Products of Volterra-type operators and composition operators from analytic Morrey spaces into Zygmund spaces

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Abstract: In recent years the composition operator C_φ has been received much attention and appear in various settings in the literature. It is interesting to provide a function theoretic characterization when φ induces a bounded or compact composition operator on various function spaces. In this paper we consider the products of Volterra-type operators and composition operators. We characterize the boundedness and compactness of the products of Volterra-type operators and composition operators T_gC_φ and I_gC_φ from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$, and the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$ over the unit disk, respectively.

Key–Words: Analytic Morrey space, Zygmund space, Volterra-type operators, Composition operators, Boundedness, Compactness

1 Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane and $H(D)$ denote the set of all analytic functions on D. Let φ be an analytic self-map of the unit disk D. Associated with φ is the composition operator C_φ defined by

$$C_\varphi f = f \circ \varphi, \qquad f \in H(D).$$

It is interesting to provide a function theoretic characterization when φ induces a bounded or compact composition operator on various function spaces. Boundedness and compactness of composition operators on various function spaces have been studied by numerous authors, for example, see [3, 4, 10, 11, 13, 14, 15, 17, 20, 25].

For an arc $I \subset \partial\mathbb{D}$, let $|I| = \frac{1}{2\pi}\int_I |d\zeta|$ be the normalized arc length of I,

$$f_I = \frac{1}{|I|}\int_I f(\zeta)\frac{|d\zeta|}{2\pi}, f \in H(D),$$

and $S(I)$ be the Carleson box based on I with

$$S(I) = \{z \in D : 1 - |I| \le |z| < 1, \frac{z}{|z|} \in I\}.$$

Clearly, if $I = \partial D$, then $S(I) = D$.

Let $\mathcal{L}^{2,\lambda}(D)$ represent the analytic $Morrey$ spaces of all analytic functions $f \in H^2$ on D such that

$$\sup_{I\subset\partial D}\big(\frac{1}{|I|^\lambda}\int_I |f(\zeta) - f_I|^2\frac{|d\zeta|}{2\pi}\big)^{1/2} < \infty,$$

where $0 < \lambda \le 1$ and the Hardy space H^2 consists of analytic functions f in D satisfying

$$\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2\,d\theta < \infty.$$

From Theorem 3.1 of [21] or Theorem 3.21 of [23], we can define the norm of function $f \in \mathcal{L}^{2,\lambda}(D)$ and its equivalent formula as follows

$$\begin{aligned}\|f\|_{\mathcal{L}^{2,\lambda}} &= |f(0)| + \\ &\sup_{I\subset\partial D}\big(\frac{1}{|I|^\lambda}\int_{S(I)}|f'(z)|^2(1-|z|^2)dm(z)\big)^{1/2} \\ \approx\ & |f(0)| + \\ &\sup_{a\in D}((1-|a|^2)^{1-\lambda}\int_D |f'(z)|^2(1-|\varphi_a(z)|^2)dm(z))^{1/2}.\end{aligned}$$

Similarly to the relation between $BMOA$ space and $VMOA$ space, we have that $f \in \mathcal{L}_0^{2,\lambda}(D)$, the little analytic $Morrey$ spaces, if $f \in \mathcal{L}^{2,\lambda}(D)$ and

$$\lim_{|I|\to 0}\big(\frac{1}{|I|^\lambda}\int_I |f(\zeta) - f_I|^2\frac{|d\zeta|}{2\pi}\big)^{1/2} = 0.$$

Clearly, $\mathcal{L}_0^{2,1}(D) = VMOA$. The following lemma gives equivalent conditions of $\mathcal{L}_0^{2,\lambda}$. The proof is similar to that of Theorem 6.3 in [6], we omit the details.

Lemma 1 *Suppose that* $0 < \lambda < 1$ *and* $f \in H(D)$. *Let* $a \in D$, $\varphi_a(z) = \dfrac{a-z}{1-\bar{a}z}$. *Then the following statements are equivalent.*

(i) $f \in \mathcal{L}_0^{2,\lambda}(D)$;

(ii) $\lim\limits_{|a|\to 1}(1-|a|^2)^{1-\lambda}$

$$\times \int_D |f'(z)|^2(1-|\varphi_a(z)|^2)dm(z) = 0; \qquad (1)$$

(iii) $\lim\limits_{|a|\to 1}(1-|a|^2)^{1-\lambda}$

$$\times \int_D |f'(z)|^2 \log \frac{1}{|\varphi_a(z)|} dm(z) = 0. \qquad (2)$$

It is known that $\mathcal{L}^{2,1}(D) = BMOA$ and if $0 < \lambda < 1$, $BMOA \subsetneq \mathcal{L}^{2,\lambda}(D)$. For more information on $BMOA$ and $VMOA$, see [6].

The Zygmund space $\mathcal{Z}$ consists of all analytic functions f defined on D such that

$$z(f) = \sup\{(1-|z|^2)|f''(z)| : z \in D\} < +\infty.$$

From a theorem of Zygmund (see [30, vol. I, p. 263] or [5, Theorem 5.3]), we see that $f \in \mathcal{Z}$ if and only if f is continuous in the close unit disk $\overline{D} = \{z : |z| \le 1\}$ and the boundary function $f(e^{i\theta})$ such that

$$\sup_{h>0,\theta} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$

An analytic function $f \in H(D)$ is said to belong to the little Zymund space $\mathcal{Z}_0$ consists of all $f \in \mathcal{Z}$ satisfying $\lim_{|z|\to 1}(1-|z|^2)|f''(z)| = 0$. It can easily proved that $\mathcal{Z}$ is a Banach space under the norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + z(f)$$

and the polynomials are norm-dense in closed subspace $\mathcal{Z}_0$ of $\mathcal{Z}$. For some other information on this space and some operators on it, see, for example, [7, 8, 26, 27].

Suppose that $g : D \longrightarrow \mathbf{C}$ is a analytic map. Let T_g and I_g denote the Volterra-type operators with the analytic symbol g on D respectively:

$$T_g f(z) = \int_0^z f(w)g'(w)\,dw$$

and

$$I_g f(z) = \int_0^z f'(w)g(w)\,dw, \qquad z \in D.$$

In [12] Pommerenke introduced the Volterra-type operator T_g and showed that T_g is a bounded operator on the Hardy space H^2 if and only if $g \in BMOA$. In [26] the author studied the boundedness and compactness of T_g between the α-Bloch spaces β_α and the logarithmic Bloch space $\mathcal{LB}^1$. Boundedness and compactness of this operators T_g acting on various function spaces have been studied in many literature. See [1, 2, 16, 18, 19, 20, 22] for more information.

Here, we consider the products of Volterra-type operators and composition operators, which are defined by

$$(T_g C_\varphi f)(z) = \int_0^z (f\circ\varphi)(\zeta)g'(\zeta)\,d\zeta,\ f \in H(D)$$

and

$$(I_g C_\varphi f)(z) = \int_0^z (f\circ\varphi)'(\zeta)g(\zeta)\,d\zeta,\ f \in H(D).$$

In [8], Li and Stević studied those operators from H^∞ and Bloch spaces to Zygmund Spaces. The author in [28] characterized the boundedness and compactness of those operators on the logarithmic Bloch space $\mathcal{LB}^1$. Xiao and Xu [24] studied the composition operators on the analytic $Morrey$ spaces $\mathcal{L}^{2,\lambda}$ spaces. Li, Liu and Lou[9] studied the Volterra-type operators on $\mathcal{L}^{2,\lambda}$ spaces. Zhuo and Ye [29] considered this operators from $\mathcal{L}^{2,\lambda}$ spaces to the classical Bloch space. In 2006, the boundedness of composition operators on the Zygmund space $\mathcal{Z}$ was first studied by Choe, Koo, and Smith in [3]. Later, many researchers have studied composition operators and weighted composition operators acting on the Zygmund space $\mathcal{Z}$. Li and Stević in [7] studied the boundedness and compactness of the generalized composition operators on Zygmund spaces and Bloch type spaces. Ye and Hu in [27] characterized boundedness and compactness of weighted composition operators on the Zygmund space $\mathcal{Z}$. In this paper theboundedness and compactness of those operators from analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ into Zygmund spaces $\mathcal{Z}$ are discussed. As some corollaries we obtain the boundedness and compactness for T_g and I_g from $\mathcal{L}^{2,\lambda}$ into $\mathcal{Z}$ spaces.

$Notations$: For two functions F and G, if there is a constant $C > 0$ dependent only on indexes p, λ... such that $F \le CG$, then we say that $F \lesssim G$. Furthermore, denote that $F \approx G$ (F is comparable with G) whenever $F \lesssim G \lesssim F$.

2 Auxiliary results

In order to prove the main results of this paper. we need some auxiliary results.

Lemma 2 *Let* $0 < \lambda < 1$. *If* $f \in \mathcal{L}^{2,\lambda}$, *then*

(i) $|f(z)| \lesssim \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{1-\lambda}{2}}}$ *for every* $z \in D$;

(ii) $|f'(z)| \lesssim \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{3-\lambda}{2}}}$ *for every* $z \in D$;

(iii) $|f''(z)| \lesssim \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{5-\lambda}{2}}}$ *for every* $z \in D$.

Proof: (i) and (ii) are from Lemma 2.5 in [9].

For any $f \in \mathcal{L}^{2,\lambda}$. Fix $z \in D$ and let $\rho = \frac{1+|z|}{2}$, by the Cauchy integral formula, we obtain that

$$\begin{aligned} |f''(z)| &= |\frac{1}{2\pi i}\int_{|\xi|=\rho}\frac{f'(\xi)}{(\xi-z)^2}\,d\xi| \\ &\leq \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-\rho^2)^{\frac{3-\lambda}{2}}}\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho\,d\theta}{|\rho e^{i\theta}-z|^2} \\ &= \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-\rho^2)^{\frac{3-\lambda}{2}}}\frac{\rho}{\rho^2-|z|^2} \lesssim \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{5-\lambda}{2}}}. \end{aligned}$$

Hence (iii) holds.

Lemma 3 *Let* $0 < \lambda < 1$. *If* $f \in \mathcal{L}_0^{2,\lambda}$, *then*

(i) $\lim_{|z|\to 1}(1-|z|^2)^{\frac{3-\lambda}{2}}|f'(z)| = 0$;

(ii) $\lim_{|z|\to 1}(1-|z|^2)^{\frac{1-\lambda}{2}}|f(z)| = 0$;

(iii) $\lim_{|z|\to 1}(1-|z|^2)^{\frac{5-\lambda}{2}}|f''(z)| = 0$.

The proof of (i) is similar to that of Lemma 2.5 in [9], and we easily obtain (ii) and (iii) by (i). These details are omitted here.

Lemma 4 *Suppose* $T_gC_\varphi($ *or* $I_gC_\varphi) : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ *is a bounded operator, then* $T_gC_\varphi($ *or* $I_gC_\varphi) : \mathcal{L}^{2,\lambda} \to \mathcal{Z}$ *is a bounded operator.*

The proof is similar to that of Lemma 2.3 in [26]. The details are omitted.

3 Boundedness of T_gC_φ

In this section we characterize the boundedness of the operator T_gC_φ from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$, and the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$, respectively.

Theorem 5 *Let g be an analytic function on the unit disc D and φ an analytic self-map of D. Then T_gC_φ is a bounded operator from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$ if and only if the following are satisfied:*

$$\sup_{z\in D}\frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \infty; \tag{3}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \infty. \tag{4}$$

Proof: Suppose T_gC_φ is bounded from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$. Using functions $f(z) = 1$ and $f(z) = z$ in $\mathcal{L}^{2,\lambda}$, we have

$$g \in \mathcal{Z}, \tag{5}$$

and

$$\sup_{z\in D}(1-|z|^2)|\varphi'(z)g'(z)+\varphi(z)g''(z)| < +\infty. \tag{6}$$

Since $\varphi(z)$ is a self-map, we get

$$K_1 = \sup_{z\in D}(1-|z|^2)|\varphi'(z)g'(z)| < +\infty. \tag{7}$$

Fix $a \in D$ with $|a| > \frac{1}{2}$, we take the test functions:

$$f_a(z) = \frac{1-|a|^2}{(1-\bar{a}z)^{\frac{3-\lambda}{2}}} - \frac{(1-|a|^2)^2}{(1-\bar{a}z)^{\frac{5-\lambda}{2}}} \tag{8}$$

for $z \in D$. Then, arguing as the proof of Lemma 3.2 in [9] we obtain that $f_a \in \mathcal{L}^{2,\lambda}$ and $\sup_a \|f_a\|_{\mathcal{L}^{2,\lambda}} \lesssim 1$. Since $f_a(a) = 0$, $f_a'(a) = \frac{-\bar{a}}{(1-|a|^2)^{\frac{3-\lambda}{2}}}$, therefore, for all $\lambda \in D$ with $|\varphi(\lambda)| > \frac{1}{2}$, we have

$$\begin{aligned} &\|f_a\|_{\mathcal{L}^{2,\lambda}} \gtrsim \|T_gC_\varphi f_a\|_{\mathcal{Z}} \\ \geq\ & \sup_{z\in D}(1-|z|^2)|(T_gC_\varphi f_a)''(z)| \\ =\ & \sup_{z\in D}(1-|z|^2)|\varphi'(z)g'(z)f_a'(\varphi(z))+g''(z)f_a(\varphi(z))|. \end{aligned}$$

Let $a=\varphi(\lambda)$, it follows that

$$\begin{aligned}\|f_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim (1-|\lambda|^2)|\varphi'(\lambda)g'(\lambda)f'_{\varphi(\lambda)}(\varphi(\lambda))\\ &+ g''(\lambda)f_{\varphi(\lambda)}(\varphi(\lambda))|\\ &= (1-|\lambda|^2)|\varphi'(\lambda)g'(\lambda)\frac{-\overline{\varphi(\lambda)}}{(1-|\varphi(\lambda)|^2)^{\frac{3-\lambda}{2}}}|\\ &\geq \frac{1}{2}\frac{(1-|\lambda|^2)|\varphi'(\lambda)g'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{3-\lambda}{2}}}.\end{aligned}$$

For $\forall\lambda\in D$ with $|\varphi(\lambda)|\leq\frac{1}{2}$, by (7), we have

$$\begin{aligned}&\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|\varphi'(\lambda)g'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{3-\lambda}{2}}}\\ \leq\ &(\frac{4}{3})^{\frac{3-\lambda}{2}}\sup_{\lambda\in D}(1-|\lambda|^2)|\varphi'(\lambda)g'(\lambda)|<+\infty.\end{aligned}$$

Hence (4) holds.

Next we will show (3) holds. Let

$$h_a(z)=\frac{1-|a|^2}{(1-\bar{a}z)^{\frac{3-\lambda}{2}}}-\frac{3-\lambda}{5-\lambda}\frac{(1-|a|^2)^2}{(1-\bar{a}z)^{\frac{5-\lambda}{2}}} \quad (9)$$

for $z\in D$. Similar to the case of f_a, we have $h_a\in\mathcal{L}^{2,\lambda}$ and $\sup_{\frac{1}{2}<|a|<1}\|h_a\|_{\mathcal{L}^{2,\lambda}}\lesssim 1$. From this and by that facts that $h'_a(a)=0$ and $h_a(a)=\dfrac{2}{5-\lambda}\dfrac{1}{(1-|a|^2)^{\frac{1-\lambda}{2}}}$, it follows that for all $\lambda\in D$ with $|\varphi(\lambda)|>\frac{1}{2}$,

$$\begin{aligned}\|h_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim (1-|\lambda|^2)|\varphi'(\lambda)g'(\lambda)h'_{\varphi(\lambda)}(\varphi(\lambda))\\ &+ g''(\lambda)h_{\varphi(\lambda)}(\varphi(\lambda))|\\ &= (1-|\lambda|^2)|g''(\lambda)\frac{2}{(5-\lambda)(1-|\varphi(\lambda)|^2)^{\frac{1-\lambda}{2}}}|\\ &= \frac{2}{5-\lambda}\frac{(1-|\lambda|^2)|g''(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{1-\lambda}{2}}}.\end{aligned}$$

For $\forall\lambda\in D$ with $|\varphi(\lambda)|\leq\frac{1}{2}$, by (5), we have

$$\begin{aligned}&\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|g''(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{1-\lambda}{2}}}\\ =\ &(\frac{4}{3})^{\frac{1-\lambda}{2}}\sup_{\lambda\in D}(1-|\lambda|^2)|g''(\lambda)|<\infty.\end{aligned}$$

Hence (3) holds.

Conversely, suppose that (3) and (4) hold. For $f\in\mathcal{L}^{2,\lambda}$, by Lemma 2, we have the following inequality:

$$\begin{aligned}&(1-|z|^2)|(T_gC_\varphi f)''(z)|\\ =\ &(1-|z|^2)|\varphi'(z)g'(z)f'(\varphi(z))+g''(z)f(\varphi(z))|\\ \leq\ &(1-|z|^2)|\varphi'(z)g'(z)f'(\varphi(z))|\\ +\ &(1-|z|^2)|g''(z)f(\varphi(z))|\\ \lesssim\ &\frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}}\\ +\ &\frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}}\\ \lesssim\ &\|f\|_{\mathcal{L}^{2,\lambda}},\end{aligned}$$

and

$$\begin{aligned}&|(T_gC_\varphi f)(0))|+|(T_gC_\varphi f)'(0)|\\ =\ &|f(\varphi(0))g'(0)|\\ \lesssim\ &\frac{|g'(0)|}{(1-|\varphi(0)|^2)^{\frac{1-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}}.\end{aligned}$$

This shows that T_gC_φ is bounded. This completes the proof of Theorem 5.

Theorem 6 *Let g be an analytic function on the unit disc D and φ an analytic self-map of D. Then T_gC_φ is bounded from the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$ if and only if (3) and (4) hold, and the following are satisfied:*

$$g\in\mathcal{Z}_0;$$

$$\lim_{|z|\to 1}(1-|z|^2)|\varphi'(z)g'(z)|=0. \quad (10)$$

Proof: Suppose that T_gC_φ is bounded from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$. Then $g(z)-g(0)=T_gC_\varphi 1\in\mathcal{Z}_0$. Also $T_gC_\varphi z\in\mathcal{Z}_0$, thus

$$(1-|z|^2)|\varphi'(z)g'(z)+\varphi(z)g''(z)|\longrightarrow 0 \quad (|z|\to 1^-).$$

Since $|\varphi|\leq 1$ and $g\in\mathcal{Z}_0$, we have $\lim_{|z|\to}(1-|z|^2)|\varphi'(z)g'(z)|=0$. Hence (10) holds.

On the other hand, by Lemma 4 and Theorem 5, we obtain that (3) and (4) hold.

Conversely, let

$$M_1=\sup_{z\in D}\frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}<\infty;$$

$$M_2 = \sup_{z\in D} \frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \infty.$$

For $\forall f \in \mathcal{L}_0^{2,\lambda}$, by Lemma 3, given $\epsilon > 0$ there is a $0 < \delta < 1$ such that $(1-|z|^2)^{\frac{3-\lambda}{2}}|f'(z)| < \frac{\epsilon}{2M_1}$ and $(1-|z|^2)^{\frac{1-\lambda}{2}}|f(z)| < \frac{\epsilon}{2M_2}$ for all z with $\delta < |z| < 1$.

If $|\varphi(z)| > \delta$, it follows that

$$\begin{aligned}
&(1-|z|^2)|(T_gC_\varphi f)''(z)| \\
=\ & (1-|z|^2)|\varphi'(z)g'(z)f'(\varphi(z)) + g''(z)f(\varphi(z))| \\
\le\ & (1-|z|^2)|\varphi'(z)g'(z)f'(\varphi(z))| \\
+\ & (1-|z|^2)|g''(z)f(\varphi(z))| \\
<\ & \frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\frac{\epsilon}{2M_1} \\
+\ & \frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}\frac{\epsilon}{2M_2} \\
<\ & \epsilon,
\end{aligned}$$

We know that there exists a constant K such that $|f(z)| \le K$ and $|f'(z)| \le K$ for all $|z| \le \delta$.

If $|\varphi(z)| \le \delta$, it follows that

$$\begin{aligned}
&(1-|z|^2)|(T_gC_\varphi f)''(z)| \\
=\ & (1-|z|^2)|\varphi'(z)g'(z)f'(\varphi(z)) + g''(z)f(\varphi(z))| \\
\le\ & (1-|z|^2)|\varphi'(z)g'(z)f'(\varphi(z))| \\
+\ & (1-|z|^2)|g''(z)f(\varphi(z))| \\
\le\ & K(1-|z|^2)|\varphi'(z)g'(z)| + K(1-|z|^2)|g''(z)|.
\end{aligned}$$

Thus we conclude that $(1-|z|^2)|(T_gC_\varphi(f))''(z)| \to 0$ as $|z| \to 1^-$. Hence $T_gC_\varphi f \in \mathcal{Z}_0$ for all $f \in \mathcal{L}_0^{2,\lambda}$. On the other hand, T_gC_φ is bounded from $\mathcal{L}^{2,\lambda}$ to $\mathcal{Z}$ by Theorem 5. Hence T_gC_φ is a bounded operator from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$.

Corollary 7 *The Volterra-type operator $T_g : \mathcal{L}^{2,\lambda} \to \mathcal{Z}$ is a bounded operator if and only if $g = 0$.*

4 Boundedness of I_gC_φ

In this section we study the boundedness of the operator

$$I_gC_\varphi : \mathcal{L}^{2,\lambda}(\text{ or } \mathcal{L}_0^{2,\lambda}) \to \mathcal{Z}(\text{ or } \mathcal{Z}_0).$$

Theorem 8 *Let g be an analytic function on the unit disc D and φ an analytic self-map of D. Then I_gC_φ is a bounded operator from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$ if and only if the following are satisfied:*

$$\sup_{z\in D} \frac{(1-|z|^2)|\varphi'(z)g'(z) + \varphi''(z)g(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \infty; \tag{11}$$

$$\sup_{z\in D} \frac{(1-|z|^2)|g(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} < \infty. \tag{12}$$

Proof: Suppose I_gC_φ is bounded from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$. Using functions $f(z) = z$ and $f(z) = z^2$ in $\mathcal{L}^{2,\lambda}$, we have

$$\sup_{z\in D}(1-|z|^2)|\varphi'(z)g'(z) + \varphi''(z)g(z)| < +\infty, \tag{13}$$

and

$$\begin{aligned}
&\sup_{z\in D}(1-|z|^2)|2\varphi(z)\varphi'(z)g'(z) \\
+\ & 2g(z)\varphi(z)\varphi''(z) + 2g(z)(\varphi'(z))^2| < \infty.
\end{aligned}$$

Since $\varphi(z)$ is a self-map, we get

$$\sup_{z\in D}(1-|z|^2)|(\varphi'(z))^2g(z)| < +\infty. \tag{14}$$

Fix $a \in D$ with $|a| > \frac{1}{2}$, we still take the test functions h_a in (9). Noting that $h_a'(a) = 0$, $h_a''(a) = \frac{(\lambda-5)\bar{a}^2}{(1-|a|^2)^{\frac{5-\lambda}{2}}}$, it follows that for all $\lambda \in D$ with $|\varphi(\lambda)| > \frac{1}{2}$, we have

$$\begin{aligned}
&\|h_a\|_{\mathcal{L}^{2,\lambda}} \gtrsim \|I_gC_\varphi h_a\|_{\mathcal{Z}} \\
\ge\ & \sup_{z\in D}(1-|z|^2)|(I_gC_\varphi h_a)''(z)| \\
=\ & \sup_{z\in D}(1-|z|^2)|\big(\varphi'(z)g'(z) + \varphi''(z)g(z)\big)h_a'(\varphi(z)) \\
+\ & h_a''(\varphi(z))(\varphi'(z))^2g(z)|.
\end{aligned}$$

Let $a=\varphi(\lambda)$, it follows that

$$\begin{aligned}
&\|h_a\|_{\mathcal{L}^{2,\lambda}}\\
\gtrsim\ & (1-|\lambda|^2)|\big(\varphi'(\lambda)g'(\lambda)+\varphi''(\lambda)g(\lambda)\big)h'_{\varphi(\lambda)}(\varphi(\lambda))\\
+\ & h''_{\varphi(\lambda)}(\varphi(\lambda))(\varphi'(\lambda))^2g(\lambda)|\\
=\ & (1-|\lambda|^2)|(\varphi'(\lambda))^2g(\lambda)\frac{(\lambda-5)\overline{\varphi(\lambda)^2}}{(1-|\varphi(\lambda)|^2)^{\frac{5-\lambda}{2}}}|\\
\geq\ & \frac{5-\lambda}{4}\frac{(1-|\lambda|^2)|\varphi'(\lambda))^2g(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{5-\lambda}{2}}}.
\end{aligned}$$

For $\forall\lambda\in D$ with $|\varphi(\lambda)|\leq\frac{1}{2}$, by (14), we have

$$\begin{aligned}
&\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|\varphi'(\lambda))^2g(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{5-\lambda}{2}}}\\
\leq\ & (\frac{4}{3})^{\frac{5-\lambda}{2}}\sup_{\lambda\in D}(1-|\lambda|^2)|\varphi'(\lambda))^2g(\lambda)|<+\infty.
\end{aligned}$$

Hence (12) holds.

Next, we take

$$r_a(z)=\frac{1-|a|^2}{(1-\bar{a}z)^{\frac{3-\lambda}{2}}} \qquad (15)$$

for $z\in D$. Similar to the case of f_a, we have $r_a\in\mathcal{L}^{2,\lambda}$ and $\sup_{\frac{1}{2}<|a|<1}\|r_a\|_{\mathcal{L}^{2,\lambda}}\lesssim 1$. Then,

$$\begin{aligned}
&\|r_a\|_{\mathcal{L}^{2,\lambda}}\gtrsim\|I_gC_\varphi r_a\|_{\mathcal{Z}}\\
\geq\ & (1-|z|^2)|(I_gC_\varphi r_a)''(z)|\\
\geq\ & (1-|z|^2)|(\varphi'(z)g'(z)+\varphi''(z)g(z))r'_a(\varphi(z))|\\
-\ & (1-|z|^2)|r''_a(\varphi(z))(\varphi'(z))^2g(z)|.
\end{aligned}$$

Therefore, by Lemma 2 and (12), we obtain that

$$\begin{aligned}
&\sup_{z\in D}(1-|z|^2)|(\varphi'(z)g'(z)+\varphi''(z)g(z))r'_a(\varphi(z))|\\
\leq\ & \sup_{z\in D}(1-|z|^2)|r''_a(\varphi(z))(\varphi'(z))^2g(z)|+C\|r_a\|_{\mathcal{L}^{2,\lambda}}\\
\lesssim\ & \sup_{z\in D}\frac{(1-|z|^2)|g(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}\|r_a\|_{\mathcal{L}^{2,\lambda}}\\
+\ & \|r_a\|_{\mathcal{L}^{2,\lambda}}<\infty.
\end{aligned}$$

Let $a=\varphi(z)$, it follows that

$$\begin{aligned}
&\sup_{z\in D}(1-|z|^2)\frac{|\varphi'(z)g'(z)+\varphi''(z)g(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\\
\lesssim\ & \sup_{z\in D}(1-|z|^2)|(\varphi'(z)g'(z)+\varphi''(z)g(z))r'_a(\varphi(z))|\\
<\ & \infty.
\end{aligned}$$

For $\forall\lambda\in D$ with $|\varphi(\lambda)|\leq\frac{1}{2}$, by (13), we have

$$\begin{aligned}
&\sup_{z\in D}(1-|\lambda|^2)\frac{|\varphi'(\lambda)g'(\lambda)+\varphi''(\lambda)g(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{3-\lambda}{2}}}\\
\leq\ & (\frac{4}{3})^{\frac{3-\lambda}{2}}\sup_{\lambda\in D}(1-|\lambda|^2)|\varphi'(\lambda)g'(\lambda)+\varphi''(\lambda)g(\lambda)|\\
<\ & \infty.
\end{aligned}$$

Hence (11) holds.

Conversely, suppose that (11) and (12) hold. For $f\in\mathcal{L}^{2,\lambda}$, by Lemma 2, we have the following inequality:

$$\begin{aligned}
&(1-|z|^2)|(I_gC_\varphi f)''(z)|\\
=\ & (1-|z|^2)|\big(\varphi'(z)g'(z)+\varphi''(z)g(z)\big)f'(\varphi(z))\\
+\ & f''(\varphi(z))(\varphi'(z))^2g(z)|\\
\leq\ & (1-|z|^2)|\big(\varphi'(z)g'(z)+\varphi''(z)g(z)\big)f'(\varphi(z))|\\
+\ & (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2g(z)|\\
\lesssim\ & \frac{(1-|z|^2)|\varphi'(z)g'(z)+\varphi''(z)g(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}}\\
+\ & \frac{(1-|z|^2)|(\varphi'(z))^2g(z)|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}}\\
\lesssim\ & \|f\|_{\mathcal{L}^{2,\lambda}},
\end{aligned}$$

and

$$\begin{aligned}
&|(I_gC_\varphi f)(0))|+|(I_gC_\varphi f)'(0)|\\
=\ & |f(\varphi(0))\varphi'(0)g(0)|\\
\lesssim\ & \frac{|\varphi'(0)g(0)|}{(1-|\varphi(0)|^2)^{\frac{1-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}}.
\end{aligned}$$

This shows that I_gC_φ is bounded. This completes the proof of Theorem 8.

Theorem 9 *Let g be an analytic function on the unit disc D and φ an analytic self-map of D. Then I_gC_φ*

is bounded from the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$ if and only if (11) and (12) hold, and the following are satisfied:

$$\lim_{|z|\to 1}(1-|z|^2)|\varphi'(z)g'(z)+\varphi''(z)g(z)|=0; \quad (16)$$

$$\lim_{|z|\to 1}(1-|z|^2)|g(z)(\varphi'(z))^2|=0. \quad (17)$$

Proof: Suppose that I_gC_φ is bounded from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$. Then $I_gC_\varphi z\in\mathcal{Z}_0$, then

$$\lim_{|z|\to 1}(1-|z|^2)|\varphi'(z)g'(z)+\varphi''(z)g(z)|=0,$$

i. e. that (16) holds. Also, $I_gC_\varphi z^2\in\mathcal{Z}_0$, thus

$$\begin{aligned}&\lim_{|z|\to 1}(1-|z|^2)|2\varphi(z)\varphi'(z)g'(z)\\ +\;& 2g(z)\varphi(z)\varphi''(z)+2g(z)(\varphi'(z))^2|=0\end{aligned}$$

Since $|\varphi|\le 1$, we get

$$\lim_{|z|\to 1}(1-|z|^2)|g(z)(\varphi'(z))^2|=0.$$

Hence (17) holds.

On the other hand, by Lemma 4 and Theorem 8, we obtain that (11) and (12) hold.

Conversely, let

$$M_1=\sup_{z\in D}\frac{(1-|z|^2)|\varphi'(z)g'(z)+\varphi''(z)g(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}<\infty;$$

$$M_2=\sup_{z\in D}\frac{(1-|z|^2)|g(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}<\infty.$$

For $\forall f\in\mathcal{L}_0^{2,\lambda}$, by Lemma 3, given $\epsilon>0$ there is a $0<\delta<1$ such that $(1-|z|^2)^{\frac{3-\lambda}{2}}|f'(z)|<\dfrac{\epsilon}{2M_1}$ and $(1-|z|^2)^{\frac{5-\lambda}{2}}|f''(z)|<\dfrac{\epsilon}{2M_2}$ for all z with $\delta<|z|<1$.

If $|\varphi(z)|>\delta$, it follows that

$$\begin{aligned}&(1-|z|^2)|(I_gC_\varphi f)''(z)|\\ =\;&(1-|z|^2)|\big(\varphi'(z)g'(z)+\varphi''(z)g(z)\big)f'(\varphi(z))\\ +\;&f''(\varphi(z))(\varphi'(z))^2g(z)|\\ \le\;&(1-|z|^2)|\big(\varphi'(z)g'(z)+\varphi''(z)g(z)\big)f'(\varphi(z))|\\ +\;&(1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2g(z)|\\ <\;&\frac{(1-|z|^2)|\varphi'(z)g'(z)+\varphi''(z)g(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\frac{\epsilon}{2M_1}\\ +\;&\frac{(1-|z|^2)|(\varphi'(z))^2g(z)|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}\frac{\epsilon}{2M_2}\\ <\;&\epsilon,\end{aligned}$$

We know that there exists a constant K such that $|f'(z)|\le K$ and $|f''(z)|\le K$ for all $|z|\le\delta$.

If $|\varphi(z)|\le\delta$, it follows that

$$\begin{aligned}&(1-|z|^2)|(I_gC_\varphi f)''(z)|\\ =\;&(1-|z|^2)|\big(\varphi'(z)g'(z)+\varphi''(z)g(z)\big)f'(\varphi(z))\\ +\;&f''(\varphi(z))(\varphi'(z))^2g(z)|\\ \le\;&(1-|z|^2)|\big(\varphi'(z)g'(z)+\varphi''(z)g(z)\big)f'(\varphi(z))|\\ +\;&(1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2g(z)|\\ <\;&K(1-|z|^2)|\varphi'(z)g'(z)+\varphi''(z)g(z)|\\ +\;&K(1-|z|^2)|(\varphi'(z))^2g(z)|,\end{aligned}$$

Thus we conclude that $(1-|z|^2)|(I_gC_\varphi(f))''(z)|\to 0$ as $|z|\to 1^-$. Hence $I_gC_\varphi f\in\mathcal{Z}_0$ for all $f\in\mathcal{L}_0^{2,\lambda}$. On the other hand, I_gC_φ is bounded from $\mathcal{L}^{2,\lambda}$ to $\mathcal{Z}$ by Theorem 5. Hence I_gC_φ is a bounded operator from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$.

Corollary 10 *The Volterra-type operator $I_g:\mathcal{L}^{2,\lambda}\to\mathcal{Z}$ is a bounded operator if and only if $g=0$.*

5 Compactness of T_gC_φ and I_gC_φ

In order to prove the compactness of T_gC_φ, we require the following lemmas.

Lemma 11 *Suppose that T_gC_φ be a bounded operator from $\mathcal{L}^{2,\lambda}$ to $\mathcal{Z}$. Then T_gC_φ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathcal{L}^{2,\lambda}$ which converges to 0 uniformly on compact subsets of D. We have $\|T_gC_\varphi(f_n)\|_{\mathcal{Z}} \to 0$, as $n \to \infty$.*

The proof is similar to that of Proposition 3.11 in [4]. The details are omitted.

Lemma 12 *Let $U \subset \mathcal{Z}_0$. Then U is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in U} (1-|z|^2)|f''(z)| = 0.$$

The proof is similar to that of Lemma 1 in [10], we omit it.

Theorem 13 *Let g be an analytic function on the unit disc D and φ an analytic self-map of D. Suppose that T_gC_φ is a bounded operator from $\mathcal{L}^{2,\lambda}$ to $\mathcal{Z}$. Then T_gC_φ is compact if and only if the following are satisfied:*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = 0; \tag{18}$$

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0. \tag{19}$$

Proof: Suppose that T_gC_φ is compact from $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$. Let $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. If such a sequence does not exist, then (18) and (19) are automatically satisfied. Without loss of generality we may suppose that $|\varphi(z_n)| > \frac{1}{2}$ for all n. We take the test functions

$$f_n(z) = \frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{3-\lambda}{2}}} - \frac{(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\frac{5-\lambda}{2}}}. \tag{20}$$

By the proof of Theorem 5 we know that that $\sup_n \|f_n\|_{\mathcal{L}^{2,\lambda}} \le C < \infty$. Then $\{f_n\}$ is a bounded sequence in $\mathcal{L}^{2,\lambda}$ which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty} \|T_gC_\varphi(f_n)\|_{\mathcal{Z}} = 0$ by Lemma 11. Note that

$$f_n(\varphi(z_n)) \equiv 0 \quad \text{and} \quad f_n'(\varphi(z_n)) = \frac{-\overline{\varphi(z_n)}}{(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}}.$$

It follows that

$$\begin{aligned} & \|T_gC_\varphi f_n\|_{\mathcal{Z}} \\ \ge\ & (1-|z_n|^2)|g'(z_n)\varphi'(z_n)f_n'(\varphi(z_n)) \\ +\ & g''(z_n)f_n(\varphi(z_n))| \\ =\ & (1-|z_n|^2)|g'(z_n)\varphi'(z_n)|\frac{\overline{|\varphi(z_n)|}}{(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}} \\ \ge\ & \frac{(1-|z_n|^2)|g'(z_n)\varphi'(z_n)|}{2(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}}. \end{aligned}$$

Then $\lim_{n\to\infty} \frac{(1-|z_n|^2)|g'(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}} = 0$. Thus (19) holds.

Next, let

$$h_n(z) = \frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{3-\lambda}{2}}} - \frac{3-\lambda}{5-\lambda}\frac{(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\frac{5-\lambda}{2}}}. \tag{21}$$

We know that $\{h_n\}$ is a bounded sequence in $\mathcal{L}^{2,\lambda}$ which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty} \|T_gC_\varphi(h_n)\|_{\mathcal{Z}} = 0$ by Lemma 11. Note that $h_n(\varphi(z_n)) = \dfrac{2}{(5-\lambda)(1-|\varphi(z_n)|^2)^{\frac{1-\lambda}{2}}}$ and $h_n'(\varphi(z_n)) = 0$. Then

$$\begin{aligned} \|T_gC_\varphi h_n\|_{\mathcal{Z}} &\ge (1-|z_n|^2)|g''(z_n)h_n(\varphi(z_n))| \\ &= \frac{2}{5-\lambda}\frac{(1-|z_n|^2)|g''(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{1-\lambda}{2}}}, \end{aligned}$$

hence (18) holds. The proof of the necessary is completed.

Conversely, suppose that (18) and (19) hold. Since T_gC_φ is a bounded operator, by Theorem 5, we have

$$M_1 = \sup_{z\in D} \frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \infty;$$

$$M_2 = \sup_{z\in D} \frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \infty.$$

Let $\{f_n\}$ be a bounded sequence in $\mathcal{L}^{2,\lambda}$ with $\|f_n\|_{\mathcal{L}^{2,\lambda}} \le 1$ and $f_n \to 0$ uniformly on compact subsets of D. We only prove $\lim_{n\to\infty} \|T_gC_\varphi(f_n)\|_{\mathcal{Z}} = 0$ by Lemma 11. By the assumption, for any $\epsilon > 0$, there is a constant δ, $0 < \delta < 1$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \epsilon$$

and

$$\frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \epsilon.$$

Let $K = \{w \in D : |w| \leq \delta\}$. Noting that K is a compact subset of D, we get that

$$\begin{aligned}
z(T_g C_\varphi f_n) &= \sup_{z\in D}(1-|z|^2)|(T_g C_\varphi f_n)''(z)| \\
&\leq \sup_{z\in D}(1-|z|^2)|\varphi'(z)g'(z)f_n'(\varphi(z))| \\
&+ \sup_{z\in D}(1-|z|^2)|g''(z)f_n(\varphi(z))| \\
&\lesssim 2\epsilon + \sup_{|\varphi(z)|\leq\delta}(1-|z|^2)|\varphi'(z)g'(z)f_n'(\varphi(z))| \\
&+ \sup_{|\varphi(z)|\leq\delta}(1-|z|^2)|g''(z)f_n(\varphi(z))| \\
&\leq 2\epsilon + M_1\sup_{w\in K}|f_n'(w)| + M_2\sup_{w\in K}|f_n(w)|.
\end{aligned}$$

As $n\to\infty$,

$$\|T_g C_\varphi f_n\|_{\mathcal{Z}} \to 0.$$

Hence $T_g C_\varphi$ is compact. This completes the proof of Theorem 13.

Theorem 14 *Let g be an analytic function on the unit disc D, and φ an analytic self-map of D. Then $T_g C_\varphi$ is compact from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$ if and only if the following are satisfied:*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0; \qquad (22)$$

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = 0. \qquad (23)$$

Proof: Assume (22) and (23) hold. From Theorem 6, we know that $T_g C_\varphi$ is bounded from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$. Suppose that $f \in \mathcal{L}_0^{2,\lambda}$ with $\|f\|_{\mathcal{L}^{2,\lambda}} \leq 1$. We obtain that

$$\begin{aligned}
&(1-|z|^2)|(T_g C_\varphi f)''(z)| \\
= \; &(1-|z|^2)|\varphi'(z)g'(z)f'(\varphi(z)) + g''(z)f(\varphi(z))| \\
\leq \; &(1-|z|^2)|\varphi'(z)g'(z)f'(\varphi(z))| \\
+ \; &(1-|z|^2)|g''(z)f(\varphi(z))| \\
\lesssim \; &\frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}} \\
+ \; &\frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}},
\end{aligned}$$

thus

$$\begin{aligned}
&\sup\{(1-|z|^2)|(T_g C_\varphi f)''(z)| \\
: \; &f \in \mathcal{L}_0^{2,\lambda}, \|f\|_{\mathcal{L}^{2,\lambda}} \leq 1\} \\
\lesssim \; &\frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} + \frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}},
\end{aligned}$$

and it follows that

$$\begin{aligned}
&\lim_{|z|\to 1}\sup\{|(1-|z|^2)(T_g C_\varphi f)''(z)| \\
: \; &f \in \mathcal{L}_0^{2,\lambda}, \|f\|_{\mathcal{L}^{2,\lambda}} \leq 1\} = 0,
\end{aligned}$$

hence $T_g C_\varphi : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ is compact by Lemma 12.

Conversely, suppose that $T_g C_\varphi : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ is compact.

First, it is obvious that $T_g C_\varphi : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ is bounded, then by Theorem 6, we have $g \in \mathcal{Z}_0$ and that (10) holds. On the other hand, by Lemma 12 we have

$$\begin{aligned}
&\lim_{|z|\to 1}\sup\{(1-|z|^2)|(T_g C_\varphi f)''(z)| \\
: \; &f \in \mathcal{L}_0^{2,\lambda}, \|f\|_{\mathcal{L}^{2,\lambda}} \leq M\} = 0,
\end{aligned}$$

for some $M > 0$.

Next, noting that the proof of Theorem 5 and the fact that the functions given in (8) are in $\mathcal{L}_0^{2,\lambda}$ and have norms bounded independently of a, we obtain that

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0$$

for $|\varphi(z)| > \frac{1}{2}$. However, if $|\varphi(z)| \le \frac{1}{2}$, by (10), we easily have

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}$$

$$\le (\frac{4}{3})^{\frac{3-\lambda}{2}} \lim_{|z|\to 1}(1-|z|^2)|\varphi'(z)g'(z)| = 0.$$

Thus (22) holds.

Similarly, noting that the functions given in (9) are in $\mathcal{L}_0^{2,\lambda}$ and have norms bounded independently of a, we obtain that

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}$$

$$\lesssim \lim_{|z|\to 1}(1-|z|^2)|(T_g C_\varphi h_a)''(z)|,$$

for $|\varphi(z)| > \frac{1}{2}$. Then

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = 0$$

for $|\varphi(z)| > \frac{1}{2}$. However, if $|\varphi(z)| \le \frac{1}{2}$, by $g \in \mathcal{Z}_0$, we easily have

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|g''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = \lim_{|z|\to 1} (\frac{4}{3})^{\frac{1-\lambda}{2}}(1-|z|^2)|g''(z)| = 0.$$

This completes the proof of Theorem 14.

Using the same methods as in the proof of Theorem 13 and 14, we can prove the following results.

Theorem 15 *Let g be an analytic function on the unit disc D and φ an analytic self-map of D. Suppose that $I_g C_\varphi$ is a bounded operator from $\mathcal{L}^{2,\lambda}$ to $\mathcal{Z}$. Then $I_g C_\varphi$ is compact if and only if the following are satisfied:*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)g'(z)+\varphi''(z)g(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0; \quad (24)$$

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|g(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} = 0. \quad (25)$$

Theorem 16 *Let g be an analytic function on the unit disc D and φ an analytic self-map of D. Then $I_g C_\varphi$ is compact from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$ if and only if the following are satisfied:*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|\varphi'(z)g'(z)+\varphi''(z)g(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0; \quad (26)$$

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|g(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} = 0. \quad (27)$$

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