Completeness of Inference Rules for New Vague Multivalued Dependencies

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Abstract: In this paper we prove that the set of the main inference rules for new vague functional and vague multivalued dependencies is complete set. More precisely, we prove that there exists a vague relation instance on given scheme, which satisfies all vague functional and vague multivalued dependencies from the set of all vague functional and vague multivalued dependencies that can be derived from given ones by repeated applications of the main inference rules, and violates given vague functional resp. vague multivalued dependency which is initially known not to be an element of the aforementioned set of derived vague dependencies. The paper can be considered as a natural continuation of our previous study, where new definitions of vague functional and vague multivalued dependencies are introduced, the corresponding inference rules are listed, and are shown to be sound.

Key–Words: Vague functional and vague multivalued dependencies, inference rules, completeness

1 Introduction and preliminaries

The main tool applied in this research is a vague set.

Recall that a vague set in some universe of discourse U is a set

$$V = \{\langle u, [t_V(u), 1 - f_V(u)]\rangle : u \in U\},$$

where $[t_V(u), 1 - f_V(u)] \subseteq [0,1]$ is the vague value joined to $u \in U$, and $t_V : U \to [0,1]$, $f_V : U \to [0,1]$ are functions such that $t_V(u) + f_V(u) \leq 1$ for all $u \in U$.

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in \{1, 2, ..., n\} = I$.

Suppose that $V(U_i)$ is the family of all vague sets in U_i, $i \in I$.

A vague relation instance r on $R(A_1, A_2, ..., A_n)$ is a subset of the cross product $V(U_1) \times V(U_2) \times ... \times V(U_n)$.

A tuple t of r is denoted by

$$(t[A_1], t[A_2], ..., t[A_n]),$$

where the vague set $t[A_i]$ may be considered as the value of the attribute A_i on tuple t.

Let $Vag(U_i)$ be the set of all vague values associated to the elements $u_i \in U_i$, $i \in I$.

A similarity measure on $Vag(U_i)$ is a mapping $SE_i : Vag(U_i) \times Vag(U_i) \to [0,1]$, such that $SE_i(x,x) = 1$, $SE_i(x,y) = SE_i(y,x)$, and $SE_i(x,z) \geq \max_{y \in Vag(U_i)} (\min(SE_i(x,y), SE_i(y,z)))$ for all x, y, $z \in Vag(U_i)$.

Suppose that SE_i is a similarity measure on $Vag(U_i)$, $i \in I$.

If

$$A_i = \{\langle u, [t_{A_i}(u), 1 - f_{A_i}(u)]\rangle : u \in U_i\} = \{a_u^i : u \in U_i\}$$

and

$$B_i = \{\langle u, [t_{B_i}(u), 1 - f_{B_i}(u)]\rangle : u \in U_i\} = \{b_u^i : u \in U_i\}$$

are two vague sets in U_i, then, the similarity measure $SE(A_i, B_i)$ between the vague sets A_i and B_i is given by

$$SE(A_i, B_i) =$$

$$\min\left\{\min_{a_u^i\in A_i}\left\{\max_{b_u^i\in B_i}\left\{SE_i\Big(\left[t_{A_i}(u),1-f_{A_i}(u)\right],\right.\right.\right.$$
$$\left.\left.\left.\left[t_{B_i}(u),1-f_{B_i}(u)\right]\Big)\right\}\right\},$$
$$\min_{b_u^i\in B_i}\left\{\max_{a_u^i\in A_i}\left\{SE_i\Big(\left[t_{B_i}(u),1-f_{B_i}(u)\right],\right.\right.$$
$$\left.\left.\left.\left[t_{A_i}(u),1-f_{A_i}(u)\right]\Big)\right\}\right\}\right\}.$$

Finally, if r is a vague relation instance on $R(A_1, A_2, ..., A_n)$, t_1 and t_2 are any two tuples in r, and X is a subset of $\{A_1, A_2, ..., A_n\}$, then, the similarity measure $SE_X(t_1, t_2)$ between the tuples t_1 and t_2 on the attribute set X is given by

$$SE_X(t_1,t_2)=\min_{A\in X}\{SE(t_1[A],t_2[A])\}.$$

Note that various authors proposed various definitions of similarity measures (see, e.g., [7], [2], [1], [5], [6]).

Through the rest of the paper, we shall assume that the similarity measures SE_i, SE and SE_X are given as above.

Recently, in [3] and [4], we introduced new formal definitions of vague functional and vague multivalued dependencies, respectively.

In particular, if X and Y are subsets of $\{A_1, A_2, ..., A_n\}$, and $\theta\in[0,1]$ is a number, then, the vague relation instance r is said to satisfy the vague functional dependency $X \overset{\theta}{\to}_V Y$, if for every pair of tuples t_1 and t_2 in r,

$$SE_Y(t_1,t_2)\geq\min\{\theta,SE_X(t_1,t_2)\}.$$

Vague relation instance r is said to satisfy the vague multivalued dependency $X \to\overset{\theta}{\to}_V Y$, if for every pair of tuples t_1 and t_2 in r, there exists a tuple t_3 in r, such that

$$SE_X(t_3,t_1)\geq\min\{\theta,SE_X(t_1,t_2)\},$$
$$SE_Y(t_3,t_1)\geq\min\{\theta,SE_X(t_1,t_2)\},$$
$$SE_{\{A_1,A_2,...,A_n\}\setminus(X\cup Y)}(t_3,t_2)$$
$$\geq\min\{\theta,SE_X(t_1,t_2)\}.$$

We write $X \to_V Y$ resp. $X \to\to_V Y$ instead of $X \overset{\theta}{\to}_V Y$ resp. $X \to\overset{\theta}{\to}_V Y$ if $\theta = 1$.

For various definitions of vague functional and vague multivalued dependencies proposed by various authors, see [7], [8], [11] and [9].

The following inference rules are the inference rules for vague functional and vague multivalued dependencies introduced above (see, [3] and [4]).

VF1 Inclusive rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ holds, and $\theta_1\geq\theta_2$, then $X \overset{\theta_2}{\longrightarrow}_V Y$ holds.

VF2 Reflexive rule for VFDs: If $X\supseteq Y$, then $X \to_V Y$ holds.

VF3 Augmentation rule for VFDs: If $X \overset{\theta}{\to}_V Y$ holds, then $X\cup Z \overset{\theta}{\to}_V Y\cup Z$ holds.

VF4 Transitivity rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $Y \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to}_V Z$ holds true.

VF5 Union rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $X \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to}_V Y\cup Z$ holds also true.

VF6 Pseudo-transitivity rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $W\cup Y \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $W\cup X \overset{\min(\theta_1,\theta_2)}{\to}_V Z$ holds true.

VF7 Decomposition rule for VFDs: If $X \overset{\theta}{\to}_V Y$ holds, and $Z\subseteq Y$, then $X \overset{\theta}{\to}_V Z$ also holds.

VM1 Inclusive rule for VMVDs: If $X \to\overset{\theta_1}{\longrightarrow}_V Y$ holds, and $\theta_1\geq\theta_2$, then $X \to\overset{\theta_2}{\longrightarrow}_V Y$ holds.

VM2 Complementation rule for VMVDs: If $X \to\overset{\theta}{\to}_V Y$ holds, then $X \to\overset{\theta}{\to}_V \{A_1, A_2, ..., A_n\}\setminus(X\cup Y)$ holds.

VM3 Augmentation rule for VMVDs: If $X \to\overset{\theta}{\to}_V Y$ holds, and $W\supseteq Z$, then $W\cup X \to\overset{\theta}{\to}_V Y\cup Z$ also holds.

VM4 Transitivity rule for VMVDs: If $X \to\overset{\theta_1}{\longrightarrow}_V Y$ and $Y \to\overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Z\setminus Y$ holds true.

VM5 Replication rule: If $X \overset{\theta}{\to}_V Y$ holds, then $X \to\overset{\theta}{\to}_V Y$ holds.

VM6 Coalescence rule for VFDs and VMVDs: If $X \rightarrow\xrightarrow{\theta_1}_V Y$ holds, $Z \subseteq Y$, and for some W disjoint from Y, we have that $W \xrightarrow{\theta_2}_V Z$ holds true, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow}_V Z$ also holds true.

VM7 Union rule for VMVDs: If $X \rightarrow\xrightarrow{\theta_1}_V Y$ and $X \rightarrow\xrightarrow{\theta_2}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow\rightarrow}_V Y \cup Z$ holds true.

VM8 Pseudo-transitivity rule for VMVDs: If $X \rightarrow\xrightarrow{\theta_1}_V Y$ and $W \cup Y \rightarrow\xrightarrow{\theta_2}_V Z$ hold true, then $W \cup X \overset{\min(\theta_1,\theta_2)}{\rightarrow\rightarrow}_V Z \setminus (W \cup Y)$ holds also true.

VM9 Decomposition rule for VMVDs: If $X \rightarrow\xrightarrow{\theta_1}_V Y$ and $X \rightarrow\xrightarrow{\theta_2}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow\rightarrow}_V Y \cap Z$, $X \overset{\min(\theta_1,\theta_2)}{\rightarrow\rightarrow}_V Y \setminus Z$, and $X \overset{\min(\theta_1,\theta_2)}{\rightarrow\rightarrow}_V Z \setminus Y$ hold also true.

VM10 Mixed pseudo-transitivity rule: If $X \rightarrow\xrightarrow{\theta_1}_V Y$ and $X \cup Y \xrightarrow{\theta_2}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow}_V Z \setminus Y$ holds true.

The inference rules $VF1 - VF4$ and $VM1 - VM6$ are the main inference rules, while the inference rules $VF5 - VF7$ and $VM7 - VM10$ are additional inference rules.

This means that the inference rules $VF5 - VF7$ resp. $VM7 - VM10$ follow from the rules $VF1 - VF4$ resp. $VF1 - VF4$ and $VM1 - VM6$ (see, [3, Th. 5] resp. [4, Th. 3]).

In [3] and [4], we have proved that the inference $VF1 - VF7$ and $VM1 - VM10$ are sound (see, Theorems 4, 5 and Theorems 2, 3).

The structure of the paper is as follows: Section 1 provides some necessary background and preliminary material. We introduce: vagues sets (over some universe of discourse), vague values (joined to the elements of some universe of discourse), vague relation instances (over some relation scheme), similarity measures between: vague values, vague sets, tuples. We recall the main and additional inference rules for vague functional and vague multivalued dependencies. Finally, we assemble those facts and results we will need. Section 2 is the main section of the paper. In this section we introduce closures, limit strengths of dependencies, dependency basis. We prove various auxiliary results related to closures and dependency basis. Ultimately, we state and prove the main result of the paper, i.e., that the set of the main inference rules for vague functional and vague multivalued dependencies is complete set.

Thus, the main purpose of the paper is to prove that the set $\{VF1 - VF4, VM1 - VM6\}$ is complete set (see, [10] in the case of fuzzy functional and fuzzy multivalued dependencies).

In order to prove this, it will be enough to prove that there exists a vague relation instance r on $R(A_1, A_2, ..., A_n)$ which satisfies $A \overset{1\theta}{\rightarrow}_V B$ resp. $A \rightarrow\overset{1\theta}{\rightarrow}_V B$ if $A \overset{1\theta}{\rightarrow}_V B$ resp. $A \rightarrow\overset{1\theta}{\rightarrow}_V B$ belongs to $(\mathcal{V}, \mathcal{M})^+$, and violates $X \xrightarrow{\theta}_V Y$ resp. $X \rightarrow\xrightarrow{\theta}_V Y$, where $R(A_1, A_2, ..., A_n)$ is a relation scheme on domains U_1, U_2,..., U_n, A_i is an attribute on the universe of discourse U_i, $i \in I$, $X \xrightarrow{\theta}_V Y$ resp. $X \rightarrow\xrightarrow{\theta}_V Y$ is a vague functional resp. vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$ which is not an element of $(\mathcal{V}, \mathcal{M})^+$, and $(\mathcal{V}, \mathcal{M})^+$ is the set of all vague functional and vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$ that can be derived from $\mathcal{V} \cup \mathcal{M}$ by repeated applications of the inference rules $VF1 - VF4$, $VM1 - VM6$, where $\mathcal{V}$ resp. $\mathcal{M}$ is some set of vague functional resp. vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$.

We close this section by noting that $SE_Y(t_1, t_2) \geq SE_X(t_1, t_2)$ for $Y \subseteq X$, and $SE_X(t_1, t_3) \geq \theta$ for $SE_X(t_1, t_2) \geq \theta$, $SE_X(t_2, t_3) \geq \theta$.

2 Main Result

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$.

Suppose that $\mathcal{V}$ resp. $\mathcal{M}$ is some set of vague functional resp. vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$.

The closure $(\mathcal{V}, \mathcal{M})^+$ of $\mathcal{V} \cup \mathcal{M}$ is the set of all vague functional dependencies and vague multivalued dependencies that can be derived from $\mathcal{V} \cup \mathcal{M}$ by repeated applications of the inference rules: $VF1 - VF4, VM1 - VM6$.

The set $(\mathcal{V}, \mathcal{M})^+$ is infinite one regardless of whether $\mathcal{V} \cup \mathcal{M}$ is finite or not.

Namely, if $X \rightarrow\xrightarrow{\theta}_V Y$ belongs to $\mathcal{V} \cup \mathcal{M}$ for example, then, by $VM1$, $X \rightarrow\xrightarrow{\theta_1}_V Y$ belongs to $(\mathcal{V}, \mathcal{M})^+$ for $\theta_1 \in [0, \theta]$.

Let $X \xrightarrow{\theta}_V Y$ be some vague functional dependency on $\{A_1, A_2, ..., A_n\}$.

The dependency $X \xrightarrow{\theta}_V Y$ may or may not belong to $(\mathcal{V}, \mathcal{M})^+$.

The limit strength of $X \xrightarrow{\theta}_V Y$ (with respect to $\mathcal{V}$ and $\mathcal{M}$) is the number $\theta_l(\mathcal{V}, \mathcal{M}) \in [0, 1]$, such that $X \overset{\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow}_V Y$ belongs to $(\mathcal{V}, \mathcal{M})^+$, and $\theta' \leq \theta_l(\mathcal{V}, \mathcal{M})$

for each $X \overset{\theta'}{\to}_V Y$ that belongs to $(\mathcal{V},\mathcal{M})^+$.

If $X \overset{\theta}{\to}_V Y$ belongs to $(\mathcal{V},\mathcal{M})^+$, then the limit strength $\theta_l(\mathcal{V},\mathcal{M})$ obviously exists.

Namely, in this case, $\theta_l(\mathcal{V},\mathcal{M})$ is given by

$$\theta_l(\mathcal{V},\mathcal{M}) = \max\left\{\theta' \,:\, X \overset{\theta'}{\to}_V Y \in (\mathcal{V},\mathcal{M})^+\right\}.$$

Otherwise, if $X \overset{\theta}{\to}_V Y$ does not belong to $(\mathcal{V},\mathcal{M})^+$, the limit strength $\theta_l(\mathcal{V},\mathcal{M})$ does not necessarily exists.

Let $X \overset{\theta}{\to\to}_V Y$ be some vague multivalued dependency on $\{A_1,A_2,...,A_n\}$.

The dependency $X \overset{\theta}{\to\to}_V Y$ may or may not belong to $(\mathcal{V},\mathcal{M})^+$.

The limit strength of $X \overset{\theta}{\to\to}_V Y$ (with respect to $\mathcal{V}$ and $\mathcal{M}$) is the number $\theta_l(\mathcal{V},\mathcal{M}) \in [0,1]$, such that $X \overset{\theta_l(\mathcal{V},\mathcal{M})}{\to\to}_V Y$ belongs to $(\mathcal{V},\mathcal{M})^+$, and $\theta' \le \theta_l(\mathcal{V},\mathcal{M})$ for each $X \overset{\theta'}{\to\to}_V Y$ that belongs to $(\mathcal{V},\mathcal{M})^+$.

Reasoning as in the case of vague functional dependencies, we conclude that $\theta_l(\mathcal{V},\mathcal{M})$ exists if $X \overset{\theta}{\to\to}_V Y$ belongs to $(\mathcal{V},\mathcal{M})^+$.

Otherwise, if $X \overset{\theta}{\to\to}_V Y$ does not belong to $(\mathcal{V},\mathcal{M})^+$, the limit strength $\theta_l(\mathcal{V},\mathcal{M})$ does not necessarily exists.

Let X be a subset of $\{A_1,A_2,...,A_n\}$, and θ be a number in $[0,1]$.

The closure $X^+(\theta,\mathcal{V},\mathcal{M})$ of X (with respect to $\mathcal{V}$ and $\mathcal{M}$) is the set of attributes $A \in \{A_1,A_2,...,A_n\}$, such that $X \overset{\theta}{\to}_V A$ belongs to $(\mathcal{V},\mathcal{M})^+$.

Suppose that $A \in X$.

By $VF2$, $X \to_V A$ belongs to $(\mathcal{V},\mathcal{M})^+$.

Now, by $VF1$, $X \overset{\theta}{\to}_V A$ belongs to $(\mathcal{V},\mathcal{M})^+$. Therefore, $A \in X^+(\theta,\mathcal{V},\mathcal{M})$.

Since $A \in X$, we obtain that $X \subseteq X^+(\theta,\mathcal{V},\mathcal{M})$.

Theorem 1. *Let $R(A_1,A_2,...,A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let $(\mathcal{V},\mathcal{M})^+$ be the closure of $\mathcal{V} \cup \mathcal{M}$, where $\mathcal{V}$ resp. $\mathcal{M}$ is some set of vague functional resp. vague multivalued dependencies on $\{A_1,A_2,...,A_n\}$. Suppose that $X \overset{\theta}{\to}_V Y$ is some vague functional dependency on $\{A_1,A_2,...,A_n\}$. Then, $X \overset{\theta}{\to}_V Y$ belongs to $(\mathcal{V},\mathcal{M})^+$ if and only if $Y \subseteq X^+(\theta,\mathcal{V},\mathcal{M})$.*

Proof. $(\Rightarrow)$ Suppose that $X \overset{\theta}{\to}_V Y$ belongs to $(\mathcal{V},\mathcal{M})^+$.

Now, by $VF7$, $X \overset{\theta}{\to}_V A$ belongs to $(\mathcal{V},\mathcal{M})^+$ for every $A \in Y$.

Thus, $A \in X^+(\theta,\mathcal{V},\mathcal{M})$ for every $A \in Y$. This means that $Y \subseteq X^+(\theta,\mathcal{V},\mathcal{M})$.

$(\Leftarrow)$ Suppose that $Y \subseteq X^+(\theta,\mathcal{V},\mathcal{M})$.

Now, $A \in X^+(\theta,\mathcal{V},\mathcal{M})$ for every $A \in Y$.

This means that $X \overset{\theta}{\to}_V A$ belongs to $(\mathcal{V},\mathcal{M})^+$ for every $A \in Y$.

Now, by $VF5$, $X \overset{\theta}{\to}_V Y$ belongs to $(\mathcal{V},\mathcal{M})^+$.

This completes the proof. □

Theorem 2. *Let $R(A_1,A_2,...,A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Suppose that X is a subset of $\{A_1,A_2,...,A_n\}$, and θ is a number in $[0,1]$. Put*

$$\mathcal{F}(X,\theta) = \left\{Z \subseteq \{A_1,A_2,...,A_n\} \,:\, X \overset{\theta}{\to\to}_V Z\right\}.$$

There is a partition Y_1, Y_2,..., Y_k of $\{A_1,A_2,...,A_n\}$, such that $Z \in \mathcal{F}(X,\theta)$ if and only if Z is the union of some of the sets Y_1, Y_2,..., Y_k. Furthermore, $X \overset{\theta}{\to\to}_V Y_i$ for $i \in \{1,2,...,k\}$.

Proof. We start with the case $k=1$, i.e., with the partition $Y_1 = \{A_1,A_2,...,A_n\}$.

Let $Z_1 \in \mathcal{F}(X,\theta)$, $Z_1 \ne \{A_1,A_2,...,A_n\}$.

We have, $X \overset{\theta}{\to\to}_V Z_1$.

Since $X \overset{\theta}{\to\to}_V Z_1$, it follows by $VM2$ that $X \overset{\theta}{\to\to}_V \{A_1,A_2,...,A_n\} \setminus (X \cup Z_1)$.

Furthermore, $X \supseteq X \setminus Z_1$ and $VF2$ yield that $X \to_V X \setminus Z_1$.

Since $1 \ge \theta$, it follows by $VF1$ that $X \overset{\theta}{\to}_V X \setminus Z_1$.

Now, by $VM5$, $X \overset{\theta}{\to\to}_V X \setminus Z_1$.

Since $X \overset{\theta}{\to\to}_V \{A_1,A_2,...,A_n\} \setminus (X \cup Z_1)$ and $X \overset{\theta}{\to\to}_V X \setminus Z_1$, it follows by $VM7$ that $X \overset{\theta}{\to\to}_V \{A_1,A_2,...,A_n\} \setminus Z_1$,

Note that $Z_1 \ne Y_1$.

Having in mind this fact, we replace Y_1 by $Y_1 \cap Z_1 = Z_1$ and $Y_1 \setminus Z_1 = \{A_1,A_2,...,A_n\} \setminus Z_1$. Denote these sets by Y_1 and Y_2, respectively.

Since $X \overset{\theta}{\to\to}_V Z_1$ and $X \overset{\theta}{\to\to}_V \{A_1,A_2,...,A_n\} \setminus Z_1$, it follows that $X \overset{\theta}{\to\to}_V Y_1$ and $X \overset{\theta}{\to\to}_V Y_2$.

Now, if $Z \in \{Z_1\} \subseteq \mathcal{F}(X,\theta)$, it follows that $Z = Z_1 = Y_1$, i.e., it follows that Z is the union of some of the sets Y_1, Y_2.

Suppose that Z is the union of some of the sets Y_1, Y_2. It follows that $Z = Y_1$ or $Z = Y_2$ or $Z = Y_1 \cup Y_2$.

Now, $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y_1$, $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y_2$ and $VM7$ imply that $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y_1 \cup Y_2$.

This means that $Z \in \mathcal{F}(X,\theta)$ in any case.

We proceed with the partition Y_1, Y_2.

Let $Z_2 \in \mathcal{F}(X,\theta)$.

We have, $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Z_2$.

Suppose that Z_2 is the union of some of the sets Y_1, Y_2.

Now, if $Z \in \{Z_1, Z_2\} \subseteq \mathcal{F}(X,\theta)$, it follows that Z is the union of some of the sets Y_1, Y_2.

If Z is the union of some of the sets Y_1, Y_2, then, as before, $Z \in \mathcal{F}(X,\theta)$.

Suppose that Z_2 is not the union of some of the sets Y_1, Y_2.

Now, reasoning as earlier, we replace each $Y_i \in \{Y_1, Y_2\}$ such that $Y_i \cap Z_2$ and $Y_i \setminus Z_2$ are both nonempty, by $Y_i \cap Z_2$ and $Y_i \setminus Z_2$.

The obtained partition we denote by Y_1, Y_2,..., Y_j. Clearly, $j = 3$ or $j = 4$.

Suppose that $Y_i \in \{Y_1, Y_2\}$ is such that $Y_i \cap Z_2$ and $Y_i \setminus Z_2$ are both nonempty.

Since $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y_i$ and $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Z_2$, it follows by $VM9$ that $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y_i \cap Z_2$ and $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y_i \setminus Z_2$.

This means that $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y_i$ for $i \in \{1, 2, ..., j\}$.

Now, if $Z \in \{Z_1, Z_2\} \subseteq \mathcal{F}(X,\theta)$, it immediately follows that Z is the union of some of the sets Y_1, Y_2,..., Y_j.

If Z is the union of some of the sets Y_1, Y_2,..., Y_j, then, $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y_i$ for $i \in \{1, 2, ..., j\}$ and $VM7$ yield that $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Z$. Therefore, $Z \in \mathcal{F}(X,\theta)$.

Proceeding with the partition Y_1, Y_2,..., Y_j in the way described above, we obtain that there exists a partition Y_1, Y_2,..., Y_k of $\{A_1, A_2, ..., A_n\}$, such that $Z \in \mathcal{F}(X,\theta)$ if and only if Z is the union of some of the sets Y_1, Y_2,..., Y_k.

Moreover, we obtain that $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y_i$ for $i \in \{1, 2, ..., k\}$.

This completes the proof. $\square$

The set $\{Y_1, Y_2, ..., Y_k\}$ of the sets Y_1, Y_2,..., Y_k that appear in Theorem 2 is called the dependency basis of X with respect to θ. The dependency basis of X with respect to θ is denoted by $dep(X,\theta)$.

Theorem 3. *The set $\{VF1 - VF4, VM1 - VM6\}$ is complete set.*

Proof. Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$.

Let $(\mathcal{V}, \mathcal{M})^+$ be the closure of $\mathcal{V} \cup \mathcal{M}$, where $\mathcal{V}$ resp. $\mathcal{M}$ is some set of vague functional resp. vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$.

Suppose that $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y$ is some vague functional resp. vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$ which is not a member of $(\mathcal{V}, \mathcal{M})^+$.

In order to prove the theorem, it is enough to prove that there exists a vague relation instance r on $R(A_1, A_2, ..., A_n)$ which satisfies $A \overset{1\theta}{\rightarrow}_V B$ resp. $A \overset{1\theta}{\rightarrow\!\!\rightarrow}_V B$ if $A \overset{1\theta}{\rightarrow}_V B$ resp. $A \overset{1\theta}{\rightarrow\!\!\rightarrow}_V B$ belongs to $(\mathcal{V}, \mathcal{M})^+$, and violates $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y$.

r can be constructed in the following way.

Suppose that $X^+(\theta, \mathcal{V}, \mathcal{M}) = \{A_1, A_2, ..., A_n\}$.

It follows that $Y \subseteq X^+(\theta, \mathcal{V}, \mathcal{M})$.

Hence, by Theorem 1, $X \overset{\theta}{\rightarrow}_V Y$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

Consequently, by $VM5$, $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

This contradicts the fact that $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y$ is not a member of $(\mathcal{V}, \mathcal{M})^+$.

We conclude, $X^+(\theta, \mathcal{V}, \mathcal{M}) \subset \{A_1, A_2, ..., A_n\}$.

Let $dep(X,\theta) = \{Y_1, Y_2, ..., Y_k\}$ be the dependency basis of X with respect to θ.

Since $X^+(\theta, \mathcal{V}, \mathcal{M}) \subseteq X^+(\theta, \mathcal{V}, \mathcal{M})$, it follows ba Theorem 1 that $X \overset{\theta}{\rightarrow}_V X^+(\theta, \mathcal{V}, \mathcal{M})$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

Thus, by $VM5$, the dependency $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V X^+(\theta, \mathcal{V}, \mathcal{M})$ exists.

Hence, by Theorem 2, $X^+(\theta, \mathcal{V}, \mathcal{M})$ is the union of some of the sets Y_1, Y_2,..., Y_k.

Since $X^+(\theta, \mathcal{V}, \mathcal{M}) \subset \{A_1, A_2, ..., A_n\}$, we have that

$$X^+(\theta, \mathcal{V}, \mathcal{M}) = \bigcup_{i=1}^{l} Y_i,$$

for some $l < k$. Therefore,

$$\{A_1, A_2, ..., A_n\} \setminus X^+(\theta, \mathcal{V}, \mathcal{M}) = \bigcup_{i=1+1}^{k} Y_i.$$

For the sake of simplicity, we shall write

$$\{A_1, A_2, ..., A_n\} \setminus X^+(\theta, \mathcal{V}, \mathcal{M}) = \bigcup_{i=1}^{m} W_i,$$

where, clearly, $m \geq 1$, and $W_1 = Y_{l+1}$, $W_2 = Y_{l+2}$,..., $W_m = Y_k$.

Thus, the sets Y_1, Y_2,..., Y_l cover $X^+(\theta, \mathcal{V}, \mathcal{M})$, while the sets Y_{l+1}, Y_{l+2},..., Y_k, i.e., the sets W_1, W_2,..., W_m cover $\{A_1, A_2, ..., A_n\} \setminus X^+(\theta, \mathcal{V}, \mathcal{M})$.

Consequently, the sets $X^+(\theta, \mathcal{V}, \mathcal{M})$, W_1, W_2,..., W_m form a partition of $\{A_1, A_2, ..., A_n\}$.

Now, if $X \overset{\theta_1}{\rightarrow\rightarrow}_V Z$ is a vague multivalued dependency such that $\theta_1 \geq \theta$, then, by $VM1$, the dependency $X \overset{\theta}{\rightarrow\rightarrow}_V Z$ exists. Therefore, by Theorem 2, Z is the union of some of the sets Y_1, Y_2,..., Y_k, i.e., the sets Y_1, Y_2,..., Y_l, W_1, W_2,..., W_m. Since

$$X^+(\theta, \mathcal{V}, \mathcal{M}) = \bigcup_{i=1}^{l} Y_i,$$

it follows that Z is the union of a subset of $X^+(\theta, \mathcal{V}, \mathcal{M})$ and some of the sets W_1, W_2,..., W_m.

Suppose that ${}_1\Delta_l(\mathcal{V}, \mathcal{M}) \neq \varnothing$, where ${}_1\Delta_l(\mathcal{V}, \mathcal{M}) \subseteq (\mathcal{V}, \mathcal{M})^+$ is given by

$$\begin{aligned}
&{}_1\Delta_l(\mathcal{V}, \mathcal{M}) \\
&= \left\{ A \overset{{}_1\theta}{\rightarrow}_V B \in (\mathcal{V}, \mathcal{M})^+ \,:\, {}_1\theta_l(\mathcal{V}, \mathcal{M}) < \theta \right\} \cup \\
&\quad \left\{ A \overset{{}_1\theta}{\rightarrow\rightarrow}_V B \in (\mathcal{V}, \mathcal{M})^+ \,:\, {}_1\theta_l(\mathcal{V}, \mathcal{M}) < \theta \right\}.
\end{aligned}$$

Fix some $\theta' \in \left(\theta'', \theta\right)$, where

$$\theta'' = \max_{{}_1\Delta_l(\mathcal{V}, \mathcal{M})} \{{}_1\theta_l(\mathcal{V}, \mathcal{M})\}.$$

If ${}_1\Delta_l(\mathcal{V}, \mathcal{M}) = \varnothing$, we put $\theta' = 0$.

Now, if $A \overset{{}_1\theta}{\rightarrow}_V B \in (\mathcal{V}, \mathcal{M})^+$ resp. $A \overset{{}_1\theta}{\rightarrow\rightarrow}_V B \in (\mathcal{V}, \mathcal{M})^+$ is a vague functional resp. vague multivalued dependency whose limit strength ${}_1\theta_l(\mathcal{V}, \mathcal{M})$ is less than θ, then

$${}_1\theta_l(\mathcal{V}, \mathcal{M}) \leq \theta'' < \theta' < \theta,$$

i.e.,

$${}_1\theta_l(\mathcal{V}, \mathcal{M}) < \theta' < \theta.$$

Otherwise, if ${}_1\theta_l(\mathcal{V}, \mathcal{M}) \geq \theta$, then

$$\theta' < \theta \leq {}_1\theta_l(\mathcal{V}, \mathcal{M}).$$

Suppose that $U_1 = U_2 = ... = U_n = \{u\} = U$. Let

$$\begin{aligned}
V_1 &= \{\langle u, [t_{V_1}(u), 1 - f_{V_1}(u)]\rangle \,:\, u \in U\} \\
&= \{\langle u, [t_{V_1}(u), 1 - f_{V_1}(u)]\rangle\} = \{\langle u, a\rangle\}
\end{aligned}$$

and

$$\begin{aligned}
V_2 &= \{\langle u, [t_{V_2}(u), 1 - f_{V_2}(u)]\rangle \,:\, u \in U\} \\
&= \{\langle u, [t_{V_2}(u), 1 - f_{V_2}(u)]\rangle\} = \{\langle u, b\rangle\}
\end{aligned}$$

be two vague sets in U, such that

$$SE_U(a, b) = \theta',$$

where $SE_U : Vag(U) \times Vag(U) \to [0, 1]$ is a similarity measure on $Vag(U)$.

We obtain,

$$\begin{aligned}
&SE(V_1, V_2) \\
&= \min \left\{ \min_{\langle u,a\rangle \in V_1} \left\{ \max_{\langle u,b\rangle \in V_2} \left\{ SE_U\Big(a, b\Big) \right\} \right\}, \right. \\
&\quad \left. \min_{\langle u,b\rangle \in V_2} \left\{ \max_{\langle u,a\rangle \in V_1} \left\{ SE_U\Big(b, a\Big) \right\} \right\} \right\} \\
&= \theta'.
\end{aligned}$$

Now, let r be the vague relation instance on $R(A_1, A_2, ..., A_n)$ given by Table 1.

Table 1:

$X^+(\theta, \mathcal{V}, \mathcal{M})$	W_1	...	W_m
V_1,..., V_1	V_1,..., V_1	...	V_1,..., V_1
V_1,..., V_1	V_1,..., V_1	...	V_2,..., V_2
$\vdots$	$\vdots$	...	$\vdots$
V_1,..., V_1	V_2,..., V_2	...	V_1,..., V_1
V_1,..., V_1	V_2,..., V_2	...	V_2,..., V_2

Table 1 obviously resembles to the Table 2.

Table 2:

W_1	W_2	...	W_{m-1}	W_m
V_1	V_1	...	V_1	V_1
V_1	V_1	...	V_1	V_2
$\vdots$	$\vdots$	...	$\vdots$	$\vdots$
V_2	V_2	...	V_2	V_1
V_2	V_2	...	V_2	V_2

Actually, the tuples of the Table 1 correspond to the m-tuples $(V_1, V_1, ..., V_1, V_1)$, $(V_1, V_1, ..., V_1, V_2)$,..., $(V_2, V_2, ..., V_2, V_1)$, $(V_2, V_2, ..., V_2, V_2)$ of the Table 2.

In other words, each m-tuple $(a_1, a_2, ..., a_m)$, where $a_i \in \{V_1, V_2\}$ for $i \in \{1, 2, ..., m\}$, determines one tuple of the Table 1.

In the obtained tuple, each of the attributes in W_i is assigned the value a_i for $i \in \{1, 2, ..., m\}$.

Furthermore, each of the attributes in $X^+(\theta, \mathcal{V}, \mathcal{M})$ is assigned the value V_1.

Obviously, Table 1 has 2^m tuples.

As we already noted, $SE(V_1, V_2) = \theta'$.

Since $SE(V_1, V_1) = SE(V_2, V_2) = 1$, it follows from

$$SE_Z(t_1, t_2) = \min_{A \in Z} \{SE(t_1[A], t_2[A])\}$$

that $SE_Z(t_1, t_2) \geq \theta'$ for any $Z \subseteq \{A_1, A_2, ..., A_n\}$, and any t_1 and t_2 in r.

Now, we prove that r satisfies $A \overset{{}_1\theta}{\rightarrow}_V B$ resp. $A \overset{{}_1\theta}{\rightarrow\rightarrow}_V B$ if $A \overset{{}_1\theta}{\rightarrow}_V B$ resp. $A \overset{{}_1\theta}{\rightarrow\rightarrow}_V B$ belongs to $(\mathcal{V}, \mathcal{M})^+$, and violates $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$.

Suppose that $A \overset{{}_1\theta}{\rightarrow}_V B$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

First, assume that ${}_1\theta_l(\mathcal{V}, \mathcal{M}) < \theta$.

Then,

$${}_1\theta \leq {}_1\theta_l(\mathcal{V}, \mathcal{M}) \leq \theta'' < \theta' < \theta.$$

Hence,

$$SE_B(t_1, t_2) \geq \theta' > {}_1\theta \geq \min\{{}_1\theta, SE_A(t_1, t_2)\}$$

for any t_1 and t_2 in r.

Therefore, r satisfies $A \overset{{}_1\theta}{\rightarrow}_V B$.

Now, assume that ${}_1\theta_l(\mathcal{V}, \mathcal{M}) \geq \theta$.

It is enough to prove that r satisfies $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow}_V B$.

Namely, in this case, the inference rule $VF1$ will yield that r also satisfies $A \overset{{}_1\theta}{\rightarrow}_V B$.

In order to prove that r satisfies $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow}_V B$, it is enough to prove that r satisfies $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow}_V B_1$, where $B_1 \in B$ is a single attribute.

Hence, $VF5$ (soundness of $VF5$), will imply that r also satisfies $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow}_V B$.

First, suppose that $B_1 \in X^+(\theta, \mathcal{V}, \mathcal{M})$.

We obtain,

$$\begin{aligned} &SE_{B_1}(t_1, t_2) \\ &= 1 \geq \min\{{}_1\theta_l(\mathcal{V}, \mathcal{M}), SE_A(t_1, t_2)\} \end{aligned}$$

for any t_1 and t_2 in r.

Consequently, r satisfies $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow}_V B_1$, i.e., r satisfies $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow}_V B$, i.e., r satisfies $A \overset{{}_1\theta}{\rightarrow}_V B$.

Second, suppose that $B_1 \notin X^+(\theta, \mathcal{V}, \mathcal{M})$.

Then, $B_1 \in W_i$ for some $i \in \{1, 2, ..., m\}$.

Suppose that $A \cap W_i = \varnothing$.

By Theorem 2, $X \overset{\theta}{\rightarrow\rightarrow}_V Y_j$ for $j \in \{1, 2, ..., k\}$. Therefore, $X \overset{\theta}{\rightarrow\rightarrow}_V W_j$ for $j \in \{1, 2, ..., m\}$. Thus, $X \overset{\theta}{\rightarrow\rightarrow}_V W_i$.

As it can be seen from the proof of Theorem 2, the dependency $X \overset{\theta}{\rightarrow\rightarrow}_V W_i$ is obtained by application of the inference rules. Therefore, $X \overset{\theta}{\rightarrow\rightarrow}_V W_i$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

By the very definition of the limit strength, we know that $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow}_V B$ belongs to $(\mathcal{V}, \mathcal{M})^+$. Therefore, by $VM7$, $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow}_V B_1$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

Now, since $X \overset{\theta}{\rightarrow\rightarrow}_V W_i$ belongs to $(\mathcal{V}, \mathcal{M})^+$, and $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow}_V B_1$ belongs to $(\mathcal{V}, \mathcal{M})^+$, it follows by $VM6$ and $A \cap W_i = \varnothing$, that $X \overset{\min({}_1\theta_l(\mathcal{V},\mathcal{M}),\theta)}{\rightarrow}_V B_1$, i.e., $X \overset{\theta}{\rightarrow}_V B_1$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

By Theorem 1, this means that $B_1 \in X^+(\theta, \mathcal{V}, \mathcal{M})$.

This is a contradiction.

We conclude, $A \cap W_i \neq \varnothing$.

Thus, we want to prove that r satisfies $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow}_V B_1$, where $B_1 \in W_i$, $A \cap W_i \neq \varnothing$, and ${}_1\theta_l(\mathcal{V}, \mathcal{M}) \geq \theta$.

In order to prove this, we shall prove the following, more general statement:

Let $P \overset{\theta_1}{\rightarrow}_V Q$ be a vague functional dependency, such that $\theta_1 \geq \theta$, and $Q \subseteq W_i$ for some $i \in \{1, 2, ..., m\}$. Then, r satisfies $P \overset{\theta_1}{\rightarrow}_V Q$ if and only if $P \cap W_i \neq \varnothing$.

Suppose that r satisfies $P \overset{\theta_1}{\to}_V Q$.

Moreover, suppose that $P \cap W_i = \varnothing$.

Let t_1 resp. t_2 be the tuple in r that corresponds to the m-tuple $(V_1, V_1, ..., V_1)$ resp. $(a_1, a_2, ..., a_m)$, where $a_i = V_2$, and $a_j = V_1$ for $j \in \{1, 2, ..., m\} \setminus \{i\}$.

Since $Q \subseteq W_i$, $\theta_1 \geq \theta$, and $P \cap W_i = \varnothing$, the construction of the instance r yields that

$$\begin{aligned} SE_Q\left(t_1, t_2\right) =& \theta^{'} < \theta \leq \theta_1 = \min\left\{\theta_1, 1\right\} \\ &= \min\left\{\theta_1, SE_P\left(t_1, t_2\right)\right\}. \end{aligned}$$

This contradicts the fact that r satisfies $P \overset{\theta_1}{\to}_V Q$.

Therefore, $P \cap W_i \neq \varnothing$.

Now, suppose that $P \cap W_i \neq \varnothing$.

Let A be any attribute in $P \cap W_i$.

Since $Q \subseteq W_i$, and $A \in W_i$, the construction of the instance r implies that

$$SE_{W_i}\left(t_1, t_2\right) = SE_Q\left(t_1, t_2\right) = SE_A\left(t_1, t_2\right)$$

for any t_1 and t_2 in r.

Moreover, $\{A\} \subseteq P$ yields that $SE_A\left(t_1, t_2\right) \geq SE_P\left(t_1, t_2\right)$ for any t_1 and t_2 in r.

Thus,

$$\begin{aligned} SE_Q\left(t_1, t_2\right) =& SE_A\left(t_1, t_2\right) \geq SE_P\left(t_1, t_2\right) \\ &\geq \min\left\{\theta_1, SE_P\left(t_1, t_2\right)\right\} \end{aligned}$$

for any t_1 and t_2 in r.

Consequently, r satisfies $P \overset{\theta_1}{\to}_V Q$.

We conclude, r satisfies $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\to}_V B_1$.

Reasoning as earlier, we obtain that r satisfies $A \overset{{}_1\theta}{\to}_V B$.

Now, we prove that r satisfies $A \overset{{}_1\theta}{\to\to}_V B$ if $A \overset{{}_1\theta}{\to\to}_V B$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

Suppose that $A \overset{{}_1\theta}{\to\to}_V B$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

First, assume that ${}_1\theta_l\left(\mathcal{V}, \mathcal{M}\right) < \theta$.

Then,

$${}_1\theta \leq_1 \theta_l\left(\mathcal{V}, \mathcal{M}\right) \leq \theta^{''} < \theta^{'} < \theta.$$

Let $t_1, t_2 \in r$.

Now, there exists a tuple t_3 in r, $t_3 = t_1$, such that

$$\begin{aligned} SE_A\left(t_3, t_1\right) =& 1 \geq \min\left\{{}_1\theta, SE_A\left(t_1, t_2\right)\right\}, \\ SE_B\left(t_3, t_1\right) =& 1 \geq \min\left\{{}_1\theta, SE_A\left(t_1, t_2\right)\right\}, \\ &SE_{\{A_1, A_2, ..., A_n\} \setminus (A \cup B)}\left(t_3, t_2\right) \\ &\geq \theta^{'} >_1 \theta \geq \min\left\{{}_1\theta, SE_A\left(t_1, t_2\right)\right\}. \end{aligned}$$

This means that r satisfies $A \overset{{}_1\theta}{\to\to}_V B$.

Suppose that ${}_1\theta_l\left(\mathcal{V}, \mathcal{M}\right) \geq \theta$.

It is enough to prove that r satisfies $A \overset{{}_1\theta_l(\mathcal{V},\mathcal{M})}{\to\to}_V B$.

Then, the soundness of $VM1$ will imply that r also satisfies $A \overset{{}_1\theta}{\to\to}_V B$.

By construction of r, it follows that

$$\begin{aligned} &SE_{B \cap X^+(\theta, \mathcal{V}, \mathcal{M})}\left(t_1, t_2\right) \\ &= 1 \geq \min\left\{1, SE_A\left(t_1, t_2\right)\right\} \end{aligned}$$

for any t_1 and t_2 in r.

This means that r satisfies the vague functional dependency $A \to_V B \cap X^+\left(\theta, \mathcal{V}, \mathcal{M}\right)$.

Hence, by $VM5$, r satisfies the vague multivalued dependency $A \to\to_V B \cap X^+\left(\theta, \mathcal{V}, \mathcal{M}\right)$.

If we prove that r satisfies $A \to\to_V B \cap W_i$ for every $i \in \{1, 2, ..., m\}$ such that $B \cap W_i \neq \varnothing$, then, $VM7$ will yield that r also satisfies $A \to\to_V B$.

Suppose that $i \in \{1, 2, ..., m\}$ is such that $B \cap W_i \neq \varnothing$.

First, suppose that $B \cap W_i = W_i$.

We have to prove that r satisfies $A \to\to_V W_i$.

In order to prove this, we shall prove the following, more general statement:

Let $i \in \{1, 2, ..., m\}$. Then, r satisfies $P \overset{\theta_1}{\to\to}_V W_i$ for any $P \subseteq \{A_1, A_2, ..., A_n\}$, and any $\theta_1 \in [0, 1]$.

Indeed, let $t_1, t_2 \in r$.

Suppose that $(a_1, a_2, ..., a_m)$ and $(b_1, b_2, ..., b_m)$, where $a_i, b_i \in \{V_1, V_2\}$ for $i \in \{1, 2, ..., m\}$, are the m-tuples that determine t_1 and t_2, respectively.

Let $t_3 \in r$ be the tuple that corresponds to the m-tuple $(c_1, c_2, ..., c_m)$, such that $c_i = a_i$, and $c_j = b_j$ for $j \in \{1, 2, ..., m\} \setminus \{i\}$.

It follows by construction of r that

$$\begin{aligned} SE_{W_i}\left(t_3, t_1\right) =& 1 \geq \min\left\{\theta_1, SE_P\left(t_1, t_2\right)\right\}, \\ &SE_{\{A_1, A_2, ..., A_n\} \setminus (P \cup W_i)}\left(t_3, t_2\right) \\ &= 1 \geq \min\left\{\theta_1, SE_P\left(t_1, t_2\right)\right\}. \end{aligned}$$

Since $SE_{W_i}(t_3,t_1)=1$ and $SE_{\{A_1,A_2,...,A_n\}\setminus W_i}(t_3,t_2)=1$, it follows from $P \cap W_i \subseteq W_i$ and $P \cap (\{A_1,A_2,...,A_n\}\setminus W_i) \subseteq \{A_1,A_2,...,A_n\}\setminus W_i$, that

$$SE_{P\cap W_i}(t_3,t_1) \geq SE_{W_i}(t_3,t_1)=1$$

and

$$\begin{aligned} &SE_{P\cap(\{A_1,A_2,...,A_n\}\setminus W_i)}(t_3,t_2) \\ \geq &SE_{\{A_1,A_2,...,A_n\}\setminus W_i}(t_3,t_2)=1. \end{aligned}$$

Therefore,

$$\begin{aligned} &SE_{P\cap W_i}(t_3,t_1)=1, \\ &SE_{P\cap(\{A_1,A_2,...,A_n\}\setminus W_i)}(t_3,t_2)=1. \end{aligned}$$

Furthermore, $P \cap (\{A_1,A_2,...,A_n\}\setminus W_i) \subseteq P$. Hence,

$$SE_{P\cap(\{A_1,A_2,...,A_n\}\setminus W_i)}(t_1,t_2) \geq SE_P(t_1,t_2).$$

Now,

$$SE_{P\cap(\{A_1,A_2,...,A_n\}\setminus W_i)}(t_3,t_2)=1 \geq SE_P(t_1,t_2)$$

and

$$SE_{P\cap(\{A_1,A_2,...,A_n\}\setminus W_i)}(t_1,t_2) \geq SE_P(t_1,t_2)$$

yield that

$$SE_{P\cap(\{A_1,A_2,...,A_n\}\setminus W_i)}(t_3,t_1) \geq SE_P(t_1,t_2).$$

Finally,

$$SE_{P\cap W_i}(t_3,t_1)=1 \geq SE_P(t_1,t_2)$$

and

$$SE_{P\cap(\{A_1,A_2,...,A_n\}\setminus W_i)}(t_3,t_1) \geq SE_P(t_1,t_2)$$

imply that

$$\begin{aligned} &SE_P(t_3,t_1) \\ &=\min_{A\in P}\{SE(t_3[A],t_1[A])\} \\ &=\min\Big(\min_{A\in P\cap W_i}\{SE(t_3[A],t_1[A])\}, \\ &\min_{A\in P\cap(\{A_1,A_2,...,A_n\}\setminus W_i)}\{SE(t_3[A],t_1[A])\}\Big) \\ &=\min\Big(SE_{P\cap W_i}(t_3,t_1), \\ &SE_{P\cap(\{A_1,A_2,...,A_n\}\setminus W_i)}(t_3,t_1)\Big) \\ &\geq \min(SE_P(t_1,t_2),SE_P(t_1,t_2)) \\ &=SE_P(t_1,t_2) \geq \min\{\theta_1,SE_P(t_1,t_2)\}. \end{aligned}$$

Thus, for t_1 and t_2 in r, there exists the tuple $t_3 \in r$, such that

$$\begin{aligned} SE_P(t_3,t_1) &\geq \min\{\theta_1,SE_P(t_1,t_2)\}, \\ SE_{W_i}(t_3,t_1) &\geq \min\{\theta_1,SE_P(t_1,t_2)\}, \\ &SE_{\{A_1,A_2,...,A_n\}\setminus(P\cup W_i)}(t_3,t_2) \\ &\geq \min\{\theta_1,SE_P(t_1,t_2)\}. \end{aligned}$$

Therefore, r satisfies $P \overset{\theta_1}{\rightarrow\rightarrow}_V W_i$.

Consequently, r satisfies $A \rightarrow\rightarrow_V W_i$, i.e., $A \rightarrow\rightarrow_V B \cap W_i$.

Now, suppose that $B \cap W_i \subset W_i$.

Suppose that $A \cap W_i = \varnothing$.

By Theorem 2, $X \overset{\theta}{\rightarrow\rightarrow}_V Y_j$ for $j \in \{1,2,...,k\}$.

Therefore, $X \overset{\theta}{\rightarrow\rightarrow}_V W_j$ for $j \in \{1,2,...,m\}$. Thus, $X \overset{\theta}{\rightarrow\rightarrow}_V W_i$.

The dependencies $X \overset{\theta}{\rightarrow\rightarrow}_V Y_j$, $j \in \{1,2,...,k\}$ (and hence the dependencies $X \overset{\theta}{\rightarrow\rightarrow}_V W_j$, $j \in \{1,2,...,m\}$) are obtained by application of the inference rules. Therefore, $X \overset{\theta}{\rightarrow\rightarrow}_V Y_j$ belongs to $(\mathcal{V},\mathcal{M})^+$ for $j \in \{1,2,...,k\}$.

In particular, $X \overset{\theta}{\rightarrow\rightarrow}_V W_i$ belongs to $(\mathcal{V},\mathcal{M})^+$.

Since $X \overset{\theta}{\rightarrow\rightarrow}_V Y_j$ belongs to $(\mathcal{V},\mathcal{M})^+$ for $j \in \{1,2,...,k\}$, it follows by $VM7$ that

$$X \overset{\theta}{\rightarrow\rightarrow}_V \{A_1,A_2,...,A_n\}$$

also belongs to $(\mathcal{V},\mathcal{M})^+$.

Hence, $VM9$ and the fact that $X \overset{\theta}{\rightarrow\rightarrow}_V W_i$ belongs to $(\mathcal{V},\mathcal{M})^+$, imply that

$$X \overset{\theta}{\rightarrow\rightarrow}_V \{A_1, A_2, ..., A_n\} \setminus W_i$$

belongs to $(\mathcal{V}, \mathcal{M})^+$.

By the very definition of the limit strength, we know that $A \overset{1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow\rightarrow}_V B$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

Hence, $VM3$ and the fact that

$$\{A_1, A_2, ..., A_n\} \setminus W_i \supseteq B \setminus W_i,$$

yield that

$$\{A_1, A_2, ..., A_n\} \setminus W_i \overset{1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow\rightarrow}_V B$$

belongs to $(\mathcal{V}, \mathcal{M})^+$.

Now,

$$X \overset{\theta}{\rightarrow\rightarrow}_V \{A_1, A_2, ..., A_n\} \setminus W_i$$

belongs to $(\mathcal{V}, \mathcal{M})^+$,

$$\{A_1, A_2, ..., A_n\} \setminus W_i \overset{1\theta_l(\mathcal{V},\mathcal{M})}{\rightarrow\rightarrow}_V B$$

belongs to $(\mathcal{V}, \mathcal{M})^+$, and $VM4$, imply that

$$X \overset{\min(\theta, 1\theta_l(\mathcal{V},\mathcal{M}))}{\rightarrow\rightarrow}_V B \setminus (\{A_1, A_2, ..., A_n\} \setminus W_i),$$

i.e.,

$$X \overset{\theta}{\rightarrow\rightarrow}_V B \cap W_i$$

belongs to $(\mathcal{V}, \mathcal{M})^+$.

Thus, the dependency $X \overset{\theta}{\rightarrow\rightarrow}_V B \cap W_i$ exists.

As we noted at the beginning of the proof, this means that $B \cap W_i$ is the union of a subset of $X^+(\theta, \mathcal{V}, \mathcal{M})$, and some of the sets W_1, W_2,..., W_m.

This is a contradiction, however, since $B \cap W_i \subset W_i$.

We conclude, $A \cap W_i \neq \varnothing$.

Thus, it remains to prove that r satisfies $A \rightarrow\rightarrow_V B \cap W_i$, where $B \cap W_i \subset W_i$, and $A \cap W_i \neq \varnothing$.

In order to prove this, we shall prove the following, more general statement:

Let $P \overset{\theta_1}{\rightarrow\rightarrow}_V Q$ be a vague multivalued dependency such that $\theta_1 \geq \theta$, and $Q \subset W_i$ for some $i \in \{1, 2, ..., m\}$. Then, r satisfies $P \overset{\theta_1}{\rightarrow\rightarrow}_V Q$ if and only if $P \cap W_i \neq \varnothing$.

Suppose that r satisfies $P \overset{\theta_1}{\rightarrow\rightarrow}_V Q$.

Moreover, suppose that $P \cap W_i = \varnothing$.

Note that $Q \subset W_i$. Hence, $W_i \setminus Q \neq \varnothing$.

Let t_1 resp. t_2 be the tuple in r that corresponds to the m-tuple $(V_1, V_1, ..., V_1)$ resp. $(a_1, a_2, ..., a_m)$, where $a_i = V_2$, and $a_j = V_1$ for $j \in \{1, 2, ..., m\} \setminus \{i\}$.

Since $P \cap W_i = \varnothing$, it immediately follows that $SE_P(t_1, t_2) = 1$.

Since r satisfies $P \overset{\theta_1}{\rightarrow\rightarrow}_V Q$, and t_1, $t_2 \in r$, we have that there exists a tuple $t_3 \in r$, such that

$$\begin{aligned} SE_P(t_3, t_1) &\geq \min\{\theta_1, SE_P(t_1, t_2)\}, \\ &= \min\{\theta_1, 1\} = \theta_1, \\ SE_Q(t_3, t_1) &\geq \min\{\theta_1, SE_P(t_1, t_2)\} = \theta_1, \\ &SE_{\{A_1, A_2, ..., A_n\} \setminus (P \cup Q)}(t_3, t_2) \\ &\geq \min\{\theta_1, SE_P(t_1, t_2)\} = \theta_1. \end{aligned}$$

Note that $\theta_1 \geq \theta > \theta'$.

Hence,

$$SE_Q(t_3, t_1) \geq \theta_1 > \theta'$$

and

$$SE_{\{A_1, A_2, ..., A_n\} \setminus (P \cup Q)}(t_3, t_2) \geq \theta_1 > \theta'$$

yield that

$$SE_Q(t_3, t_1) = 1$$

and

$$SE_{\{A_1, A_2, ..., A_n\} \setminus (P \cup Q)}(t_3, t_2) = 1.$$

Since $SE_Q(t_3, t_1) = 1$, and in the tuple t_1 each of the attributes is assigned the value V_1, it follows that in the tuple t_3 each of the attributes in Q is assigned the value V_1.

Similarly, $SE_{\{A_1, A_2, ..., A_n\} \setminus (P \cup Q)}(t_3, t_2) = 1$ implies that in the tuples t_2 and t_3 each of the attributes in $\{A_1, A_2, ..., A_n\} \setminus (P \cup Q)$ has the same value.

In particular, this means that in the tuple t_3 each of the attributes in $W_i \setminus Q$ is assigned the value V_2, and each of the attributes in

$$(\{A_1, A_2, ..., A_n\} \setminus (P \cup Q)) \setminus (W_i \setminus Q)$$

is assigned the value V_1.

Thus, in the tuple t_3, each of the attributes in Q is assigned the value V_1, while, at the same time, each of the attributes in $W_i \setminus Q$ is assigned the value V_2.

This, however, is a contradiction.

Namely, according to the construction of the instance r, in each tuple of r, each of the attributes in W_i, $i \in \{1, 2, ..., m\}$ has the same value.

We conclude, $P \cap W_i \neq \varnothing$.

Now, suppose that $P \cap W_i \neq \varnothing$.

We have: $\theta_1 \geq \theta$, $Q \subset W_i$, and $P \cap W_i \neq \varnothing$.

Hence, the first additional statement (derived in the proof of theorem) yields that r satisfies the vague functional dependency $P \overset{\theta_1}{\to}_V Q$.

Consequently, $VM5$ yields that r also satisfies the vague multivalued dependency $P \overset{\theta_1}{\to\to}_V Q$.

We obtain, r satisfies $A \to\to_V B \cap W_i$.

Thus, r satisfies $A \to\to_V B \cap X^+ (\theta, \mathcal{V}, \mathcal{M})$ and $A \to\to_V B \cap W_i$ for $i \in \{1, 2, ..., m\}$ such that $B \cap W_i \neq \varnothing$.

By $VM7$, r satisfies $A \to\to_V B$.

Finally, by $VM1$, r satisfies $A \overset{1\theta_l(\mathcal{V},\mathcal{M})}{\to\to}_V B$, i.e., $A \overset{1\theta}{\to\to}_V B$.

It remains to prove that r violates $X \overset{\theta}{\to}_V Y$ resp. $X \overset{\theta}{\to\to}_V Y$.

First, we prove that r violates $X \overset{\theta}{\to}_V Y$.

Suppose that $Y \subseteq X^+ (\theta, \mathcal{V}, \mathcal{M})$.

Then, by Theorem 1, it follows that $X \overset{\theta}{\to}_V Y$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

This is a contradiction.

Hence, $Y \setminus X^+ (\theta, \mathcal{V}, \mathcal{M}) \neq \varnothing$.

This means that there exists $i \in \{1, 2, ..., m\}$ such that $Y \cap W_i \neq \varnothing$.

Suppose that r satisfies $X \overset{\theta}{\to}_V Y$.

Hence, by $VF7$, r satisfies $X \overset{\theta}{\to}_V Y \cap W_i$.

Now, $\theta \geq \theta$, and $Y \cap W_i \subseteq W_i$ yield that $X \cap W_i \neq \varnothing$.

This, however, is a contradiction since $X \subseteq X^+ (\theta, \mathcal{V}, \mathcal{M})$, and $X^+ (\theta, \mathcal{V}, \mathcal{M}) \cap W_j = \varnothing$ for all $j \in \{1, 2, ..., m\}$.

We obtain, r violates $X \overset{\theta}{\to}_V Y$.

Now, we prove that r violates $X \overset{\theta}{\to\to}_V Y$.

Suppose that $Y \subseteq X^+ (\theta, \mathcal{V}, \mathcal{M})$.

It follows by Theorem 1 that $X \overset{\theta}{\to}_V Y$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

Then, by $VM5$, $X \overset{\theta}{\to\to}_V Y$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

This is a contradiction.

Therefore, $Y \setminus X^+ (\theta, \mathcal{V}, \mathcal{M}) \neq \varnothing$.

This means that there is $k \in \{1, 2, ..., m\}$ such that $Y \cap W_k \neq \varnothing$.

Suppose that $Y \cap W_i = W_i$ for each $i \in \{1, 2, ..., m\}$ such that $Y \cap W_i \neq \varnothing$.

Thus, either $Y \cap W_i = W_i$ or $Y \cap W_i = \varnothing$ for all $i \in \{1, 2, ..., m\}$.

Suppose that $Y \cap W_i = W_i$ for some $i \in \{1, 2, ..., m\}$.

As noted earlier, $X \overset{\theta}{\to\to}_V W_j$ belongs to $(\mathcal{V}, \mathcal{M})^+$ for all $j \in \{1, 2, ..., m\}$. In particular, $X \overset{\theta}{\to\to}_V W_i$, i.e., $X \overset{\theta}{\to\to}_V Y \cap W_i$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

Since $X^+ (\theta, \mathcal{V}, \mathcal{M}) \subseteq X^+ (\theta, \mathcal{V}, \mathcal{M})$, it follows by Theorem 1 that $X \overset{\theta}{\to}_V X^+ (\theta, \mathcal{V}, \mathcal{M})$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

Hence, $VF7$ and the fact that $Y \cap X^+ (\theta, \mathcal{V}, \mathcal{M}) \subseteq X^+ (\theta, \mathcal{V}, \mathcal{M})$ yield that

$$X \overset{\theta}{\to}_V Y \cap X^+ (\theta, \mathcal{V}, \mathcal{M})$$

belongs to $(\mathcal{V}, \mathcal{M})^+$.

Hence, by $VM5$,

$$X \overset{\theta}{\to\to}_V Y \cap X^+ (\theta, \mathcal{V}, \mathcal{M})$$

belongs to $(\mathcal{V}, \mathcal{M})^+$.

Now, $X \overset{\theta}{\to\to}_V Y \cap X^+ (\theta, \mathcal{V}, \mathcal{M})$ belongs to $(\mathcal{V}, \mathcal{M})^+$, $X \overset{\theta}{\to\to}_V Y \cap W_i$ belongs to $(\mathcal{V}, \mathcal{M})^+$ for every $i \in \{1, 2, ..., m\}$ such that $Y \cap W_i \neq \varnothing$, and $VM7$, yield that $X \overset{\theta}{\to\to}_V Y$ belongs to $(\mathcal{V}, \mathcal{M})^+$.

This is a contradiction.

We conclude, $Y \cap W_i \subset W_i$ for some $i \in \{1, 2, ..., m\}$ such that $Y \cap W_i \neq \varnothing$.

Let $i \in \{1, 2, ..., m\}$ be such that $Y \cap W_i \subset W_i$.

Suppose that r satisfies $X \overset{\theta}{\to\to}_V Y \cap W_i$.

Since $\theta \geq \theta$, and $Y \cap W_i \subset W_i$, it follows that $X \cap W_i \neq \varnothing$.

Reasoning as in the previous case, we conclude that this is a contradiction.

We conclude, r violates $X \overset{\theta}{\to\to}_V Y \cap W_i$.

As we proved above, r satisfies any vague multivalued dependency that has W_j, $j \in \{1, 2, ..., m\}$ as its right side.

Hence, r satisfies $X \to\to_V W_i$.

Suppose that r satisfies $X \overset{\theta}{\to\!\!\to}_V Y$.
By $VM9$, r satisfies $X \overset{\theta}{\to\!\!\to}_V Y \cap W_i$.
This is a contradiction.
Therefore, r violates $X \overset{\theta}{\to\!\!\to}_V Y$.
This completes the proof. $\square$

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