphi$ is a bounded operator from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$ if and only if the following are satisfied:

\[ \sup_{z \in D} \frac{(1 - |z|^2)|g''(z)|}{(1 - |\varphi(z)|^2)^{\frac{\lambda}{2}}} < \infty; \]

\[ \sup_{z \in D} \frac{(1 - |z|^2)|\varphi'(z)g'(z)|}{(1 - |\varphi(z)|^2)^{\frac{\lambda}{2}}} < \infty. \]

Proof: Suppose $T_y C_\varphi$ is bounded from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$. Using functions $f(z) = 1$ and $f(z) = z$ in $\mathcal{L}^{2,\lambda}$, we have

\[ g \in \mathcal{Z}, \]

and

\[ \sup_{z \in D} (1 - |z|^2)|\varphi'(z)g'(z) + \varphi(z)g''(z)| < +\infty. \]

Since $\varphi(z)$ is a self-map, we get

\[ K_1 = \sup_{z \in D} (1 - |z|^2)|\varphi'(z)g'(z)| < +\infty. \]

Fix $a \in D$ with $|a| > \frac{1}{2}$, we take the test functions:

\[ f_a(z) = \frac{1 - |a|^2}{(1 - \bar{a}z)^{\frac{1-\lambda}{2}}} - \frac{(1 - |a|^2)^2}{(1 - \bar{a}z)^{\frac{\lambda}{2}}}. \]

for $z \in D$. Then, arguing as the proof of Lemma 3.2 in [9] we obtain that $f_a \in \mathcal{L}^{2,\lambda}$ and $\|f_a\|_{L^{2,\lambda}} \lesssim 1$. Since $f_a(a) = 0$, $f'_a(a) = \frac{(1 - |a|^2)^{\frac{1+\lambda}{2}}}{(1 - \bar{a}z)^{\frac{1-\lambda}{2}}}$, therefore, for all $\lambda \in D$ with $|\varphi(\lambda)| > \frac{1}{2}$, we have

\[ \|f_a\|_{L^{2,\lambda}} \gtrsim \|T_y C_\varphi f_a\|_{\mathcal{Z}} \]

\[ \gtrsim \sup_{z \in D} (1 - |z|^2)|f_a(\varphi(z)) + g''(z)f_a(\varphi(z))| \]

\[ = \sup_{z \in D} (1 - |z|^2)|\varphi'(z)g'(z) + \varphi(z)g''(z)| \]
Let $a = \varphi(\lambda)$, it follows that

$$
\|f_a\|_{L^{2,\lambda}} \geq \frac{(1 - |\lambda|^2)|\varphi'(\lambda)g'(\lambda)|f'_\varphi(\lambda)(\varphi(\lambda))}{g''(\lambda)f_\varphi(\lambda)(\varphi(\lambda))} = \frac{(1 - |\lambda|^2)|\varphi'(\lambda)g'(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{\frac{1}{2}}}.
$$

For $\forall \lambda \in D$ with $|\varphi(\lambda)| \leq \frac{1}{2}$, by (7), we have

$$
\sup_{\lambda \in D} \frac{(1 - |\lambda|^2)|\varphi'(\lambda)g'(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{\frac{1}{2}}} \leq \frac{(4\lambda^{3-\lambda})}{3} \sup_{\lambda \in D} (1 - |\lambda|^2)|\varphi'(\lambda)g'(\lambda)| < +\infty.
$$

Hence (4) holds.

Next we will show (3) holds. Let

$$
h_a(z) = \frac{1 - |a|^2}{(1 - az)^{\frac{1}{2}}} - \frac{3 - \lambda}{5 - \lambda} \frac{(1 - |a|^2)}{(1 - az)^{\frac{1}{2}}}, \quad (9)
$$

for $z \in D$. Similar to the case of $f_a$, we have $h_a \in L^{2,\lambda}$ and $\sup_{\frac{1}{4} < |a| < 1} \|h_a\|_{L^{2,\lambda}} \leq 1$. From this and by that fact that $h_a(a) = 0$ and $h_a(0) = \frac{2}{5 - \lambda} \frac{(1 - |a|^2)}{(1 - az)^{\frac{1}{2}}}$, it follows that for all $\lambda \in D$ with $|\varphi(\lambda)| > \frac{1}{2}$,

$$
\|h_a\|_{L^{2,\lambda}} \geq \frac{(1 - |\lambda|^2)|\varphi'(\lambda)g'(\lambda)|}{h'_{\varphi}(\lambda)(\varphi(\lambda))} + \frac{2}{5 - \lambda} \frac{(1 - |\lambda|^2)}{(1 - |\varphi(\lambda)|^2)^{\frac{1}{2}}}.
$$

For $\forall \lambda \in D$ with $|\varphi(\lambda)| \leq \frac{1}{2}$, by (5), we have

$$
\sup_{\lambda \in D} \frac{(1 - |\lambda|^2)|g''(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{\frac{1}{2}}} = \left(\frac{4}{3}\right)^{\frac{1}{2}} \sup_{\lambda \in D} (1 - |\lambda|^2)|g''(\lambda)| < \infty.
$$

Hence (3) holds.

Conversely, suppose that (3) and (4) hold. For $f \in L^{2,\lambda}$, by Lemma 2, we have the following inequality:

$$
(1 - |z|^2)(T_gC_\varphi f)'(z) \geq (1 - |z|^2)|\varphi'(\lambda)g'(\lambda)|f'(\lambda) + g''(\lambda)f(\varphi(z))|
$$

$$
\leq (1 - |z|^2)|\varphi'(\lambda)g'(\lambda)|f'(\lambda) + (1 - |z|^2)|g''(\lambda)f(\varphi(z))| + (1 - |z|^2)|g''(\lambda)f(\varphi(z))| + \sup_{\lambda \in D} \frac{(1 - |\lambda|^2)|\varphi'(\lambda)g'(\lambda)|}{g''(\lambda)f(\varphi(z))} \|f\|_{L^{2,\lambda}}
$$

and

$$
\|T_gC_\varphi f(0)\| + \|T_gC_\varphi f)'(0)\| \leq \frac{|g(0)|}{(1 - |\varphi(0)|^2)^{\frac{1}{2}}} \|f\|_{L^{2,\lambda}}.
$$

This shows that $T_gC_\varphi$ is bounded. This completes the proof of Theorem 5.

**Theorem 6** Let $g$ be an analytic function on the unit disc $D$ and $\varphi$ an analytic self-map of $D$. Then $T_gC_\varphi$ is bounded from the little analytic Morrey spaces $L^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$ if and only if (3) and (4) hold, and the following are satisfied:

$$
g \in \mathcal{Z}_0; \quad \lim_{|z| \to 1} (1 - |z|^2)|\varphi'(\lambda)g'(\lambda)| = 0. \quad (10)
$$

**Proof:** Suppose that $T_gC_\varphi$ is bounded from $L^{2,\lambda}$ to $\mathcal{Z}_0$. Then $g(z) - g(0) = T_gC_\varphi 1 \in \mathcal{Z}_0$. Also $T_gC_\varphi z \in \mathcal{Z}_0$, thus

$$
(1 - |z|^2)|\varphi'(\lambda)g'(\lambda)| \to 0 \quad (|z| \to 1^-).
$$

Since $|\varphi| \leq 1$ and $g \in \mathcal{Z}_0$, we have $\lim_{|z| \to 1} (1 - |z|^2)|\varphi'(\lambda)g'(\lambda)| = 0$. Hence (10) holds.

On the other hand, by Lemma 4 and Theorem 5, we obtain that (3) and (4) hold.

Conversely, let

$$
M_1 = \sup_{z \in D} \frac{(1 - |\lambda|^2)|\varphi'(\lambda)g'(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{\frac{1}{2}}} < \infty;
$$

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\[ M_2 = \sup_{z \in D} \frac{(1 - |z|^2)|g''(z)|}{(1 - |\varphi(z)|^2)^{\frac{\lambda - 1}{2}}} < \infty. \]

For all \( f \in L^{2,\lambda}_0 \), by Lemma 3, given \( \epsilon > 0 \) there is a \( 0 < \delta < 1 \) such that \( (1 - |z|^2)^{\frac{\lambda - 1}{2}}|f'(z)| < \frac{\epsilon}{2M_1} \) and \( (1 - |z|^2)^{\frac{\lambda - 1}{2}}|f(z)| < \frac{\epsilon}{2M_2} \) for all \( z \) with \( \delta < |z| < 1 \).

If \( |\varphi(z)| > \delta \), it follows that

\[
(1 - |z|^2)|T_g C_{\varphi} f''(z)| \leq (1 - |z|^2)|\varphi'(z)g'(z)f'(\varphi(z))| + g''(z)f(\varphi(z))| \leq (1 - |z|^2)|\varphi'(z)g'(z)f'(\varphi(z))| + (1 - |z|^2)|g''(z)f(\varphi(z))| \leq \frac{(1 - |z|^2)|\varphi'(z)g'(z)f'(\varphi(z))|}{(1 - |\varphi(z)|^2)^{\frac{\lambda - 1}{2}}} \frac{\epsilon}{2M_1} + \frac{(1 - |z|^2)|g''(z)|}{(1 - |\varphi(z)|^2)^{\frac{\lambda - 1}{2}}} \frac{\epsilon}{2M_2} < \epsilon.
\]

We know that there exists a constant \( K \) such that \( |f(z)| \leq K \) and \( |f'(z)| \leq K \) for all \( |z| \leq \delta \).

If \( |\varphi(z)| \leq \delta \), it follows that

\[
(1 - |z|^2)|(T_g C_{\varphi} f''(z)| = (1 - |z|^2)|\varphi'(z)g'(z)f'(\varphi(z))| + g''(z)f(\varphi(z))| \leq (1 - |z|^2)|\varphi'(z)g'(z)f'(\varphi(z))| + (1 - |z|^2)|g''(z)f(\varphi(z))| \leq K(1 - |z|^2)|\varphi'(z)g'(z)| + K(1 - |z|^2)|g''(z)|.
\]

Thus we conclude that \( (1 - |z|^2)|(T_g C_{\varphi} f''(z)| \to 0 \) as \( |z| \to 1^- \). Hence \( T_g C_{\varphi} f \in Z_0 \) for all \( f \in L^{2,\lambda}_0 \).

On the other hand, \( T_g C_{\varphi} \) is bounded from \( L^{2,\lambda} \) to \( Z \) by Theorem 5. Hence \( T_g C_{\varphi} \) is a bounded operator from \( L^{2,\lambda}_0 \) to \( Z_0 \).

**Corollary 7** The Volterra-type operator \( T_g : L^{2,\lambda} \to Z \) is a bounded operator if and only if \( g = 0 \).

### 4 Boundedness of \( I_g C_{\varphi} \)

In this section we study the boundedness of the operator

\[ I_g C_{\varphi} : L^{2,\lambda}(or \ L^{2,\lambda}_0) \to Z \ (or \ Z_0). \]

**Theorem 8** Let \( g \) be an analytic function on the unit disc \( D \) and \( \varphi \) an analytic self-map of \( D \). Then \( I_g C_{\varphi} \) is a bounded operator from the analytic Morrey spaces \( L^{2,\lambda} \) to the Zygmund space \( Z \) if and only if the following are satisfied:

\[
\sup_{z \in D} \frac{(1 - |z|^2)|\varphi'(z)g'(z) + g''(z)|}{(1 - |\varphi(z)|^2)^{\frac{\lambda - 1}{2}}} < \infty;
\]

\[
\sup_{z \in D} \frac{(1 - |z|^2)|\varphi(z)g'(z)|}{(1 - |\varphi(z)|^2)^{\frac{\lambda - 1}{2}}} < \infty.
\]

**Proof:** Suppose \( I_g C_{\varphi} \) is bounded from the analytic Morrey spaces \( L^{2,\lambda} \) to the Zygmund space \( Z \). Using functions \( f(z) = z \) and \( f(z) = z^2 \) in \( L^{2,\lambda} \), we have

\[
\sup_{z \in D} (1 - |z|^2)|\varphi'(z)g'(z) + g''(z)| < +\infty, \quad (13)
\]

and

\[
\sup_{z \in D} (1 - |z|^2)|2\varphi(z)g'(z) + 2g(z)\varphi''(z) + 2g(z)(\varphi'(z))^2| < \infty.
\]

Since \( \varphi(z) \) is a self-map, we get

\[
\sup_{z \in D} (1 - |z|^2)|(\varphi'(z))^2 g(z)| < +\infty. \quad (14)
\]

Fix \( a \in D \) with \( |a| > \frac{1}{2} \), we still take the test functions \( h_a \) in (9). Noting that \( h_a''(a) = 0 \), \( h_a'(a) = \frac{(\lambda - 5)a^2}{(1 - |a|^2)^{\frac{\lambda - 1}{2}}} \), it follows that for all \( \lambda \in D \) with \( |\varphi(\lambda)| > \frac{1}{2} \), we have

\[
\|h_a\|_{L^{2,\lambda}} \geq \|I_g C_{\varphi} h_a\|_Z \geq \sup_{z \in D} (1 - |z|^2)|(I_g C_{\varphi} h_a)'(z)| = \sup_{z \in D} (1 - |z|^2)|(\varphi'(z)g'(z) + g''(z))h_a'(\varphi(z)) + h_a''(\varphi(z))(\varphi'(z))^2 g(z)|.
\]
Let \( a = \varphi(\lambda) \), it follows that
\[
\|h_a\|_{L^2,\lambda} \geq (1 - |\lambda|^2)(|\varphi'(\lambda)g'\lambda + \varphi''(\lambda)g(\lambda)|)h'_\varphi(\lambda)(\varphi(\lambda))
\]
\[
+ \ h''_\varphi(\lambda)(\varphi(\lambda))(\varphi'(\lambda))^2 g(\lambda)
\]
\[
= (1 - |\lambda|^2)(|\varphi'(\lambda)|^2)(\lambda - 5)\frac{|\varphi(\lambda)|^2}{(1 - |\varphi(\lambda)|^2)^{\frac{3}{2}}} \geq \frac{5 - \lambda}{4} \frac{(1 - |\lambda|^2)|\varphi'(\lambda)|^2 g(\lambda)}{(1 - |\varphi(\lambda)|^2)^{\frac{3}{2}}}.
\]

For \( \forall \lambda \in D \) with \( |\varphi(\lambda)| \leq \frac{1}{2} \), by (14), we have
\[
\sup_{|\lambda| < 1} \frac{(1 - |\lambda|^2)|\varphi'(\lambda)|^2 g(\lambda)}{(1 - |\varphi(\lambda)|^2)^{\frac{3}{2}}} \leq \left( \frac{4}{3} \right) \frac{5 - \lambda}{4} \sup_{|\lambda| < 1} (1 - |\lambda|^2)|\varphi'(\lambda)|^2 g(\lambda) < +\infty.
\]

Hence (12) holds.

Next, we take
\[
r_a(z) = \left( 1 - \frac{|a|^2}{1 - \bar{a}z} \right)^{\frac{1}{2}}
\] (15)

for \( z \in D \). Similar to the case of \( f_a \), we have \( r_a \in L^{2,\lambda} \) and sup\( \frac{1}{2} < |a| < 1 \) \( \|r_a\|_{L^2,\lambda} \leq 1 \). Then,
\[
\|r_a\|_{L^2,\lambda} \geq \|I_gC_r a\|_z \geq (1 - |a|^2)\|I_gC_r a\|_z \geq (1 - |a|^2)(|\varphi'(z)g'z + \varphi''(z)g(\lambda)|)r'_\varphi(\varphi(z))
\]
\[
- (1 - |a|^2)r'^{\prime\prime}_\varphi(\varphi(z))(\varphi'(z))^2 g(\lambda).
\]

Therefore, by Lemma 2 and (12), we obtain that
\[
\sup_{z \in D} (1 - |z|^2)(|\varphi'(z)g'z + \varphi''(z)g(\lambda)|)r'_\varphi(\varphi(z))
\]
\[
\leq \sup_{z \in D} (1 - |z|^2)r'^{\prime\prime}_\varphi(\varphi(z))(\varphi'(z))^2 g(\lambda) + C\|r_a\|_{L^2,\lambda}
\]
\[
\leq \sup_{z \in D} \frac{(1 - |z|^2)|g(\lambda)(\varphi'(z))^2|}{(1 - |\varphi(\lambda)|^2)^{\frac{3}{2}}} \|r_a\|_{L^2,\lambda} \quad \quad \text{and}
\]
\[
\|r_a\|_{L^2,\lambda} < \infty.
\]

Let \( a = \varphi(z) \), it follows that
\[
\sup_{z \in D} (1 - |z|^2)(|\varphi'(z)g'z + \varphi''(z)g(\lambda)|)r'_\varphi(\varphi(z))
\]
\[
\leq \sup_{z \in D} (1 - |z|^2)r'^{\prime\prime}_\varphi(\varphi(z))(\varphi'(z))^2 g(\lambda) + C\|r_a\|_{L^2,\lambda}
\]
\[
\leq \sup_{z \in D} \frac{(1 - |z|^2)|g(\lambda)(\varphi'(z))^2|}{(1 - |\varphi(\lambda)|^2)^{\frac{3}{2}}} \|r_a\|_{L^2,\lambda} \quad \quad \text{and}
\]
\[
\|r_a\|_{L^2,\lambda} < \infty.
\]

For \( \forall \lambda \in D \) with \( |\varphi(\lambda)| \leq \frac{1}{2} \), by (13), we have
\[
\sup_{z \in D} (1 - |z|^2)(|\varphi'(z)g'z + \varphi''(z)g(\lambda)|)r'_\varphi(\varphi(z))
\]
\[
\leq \left( \frac{4}{3} \right) \frac{5 - \lambda}{4} \sup_{\lambda \in D} (1 - |\lambda|^2)|\varphi'(\lambda)g'\lambda + \varphi''(\lambda)g(\lambda)| < \infty.
\]

Hence (11) holds.

Conversely, suppose that (11) and (12) hold. For \( f \in L^{2,\lambda} \), by Lemma 2, we have the following inequality:
\[
(1 - |z|^2)(I_gC_r f)'(z)
\]
\[
= (1 - |z|^2)(|\varphi'(z)g'z + \varphi''(z)g(\lambda)|)f'(\varphi(z))
\]
\[
+ f''(\varphi(z))(\varphi'(z))^2 g(\lambda)
\]
\[
\leq (1 - |z|^2)(|\varphi'(z)g'z + \varphi''(z)g(\lambda)|)f'(\varphi(z))
\]
\[
+ (1 - |z|^2)f''(\varphi(z))(\varphi'(z))^2 g(\lambda)
\]
\[
\leq \frac{(1 - |z|^2)(|\varphi'(z)g'z + \varphi''(z)g(\lambda)|)}{(1 - |\varphi(\lambda)|^2)^{\frac{3}{2}}} \|f\|_{L^2,\lambda}
\]
\[
\quad + \frac{(1 - |z|^2)(|\varphi'(z)g'z + \varphi''(z)g(\lambda)|)}{(1 - |\varphi(\lambda)|^2)^{\frac{3}{2}}} \|f\|_{L^2,\lambda}
\]
\[
\leq \|f\|_{L^2,\lambda},
\]
\[
\text{and}
\]
\[
\|I_gC_r f\|_0(0) + \|I_gC_r f\|_0(0)
\]
\[
= \|f(\varphi(0))\varphi'(0)g(0)\|
\]
\[
\leq \frac{|\varphi'(0)g(0)|}{(1 - |\varphi(\lambda)|^2)^{\frac{3}{2}}} \|f\|_{L^2,\lambda}.
\]

This shows that \( I_gC_r \) is bounded. This completes the proof of Theorem 8.

**Theorem 9** Let \( g \) be an analytic function on the unit disc \( D \) and \( \varphi \) an analytic self-map of \( D \). Then \( I_gC_\varphi \)
is bounded from the little analytic Morrey spaces $L_{0}^{2,\lambda}$ to the little Zygmund space $Z_0$ if and only if (11) and (12) hold, and the following are satisfied:

\[
\lim_{|z| \to 1} (1 - |z|^2)|\varphi'(z)g'(z) + \varphi''(z)g(z)| = 0; \tag{16}
\]

\[
\lim_{|z| \to 1} (1 - |z|^2)|\varphi(z)(\varphi'(z))^2| = 0. \tag{17}
\]

**Proof:** Suppose that $I_g C_\varphi$ is bounded from $L_{0}^{2,\lambda}$ to $Z_0$. Then $I_g C_\varphi z \in Z_0$, then

\[
\lim_{|z| \to 1} (1 - |z|^2)|\varphi'(z)g'(z) + \varphi''(z)g(z)| = 0,
\]

i.e. that (16) holds. Also, $I_g C_\varphi z^2 \in Z_0$, thus

\[
\lim_{|z| \to 1} (1 - |z|^2)2\varphi(z)\varphi'(z)g'(z) + 2g(z)\varphi''(z) + 2g(z)(\varphi'(z))^2 = 0
\]

Since $|\varphi| \leq 1$, we get

\[
\lim_{|z| \to 1} (1 - |z|^2)|g(z)(\varphi'(z))^2| = 0.
\]

Hence (17) holds.

On the other hand, by Lemma 4 and Theorem 8, we obtain that (11) and (12) hold.

Conversely, let

\[
M_1 = \sup_{z \in D} \frac{(1 - |z|^2)|\varphi'(z)g'(z) + \varphi''(z)g(z)|}{(1 - |\varphi(z)|^2)^{\frac{\lambda+1}{2}}} < \infty;
\]

\[
M_2 = \sup_{z \in D} \frac{(1 - |z|^2)|g(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^{\frac{\lambda+1}{2}}} < \infty.
\]

For $\forall f \in L_{0}^{2,\lambda}$, by Lemma 3, given $\epsilon > 0$ there is a $0 < \delta < 1$ such that $(1 - |z|^2)^{\frac{\lambda+1}{2}}|f'(z)| < \frac{\epsilon}{2M_1}$ and $(1 - |z|^2)^{\frac{\lambda+1}{2}}|f''(z)| < \frac{\epsilon}{2M_2}$ for all $z$ with $\delta < |z| < 1$.

If $|\varphi(z)| > \delta$, it follows that

\[
(1 - |z|^2)|(I_g C_\varphi f)'(z)|
\]

\[
= (1 - |z|^2)((\varphi'(z)g'(z) + \varphi''(z)g(z))f'(\varphi(z))
\]

\[
+ f''(\varphi(z))(\varphi'(z))^2g(z)|
\]

\[
\leq (1 - |z|^2)((\varphi'(z)g'(z) + \varphi''(z)g(z))f'(\varphi(z))
\]

\[
+ (1 - |z|^2)f''(\varphi(z))(\varphi'(z))^2g(z)|
\]

\[
< \frac{(1 - |z|^2)|\varphi'(z)g'(z) + \varphi''(z)g(z)|}{(1 - |\varphi(z)|^2)^{\frac{\lambda+1}{2}}} \frac{\epsilon}{2M_1}
\]

\[
+ \frac{(1 - |z|^2)|f''(\varphi(z))(\varphi'(z))^2g(z)|}{(1 - |\varphi(z)|^2)^{\frac{\lambda+1}{2}}} \frac{\epsilon}{2M_2}
\]

\[
< \epsilon,
\]

We know that there exists a constant $K$ such that $|f'(z)| \leq K$ and $|f''(z)| \leq K$ for all $|z| \leq \delta$.

If $|\varphi(z)| \leq \delta$, it follows that

\[
(1 - |z|^2)|(I_g C_\varphi f)'(z)|
\]

\[
= (1 - |z|^2)((\varphi'(z)g'(z) + \varphi''(z)g(z))f'(\varphi(z))
\]

\[
+ f''(\varphi(z))(\varphi'(z))^2g(z)|
\]

\[
\leq (1 - |z|^2)((\varphi'(z)g'(z) + \varphi''(z)g(z))f'(\varphi(z))
\]

\[
+ (1 - |z|^2)f''(\varphi(z))(\varphi'(z))^2g(z)|
\]

\[
< K(1 - |z|^2)|\varphi'(z)g'(z) + \varphi''(z)g(z)|
\]

\[
+ K(1 - |z|^2)|(\varphi'(z))^2g(z)|,
\]

Thus we conclude that $(1 - |z|^2)|(I_g C_\varphi (f))''(z)| \to 0$ as $|z| \to 1^-$. Hence $I_g C_\varphi f \in Z_0$ for all $f \in L_{0}^{2,\lambda}$. On the other hand, $I_g C_\varphi$ is bounded from $L^{2,\lambda}$ to $Z_0$ by Theorem 5. Hence $I_g C_\varphi$ is a bounded operator from $L_{0}^{2,\lambda}$ to $Z_0$.

**Corollary 10** The Volterra-type operator $I_g : L^{2,\lambda} \to Z$ is a bounded operator if and only if $g = 0$.

## 5 Compactness of $T_g C_\varphi$ and $I_g C_\varphi$

In order to prove the compactness of $T_g C_\varphi$, we require the following lemmas.
Lemma 11 Suppose that $T_g C_\varphi$ be a bounded operator from $L^{2,\lambda}$ to $Z$. Then $T_g C_\varphi$ is compact if and only if for any bounded sequence $\{f_n\}$ in $L^{2,\lambda}$ which converges to 0 uniformly on compact subsets of $D$. We have $\|T_g C_\varphi(f_n)\|_Z \to 0$, as $n \to \infty$.

The proof is similar to that of Proposition 3.11 in [4]. The details are omitted.

Lemma 12 Let $U \subset Z_0$. Then $U$ is compact if and only if it is closed, bounded and satisfies

$$\lim sup_{|z| \to a} (1 - |z|^2) f''(z) = 0.$$ 

The proof is similar to that of Lemma 1 in [10], we omit it.

Theorem 13 Let $g$ be an analytic function on the unit disc $D$ and $\varphi$ an analytic self-map of $D$. Suppose that $T_g C_\varphi$ is a bounded operator from $L^{2,\lambda}$ to $Z$. Then $T_g C_\varphi$ is compact if and only if the following are satisfied:

$$\lim_{|\varphi(z)| \to 1} (1 - |\varphi(z)|^2) |\varphi'(z)| g'(z) = 0;$$  

$$\lim_{|\varphi(z)| \to 1} (1 - |\varphi(z)|^2) \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{\frac{1}{2}}} = 0.$$  

**Proof:** Suppose that $T_g C_\varphi$ is compact from $L^{2,\lambda}$ to the Zygmund space $Z$. Let $\{z_n\}$ be a sequence in $D$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. If such a sequence does not exist, then (18) and (19) are automatically satisfied. Without loss of generality we may suppose that $|\varphi(z_n)| > \frac{1}{2}$ for all $n$. We take the test functions

$$f_n(z) = \frac{1 - |\varphi(z)|^2}{(1 - \varphi(z))^{\lambda}} - \frac{|\varphi'(z)|^2}{(1 - \varphi(z))^2}.$$  

By the proof of Theorem 5 we know that $\sup_{|z| \to 1} |f_n|_{L^{2,\lambda}} \leq C < \infty$. Then $\{f_n\}$ is a bounded sequence in $L^{2,\lambda}$ which converges to 0 uniformly on compact subsets of $D$. Then $\lim_{n \to \infty} \|T_g C_\varphi(f_n)\|_Z = 0$ by Lemma 11. Note that $f_n(\varphi(z_n)) = 0$ and $f'_n(\varphi(z_n)) = \frac{-\varphi(\overline{z_n})}{(1 - |\varphi(z_n)|^2)^{\frac{1}{2}}}$.

It follows that

$$\|T_g C_\varphi f_n\|_Z \geq (1 - |z_n|^2)|g'(z_n)\varphi'(z_n)| f'_n(\varphi(z_n))$$

$$+ |g''(z_n)f_n(\varphi(z_n))| \geq (1 - |z_n|^2)|\varphi'(z_n)| \frac{1 - |\varphi(z_n)|^2}{(1 - |\varphi(z_n)|^2)^{\frac{1}{2}}}$$

$$\geq \frac{(1 - |z_n|^2)|\varphi'(z_n)| |\varphi''(z_n)|}{2(1 - |\varphi(z_n)|^2)^{\frac{1}{2}}}.$$  

Then $\lim_{n \to \infty} \frac{(1 - |z_n|^2)|\varphi'(z_n)| |\varphi''(z_n)|}{(1 - |\varphi(z_n)|^2)^{\frac{1}{2}}} = 0$. Thus (19) holds.

Next, let

$$h_n(z) = \frac{1 - |\varphi(z_n)|^2}{(1 - \varphi(z_n))^{\lambda}} - \frac{3 - \lambda}{5 - \lambda} \frac{(1 - |\varphi(z_n)|^2)^{\frac{1}{2}}}{(1 - |\varphi(z_n)|^2)}.$$  

We know that $\{h_n\}$ is a bounded sequence in $L^{2,\lambda}$ which converges to 0 uniformly on compact subsets of $D$. Then $\lim_{n \to \infty} \|T_g C_\varphi(h_n)\|_Z = 0$ by Lemma 11.

Note that $h_n(\varphi(z_n)) = \frac{2}{(5 - \lambda)(1 - |\varphi(z_n)|^2)^{\frac{1}{2}}}$. Hence (18) holds. The proof of the necessary is completed.

Conversely, suppose that (18) and (19) hold. Since $T_g C_\varphi$ is a bounded operator, by Theorem 5, we have

$$M_1 = \sup_{z \in D} \frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)^{\frac{1}{2}}} < \infty;$$

$$M_2 = \sup_{z \in D} \frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)^{\frac{1}{2}}} < \infty.$$  

Let $\{f_n\}$ be a bounded sequence in $L^{2,\lambda}$ with $\|f_n\|_{L^{2,\lambda}} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $D$. We only prove $\lim_{n \to \infty} \|T_g C_\varphi(f_n)\|_Z = 0$ by Lemma 11. By the assumption, for any $\epsilon > 0$, there is a constant $\delta$, $0 < \delta < 1$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1 - |z|^2)}{(1 - |\varphi(z)|^2)^{\frac{1}{2}}} < \epsilon.$$
and
\[
\frac{(1 - |z|^2)|g''(z)|}{(1 - |\phi(z)|^2)^{1 + \frac{1}{2}}} < \epsilon.
\]

Let \( K = \{ w \in D : |w| \leq \delta \} \). Noting that \( K \) is a compact subset of \( D \), we get that
\[
z(T_g C_\phi f_n) = \sup_{z \in D} (1 - |z|^2)|(T_g C_\phi f_n)''(z)|
\]
\[
\leq \sup_{z \in D} (1 - |z|^2)|\phi'(z)g'(z)f_n'(\phi(z))|
\]
\[
+ \sup_{z \in D} (1 - |z|^2)|g''(z)f_n(\phi(z))|
\]
\[
\leq 2\epsilon + \sup_{|\phi(z)| \leq \delta} (1 - |z|^2)|\phi'(z)g'(z)f_n'(\phi(z))|
\]
\[
+ \sup_{|\phi(z)| \leq \delta} (1 - |z|^2)|g''(z)f_n(\phi(z))|
\]
\[
\leq 2\epsilon + M_1 \sup_{w \in K} |f_n'(w)| + M_2 \sup_{w \in K} |f_n(w)|.
\]

As \( n \to \infty \),
\[
\| T_g C_\phi f_n \|_2 \to 0.
\]

Hence \( T_g C_\phi \) is compact. This completes the proof of Theorem 13.

**Theorem 14** Let \( g \) be an analytic function on the unit disc \( D \), and \( \phi \) an analytic self-map of \( D \). Then \( T_g C_\phi \) is compact from \( L^{2,\lambda}_D \) to \( Z_0 \) if and only if the following are satisfied:

\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)|\phi'(z)g'(z)|}{(1 - |\phi(z)|^2)^{1 + \frac{1}{2}}} = 0; \quad (22)
\]

\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)|g''(z)|}{(1 - |\phi(z)|^2)^{1 + \frac{1}{2}}} = 0. \quad (23)
\]

**Proof:** Assume (22) and (23) hold. From Theorem 6, we know that \( T_g C_\phi \) is bounded from \( L^{2,\lambda}_D \) to \( Z_0 \). Suppose that \( f \in L^{2,\lambda}_D \) with \( \| f \|_{L^{2,\lambda}} \leq 1 \). We obtain that
\[
(1 - |z|^2)|(T_g C_\phi f)''(z)|
\]
\[
= (1 - |z|^2)|\phi'(z)g'(z)f'(\phi(z)) + g''(z)f(\phi(z))|
\]
\[
\leq (1 - |z|^2)|\phi'(z)g'(z)f'(\phi(z))|
\]
\[
+ (1 - |z|^2)|g''(z)f(\phi(z))|
\]
\[
\leq \frac{(1 - |z|^2)|\phi'(z)g'(z)|}{(1 - |\phi(z)|^2)^{1 + \frac{1}{2}}} + \frac{(1 - |z|^2)|g''(z)|}{(1 - |\phi(z)|^2)^{1 + \frac{1}{2}}},
\]

thus
\[
\sup \{(1 - |z|^2)|(T_g C_\phi f)''(z)|
\]
\[
: f \in L^{2,\lambda}_D, \| f \|_{L^{2,\lambda}} \leq 1\}
\]
\[
\leq \frac{(1 - |z|^2)|\phi'(z)g'(z)|}{(1 - |\phi(z)|^2)^{1 + \frac{1}{2}}} + \frac{(1 - |z|^2)|g''(z)|}{(1 - |\phi(z)|^2)^{1 + \frac{1}{2}}},
\]

and it follows that
\[
\lim_{|z| \to 1} \sup \{(1 - |z|^2)|(T_g C_\phi f)''(z)|
\]
\[
: f \in L^{2,\lambda}_D, \| f \|_{L^{2,\lambda}} \leq 1\} = 0,
\]

hence \( T_g C_\phi : L^{2,\lambda}_D \to Z_0 \) is compact by Lemma 12.

Conversely, suppose that \( T_g C_\phi : L^{2,\lambda}_D \to Z_0 \) is compact.

First, it is obvious that \( T_g C_\phi : L^{2,\lambda}_D \to Z_0 \) is bounded, then by Theorem 6, we have \( g \in Z_0 \) and that (10) holds. On the other hand, by Lemma 12 we have
\[
\lim_{|z| \to 1} \sup \{(1 - |z|^2)|(T_g C_\phi f)''(z)|
\]
\[
: f \in L^{2,\lambda}_D, \| f \|_{L^{2,\lambda}} \leq M\} = 0,
\]

for some \( M > 0 \).

Next, noting that the proof of Theorem 5 and the fact that the functions given in (8) are in \( L^{2,\lambda}_D \) and have norms bounded independently of \( a \), we obtain that
\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)|\phi'(z)g'(z)|}{(1 - |\phi(z)|^2)^{1 + \frac{1}{2}}} = 0
\]
\[
\frac{(1 - |z|^2)|g''(z)|}{(1 - |\phi(z)|^2)^{1 + \frac{1}{2}}} = 0
\]

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for $|\varphi(z)| > \frac{1}{2}$. However, if $|\varphi(z)| \leq \frac{1}{2}$, by (10), we easily have

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)|g''(z)|}{(1 - |\varphi(z)|^2)^{\frac{1-\alpha}{2}}} \leq \frac{4}{3} \lim_{|z| \to 1} (1 - |z|^2)|\varphi'(z)g'(z)| = 0.$$ 

Thus (22) holds.

Similarly, noting that the functions given in (9) are in $L^{2,\lambda}_0$ and have norms bounded independently of $a$, we obtain that

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)|\varphi''(z)g'(z)|}{(1 - |\varphi(z)|^2)^{\frac{1-\alpha}{2}}} \leq \lim_{|z| \to 1} (1 - |z|^2)|(T_g C \varphi h_a''(z)|,$n

for $|\varphi(z)| > \frac{1}{2}$. Then

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)|g''(z)|}{(1 - |\varphi(z)|^2)^{\frac{1-\alpha}{2}}} = 0$$

for $|\varphi(z)| > \frac{1}{2}$. However, if $|\varphi(z)| \leq \frac{1}{2}$, by $g \in \mathcal{Z}_0$, we easily have

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)|g''(z)|}{(1 - |\varphi(z)|^2)^{\frac{1-\alpha}{2}}} = \lim_{|z| \to 1} \frac{4}{3} \frac{1-\lambda}{2} (1 - |z|^2)|g''(z)| = 0.$$

This completes the proof of Theorem 14.

Using the same methods as in the proof of Theorem 13 and 14, we can prove the following results.

**Theorem 15** Let $g$ be an analytic function on the unit disc $D$ and $\varphi$ an analytic self-map of $D$. Suppose that $I_g C \varphi$ is a bounded operator from $L^{2,\lambda}_0$ to $\mathcal{Z}_0$. Then $I_g C \varphi$ is compact if and only if the following are satisfied:

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)|\varphi'(z)g'(z) + \varphi''(z)g(z)|}{(1 - |\varphi(z)|^2)^{\frac{1-\alpha}{2}}} = 0;$$

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)|g(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^{\frac{1-\alpha}{2}}} = 0. \quad (24)$$

**Theorem 16** Let $g$ be an analytic function on the unit disc $D$ and $\varphi$ an analytic self-map of $D$. Then $I_g C \varphi$ is compact from $L^{2,\lambda}_0$ to $\mathcal{Z}_0$ if and only if the following are satisfied:

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)|\varphi'(z)g'(z) + \varphi''(z)g(z)|}{(1 - |\varphi(z)|^2)^{\frac{1-\alpha}{2}}} = 0; \quad (25)$$

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)|g(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^{\frac{1-\alpha}{2}}} = 0.$$


