The number of solutions to the boundary value problem for the second order differential equation with cubic nonlinearity

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Abstract: The differential equation with cubic nonlinearity $x'' = -ax + bx^3$ is considered together with the Sturm - Liouville type boundary conditions. The number of solutions for the Sturm - Liouville boundary value p roblem is given. The equation for the initial values of solutions to boundary value problem is derived using representation by Jacobian elliptic functions. An explanatory example is given with a number of visualizations.

Key–Words: Boundary value problem, cubic nonlinearity, phase trajectory, multiplicity of solutions, Jacobian elliptic function.

1 Introduction

We consider a nonlinear the second order differential equation with a cubic nonlinearity

$$x'' = -ax + bx^3, \quad a > 0,\ b > 0 \tag{1}$$

and the respective boundary value problems (BVP in short) with the Sturm - Liouville boundary conditions. Equations with a cubic nonlinearity often appear in applications, for instance, in Ginzburg-Landau theory of superconductivity ([6], [13]). Despite of the fact that phase portrait and general behaviour of solutions for equation (1) are well known there are some difficulties in determining the number of solutions to boundary value problems and their properties.

Namely, the Sturm - Liouville boundary value problem (1),

$$\begin{array}{ll} \alpha_1 x(0) + \alpha_2 x'(0) = 0, & \alpha_1 \cdot \alpha_2 < 0, \\ \beta_1 x(1) + \beta_2 x'(1) = 0, & \beta_1 \cdot \beta_2 > 0 \end{array} \tag{2}$$

is considered. After careful analysis of Sturm - Liouville problem (1), (2) we provide the exact number of solutions for considered problem.

Since boundary conditions (2) does not contain neither Dirichlet conditions nor Neumann conditions, in the next section (Section 2) we briefly describe the results on the Dirichlet and the Neumann problems for equation (1).

In Section 3 we provide the main result on the number of solutions to the problem (1), (2). Also the Cauchy problem (1),

$$\begin{array}{l} \alpha_1 x(0) - \alpha_2 x'(0) = 0, \quad x(0) \neq 0, \quad x'(0) \neq 0, \\ x(0) = x_0 > 0 \end{array} \tag{3}$$

is considered and formulas for solutions of (1), (2) are obtained using theory of Jacobian elliptic functions ([14], [18]). Then we are able to derive equations for determining of the values x_0 (in (3)) which correspond to solutions of the BVP. Finally in Section 4 we demonstrate how all the developed technique and formulas work in a specific situation.

2 Review of the Dirichlet and Neumann problem

Consider the equation (1). There are three critical points of equation (1) at $x_1 = -\sqrt{\frac{a}{b}}$, $x_2 = 0$, $x_3 = \sqrt{\frac{a}{b}}$. The origin is a center and $x_{1,3} = \pm\sqrt{\frac{a}{b}}$ both are saddle points. Two heteroclinic trajectories connect the two saddle points. The phase portrait of equation (1) is depicted in Fig. 1.

Denote open region bounded by the two heteroclinic trajectories connecting saddle points by $G3$.

Consider the Neumann boundary value problem (1),

$$x'(0) = 0, \quad x'(T) = 0 \tag{4}$$

and the Dirichlet boundary value problem (1),

$$x(0) = 0, \quad x(T) = 0. \tag{5}$$

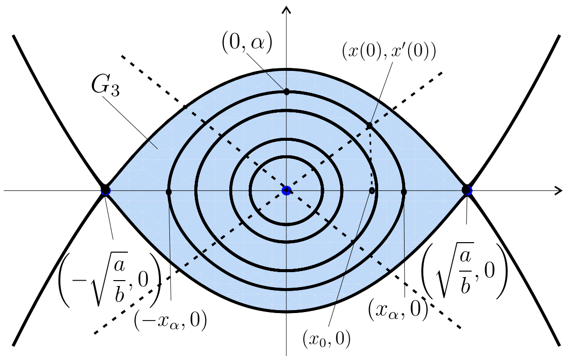

Figure 1: The phase portrait of equation (1), region $G3$

In this section first, we provide the results that give estimates of the exact number of solutions for both problems. Second, using the theory of Jacobian elliptic functions, we give expressions for solutions of the Cauchy problems (1),

$$x(0)=x_\alpha,\ x'(0)=0,\ -\sqrt{\frac{a}{b}}<x_\alpha<\sqrt{\frac{a}{b}},\ x_\alpha\neq 0, \tag{6}$$

and (1),

$$x(0)=0,\ x'(0)=\alpha,\ -\alpha_{max}<\alpha<\alpha_{max},\ \alpha\neq 0, \tag{7}$$

where $\alpha_{max} =: \dfrac{a}{\sqrt{2b}}$.
Third, we show how to find the initial values x_α of solutions of the problem (1), (4) and respective the initial values α of solutions of the problem (1), (5).

2.1 For the Neumann problem

Consider the problem (1), (4). The following statement is true.

Theorem 1 *Let i be a positive integer such that*

$$\frac{i\,\pi}{\sqrt{a}}<T<\frac{(i+1)\pi}{\sqrt{a}} \tag{8}$$

where T is the right end point of the interval in (4). The Neumann problem (1), (4) has exactly $2i$ nontrivial solutions such that $x(0)=x_\alpha\neq 0$, $x'(0)=0$, $-\sqrt{\frac{a}{b}}<x_\alpha<\sqrt{\frac{a}{b}}$.

The proof can be found in article [9].
Let us address the eigenvalue problem posed in [13]. Consider the Cauchy problem (1), (6):

$$x''=-ax+bx^3,\quad x(0)=x_\alpha,\quad x'(0)=0.$$

Let a and T (in (4)) be given. We wish to find x_α such that the respective solutions $x(t;x_\alpha)$ of the above problem satisfy the boundary condition $x'(T)=0$, i.e. $x(t;x_\alpha)$ solve the Neumann problem (1), (4). The following assertion provides the explicit formula for a solution of (1), (6).

Lemma 2 *The function*

$$x(t,a,b,x_\alpha)=x_\alpha cd\left(\sqrt{a-\frac{1}{2}b\,x_\alpha^2}\ t;k\right), \tag{9}$$

where $k=\sqrt{\frac{b\,x_\alpha^2}{2a-b\,x_\alpha^2}}$ is a solution of the Cauchy problem (1), (6).

The proof is given in article [11].

Denote $f(t,a,b,x_\alpha)=x_t'(t,a,b,x_\alpha)$. This derivative can be computed and the following formula is valid.

$$\begin{aligned}&f(t,a,b,x_\alpha)=x_\alpha cd'\,t\left(\sqrt{a-\tfrac{1}{2}b\,x_\alpha^2}\ t;k\right)=\\&=\frac{x_\alpha\sqrt{a-b\,x_\alpha^2}}{\sqrt{2}}\left(k^2-1\right)nd\left(\sqrt{a-\tfrac{1}{2}b\,x_\alpha^2}\ t;k\right)\times\\&\times sd\left(\sqrt{a-\tfrac{1}{2}b\,x_\alpha^2}\ t;k\right).\end{aligned} \tag{10}$$

The following statement is true.

Lemma 3 *[11] The eigenvalue problem (1), (4) for a and T given can be solved now by solving the equation with respect to x_α*

$$f(T,a,b,x_\alpha)=0. \tag{11}$$

Theorem 4 *[11] A solution to the Neumann problem (1), (4) is given by (9) where x_α is a solution of (11).*

2.2 For the Dirichlet problem

Consider the problem (1), (5).

Theorem 5 *Let i be a positive integer such that*

$$\frac{i\,\pi}{\sqrt{a}}<T<\frac{(i+1)\pi}{\sqrt{a}} \tag{12}$$

where T is the right end point of the interval in (5). The Dirichlet problem (1), (5) has exactly $2i$ nontrivial solutions such that $x(0)=0$, $x'(0)=\alpha\neq 0$, $-\alpha_{max}<\alpha<\alpha_{max}$.

Proof: Consider solutions of the Cauchy problem (1), (7), where $0<\alpha<\alpha_{max}$. Solutions for α small enough behave like solutions of the equation of variations $y''=-ay$ around the trivial solution

$$y(t)=\frac{\alpha}{\sqrt{a}}\sin\sqrt{a}t. \tag{13}$$

Due to the assumption $\dfrac{i\,\pi}{\sqrt{a}} < T < \dfrac{(i+1)\pi}{\sqrt{a}}$ solutions $y(t)$ along with solutions $x(t;\alpha)$ (for small enough α) have exactly i zeros in the interval $(0,T)$. These zeros move monotonically to the right as α increases. Solutions $x(t;\alpha)$ with $0 < \alpha < \alpha_{max}$ and close enough to α_{max} have not zeros in $(0,T\,]$ since the respective trajectories are close to the upper heteroclinic (and the "period" of a heteroclinic solution is infinite). Therefore there are exactly i solutions of the problem (1), (5). The additional i solutions are obtained considering solutions with $\alpha \in (-\alpha_{max}, 0)$ due to symmetry arguments. Hence the proof.

Consider representation of solution of the Dirichlet problem using Jacobian elliptic functions.
Since $\alpha^2 = ax_\alpha^2 - \frac{1}{2}bx_\alpha^4$ and $\alpha \in (0, a/\sqrt{2b})$ we obtain $a^2 - 2b\alpha^2 > 0$. Note that $0 < x_\alpha < \sqrt{\frac{a}{b}}$ we have

$$x_\alpha = \sqrt{\frac{a-\sqrt{a^2-2b\alpha^2}}{b}}. \qquad (14)$$

Lemma 6 *[10] The function*

$$x(t,a,b,\alpha) = \sqrt{\tfrac{a-\sqrt{a^2-2b\alpha^2}}{b}}\, sn\left[\sqrt{\tfrac{a+\sqrt{a^2-2b\alpha^2}}{2}}\, t, k\right],$$
$$k = \sqrt{\tfrac{a-\sqrt{a^2-2b\alpha^2}}{a+\sqrt{a^2-2b\alpha^2}}} \qquad (15)$$

is a solution of the Cauchy problem (1), (7).

Lemma 7 *[10] The eigenvalue problem (1), (5) for a and T given can be solved now by solving the equation with respect to α*

$$x(T,a,b,\alpha) = 0. \qquad (16)$$

Theorem 8 *[10] A solution to the Dirichlet problem (1), (5) is given by (15) where α is a solution of (16).*

3 Main results

3.1 Multiplicity of solutions

Equation (1) written in polar coordinates

$$x(t) = \rho(t)\sin\phi(t), \quad x'(t) = \rho(t)\cos\phi(t) \qquad (17)$$

turns to a system (18):

$$\begin{cases} \phi'(t) = \cos^2\phi(t) + a\sin^2\phi(t) - \rho^2(t)\; b\; \sin^4\phi(t), \\ \rho'(t) = \frac{1}{2}\rho(t)\sin 2\phi(t)\left(1 - a + \rho^2(t)\; b\; \sin^2\phi(t)\right). \end{cases} \qquad (18)$$

Consider any solution of equation (1) with the initial conditions $(x(t_0), x'(t_0)) \in G3$. Let initial conditions be written as

$$\phi(t_0) = \phi_0,\ \rho(t_0) = \rho_0,\ (\phi_0, \rho_0) \in G3,\ \rho_0 > 0. \qquad (19)$$

Lemma 9 *The angular function of any solution of (18), (19) is monotonically increasing.*

Proof given in paper [11], [8].

Consider the problem (1), (2). The following statement is true.

Theorem 10 *Consider linear equation $y'' = -ay$. Let i be the number of point $\tau_i \in (0,1)$, such that the solution $y(t)$ of the initial value problem $y'' = -ay$, $\alpha_1 y(0) - \alpha_2 y'(0) = 0$, $y^2(0) + y'^2(0) = 1$ $(y(0) > 0$, $y'(0) > 0)$ satisfies $\beta_1 y(\tau_i) + \beta_2 y'(\tau_i) = 0$. Then there exist at least $2i$ nontrivial solutions of the Sturm - Liouville problem (1), (2).*

Proof: Consider solutions of the Cauchy problem (1),

$$\alpha_1 x(0) - \alpha_2 x'(0) = 0, \quad x(0) \neq 0, \quad x'(0) \neq 0,$$
$$x(0) = x_0 > 0. \qquad (20)$$

Solutions of (1), (20) for $x(0) = x_0 > 0$ small enough behave like solutions of the equation of variations $y'' = -a\,y$ with conditions $\alpha_1 y(0) - \alpha_2 y'(0) = 0$, $y^2(0) + y'^2(0) = 1$ $(y(0) > 0, y'(0) > 0)$ and have exactly i points of $\tau_i \in (0,1)$ such that $\beta_1 y(\tau_i) + \beta_2 y'(\tau_i) = 0$. These zeros due to Lemma 9 move monotonically to the right as $x(0) = x_0 > 0$ increases. Solutions $x(t)$ with $x(0) = x_0 > 0$ and such that $(x(0), x'(0))$, $x(0) > 0$, $x'(0) > 0$ close enough to heteroclinic orbit have not zeros in $(0,1\,]$ since they are very slow (the "period" of a heteroclinic solution is infinite). Therefore there are at least i solutions of the problem (1), (2). The additional i solutions are obtained considering solutions with $x(0) < 0$, $x'(0) < 0$ due to symmetry arguments.

Remark 11 *We can compute τ_i in the following way: consider the linear equation $y'' = -a\,y$ with initial condition $y(0) - y'(0) = 0$, $y^2(0) + y'^2(0) = 1$. Solution of this linear equation is*

$$y(t) = \frac{1}{\sqrt{2a}}\sin\sqrt{a}t + \frac{1}{\sqrt{2}}\cos\sqrt{a}t \qquad (21)$$

and

$$y'(t) = \frac{1}{\sqrt{2}}\cos\sqrt{a}t - \sqrt{\frac{a}{2}}\sin\sqrt{a}t. \qquad (22)$$

Now we are looking for τ_i which satisfy the condition $y(\tau_i) + y'(\tau_i) = 0$:

$$y(\tau) + y'(\tau) = \frac{1-\sqrt{a}}{\sqrt{2a}}\sin\sqrt{a}\tau + \frac{2}{\sqrt{2}}\cos\sqrt{a}\tau = 0, \qquad (23)$$

$$\frac{1-a}{\sqrt{2a}}\tan\sqrt{a}\tau = -\frac{2}{\sqrt{2}}, \qquad (24)$$

$$\tan \sqrt{a}\tau = \frac{2\sqrt{a}}{a-1}, \tag{25}$$

$$\tau = \frac{1}{\sqrt{a}} \arctan \frac{2\sqrt{a}}{a-1} + \frac{\pi}{\sqrt{a}} k, \quad k \in Z. \tag{26}$$

There exist i values and respective τ_i that $y(\tau_i) + y'(\tau_i) = 0$.

Remark 12 *In fact the number of solutions to BVP (1), (2) in Theorem 10 is exact. The proof of this fact is currently not in hand.*

3.2 Formulae of solutions

Consider problem

$$\begin{array}{l} x'' = -ax + bx^3, \\ \alpha_1 x(0) - \alpha_2 x'(0) = 0,\ x(0) \neq 0,\ x'(0) \neq 0 \\ x(0) = x_0 > 0. \end{array} \tag{27}$$

The first equation in problem (27) has an integral

$$x'^2(t) = -ax^2(t) + \frac{1}{2} bx^4(t) + C, \tag{28}$$

where C is an arbitrary constant. The formula for the "upper" heteroclinic trajectory is

$$x'^2 + ax^2 - \frac{1}{2} bx^4 = \frac{a^2}{2b}, \tag{29}$$

where $x(0) = 0$, $x' = \frac{a}{\sqrt{2b}}$. Since $x' = \frac{\alpha_1}{\alpha_2} x$, then

$$\frac{b}{2} x^4 - \left(\frac{\alpha_1^2}{\alpha_2^2} + a \right) x^2 + \frac{a^2}{2b} = 0, \tag{30}$$

where

$$x_* = \sqrt{\frac{a}{b} + \frac{\alpha_1^2}{b\alpha_2^2} - \frac{\sqrt{2ab^2\alpha_1^2\alpha_2^2 + b^2\alpha_1^4}}{b^2\alpha_2^2}} \tag{31}$$

is the end value of x_0 in (27). Therefore $0 < x(0) = x_0 < x_*$.
Consider equation (28) where

$$C = \left(\frac{\alpha_1^2}{\alpha_2^2} + a \right) x_0^2 - \frac{b}{2} x_0^4 = \alpha^2 \tag{32}$$

and from (15) we have the formula of solution of the Cauchy problem (27).
Introduce a new variable

$$C(x_0) = \left(\frac{\alpha_1^2}{\alpha_2^2} + a \right) x_0^2 - \frac{b}{2} x_0^4. \tag{33}$$

Lemma 13 *The function*

$$\begin{array}{l} x(t, a, b, x_0) = \sqrt{\frac{a - \sqrt{a^2 - 2b\,C(x_0)}}{b}} \times \\ \times sn \left[\sqrt{\frac{a + \sqrt{a^2 - 2b\,C(x_0)}}{2}}\,(t + \Gamma), k \right], \end{array} \tag{34}$$

where $k = \sqrt{\frac{a - \sqrt{a^2 - 2b\,C(x_0)}}{a + \sqrt{a^2 - 2b\,C(x_0)}}}$,

$$\begin{array}{l} \Gamma = \sqrt{\frac{2}{a + \sqrt{a^2 - 2b\,C(x_0)}}} \times \\ \times sn^{-1} \left(\frac{x_0 \sqrt{b}}{a - \sqrt{a^2 - 2b\,C(x_0)}}, k \right) \end{array} \tag{35}$$

is a solution of the Cauchy problem (27).

Denote

$$F(t, a, b, x_0) = \beta_1 x(t, a, b, x_0) + \beta_2 x'_t(t, a, b, x_0), \tag{36}$$

where

$$\begin{array}{c} x'_t(t, a, b, x_0) = \\ = C(x_0) cn \left[\sqrt{\frac{a + \sqrt{a^2 - 2b\,C(x_0)}}{2}}\,(t + \Gamma), k \right] \times \\ \times dn \left[\sqrt{\frac{a + \sqrt{a^2 - 2b\,C(x_0)}}{2}}\,(t + \Gamma), k \right] \end{array} \tag{37}$$

The following statement is true.

Lemma 14 *The initial values x_0 in (27) corresponding to solutions of the Sturm - Liouville boundary value problem (1), (2) are found by solving the equation*

$$F(t, a, b, x_0) = 0 \tag{38}$$

.

Theorem 15 *A solution to the Sturm - Liouville boundary value problem (1), (2) is given by (34), where x_0 is a solution of (38).*

4 Example

Consider equation (1) with $a = 50$, $b = 25$:

$$x'' = -50x + 25x^3, \tag{39}$$

the Sturm -Liouville type conditions are

$$\begin{array}{l} x(0) - x'(0) = 0, \\ x(1) + x'(1) = 0. \end{array} \tag{40}$$

Consider linearized equation of variations with respect to the trivial solution $x \equiv 0$:

$$y'' = -50y, \tag{41}$$

with the initial conditions

$$\begin{aligned} &y(0) - y'(0) = 0, \\ &y(0) = 1. \end{aligned} \tag{42}$$

From equation (25) we have that

$$g(\tau) := \tan\sqrt{50}\tau - \frac{2\sqrt{50}}{49}. \tag{43}$$

Let us look for zeros of $g(\tau)$. For this consider Fig. 2. There exist τ_1, τ_2, τ_3 such that $y(\tau_i) + y'(\tau_i) = 0$.

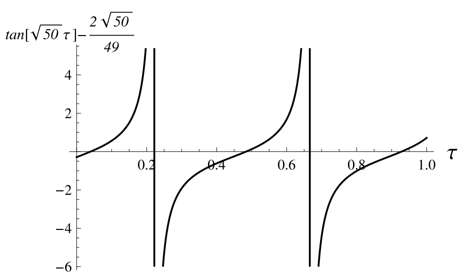

Figure 2: The graph of the function $g(\tau)$ given in (43).

Hence by Theorem 10 we will have 6 nontrivial solutions. We wish to construct three of them with positive x_0. Additional three solutions of BVP can be constructed symmetrically for $x(0) < 0$.
Consider the initial value problem (39),

$$\begin{aligned} &x(0) - x'(0) = 0, \\ &x(0) = x_0 > 0. \end{aligned} \tag{44}$$

We wish to find three initial values of x_0 such that the problem (39), (40) has solution. The graphs of solutions are depicted in Fig. 3, Fig. 5, Fig. 7. The respectively phase trajectories are depicted in Fig. 4, Fig. 6, Fig. 8.

In Fig. 9 depicted the graph of the function $F(t, a, b, x_0) = F(1, 50, 25, x_0)$ (defined in (38)). The zeros correspond to x_0 for three solutions of BVP (39), (40).

Conclusions

We proved that the number of solutions to the Sturm -Liouville problem (1), (2) is defined by the equation of variations $y'' = -a\,y$, Theorem 10. The precise estimate is given. Also equations for determining of the initial values for solutions of the Sturm -Liouville problem (1), (2) are derived.

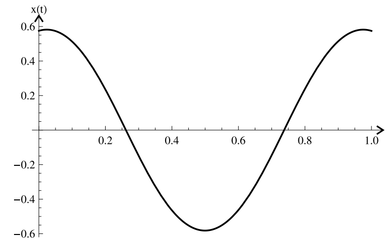

Figure 3: Solution of the problem (39), (40), $x_0 = 0.575$.

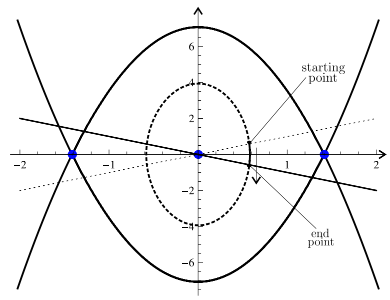

Figure 4: The phase trajectory (dashed) for solution of the problem (39), (40), $x_0 = 0.575$.

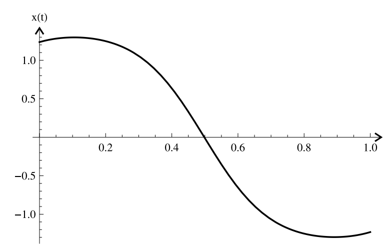

Figure 5: Solution of the problem (39), (40), $x_0 = 1.236$.

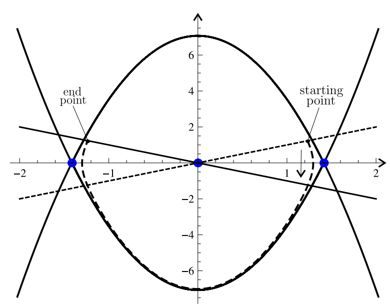

Figure 6: The phase trajectory (dashed) for solution of the problem (39), (40), $x_0 = 1.236$.

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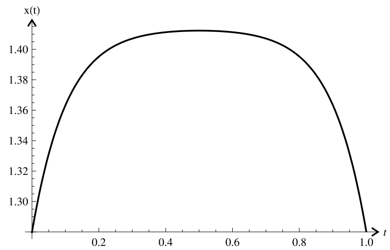

Figure 7: Solution of the problem (39), (40), $x_0 = 1.2798$.

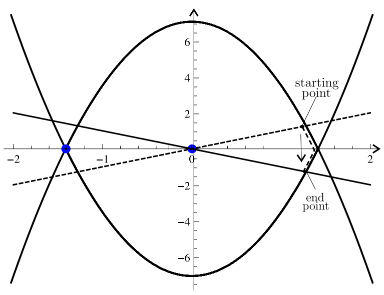

Figure 8: The phase trajectory (dashed) for solution of the problem (39), (40), $x_0 = 1.2798$.

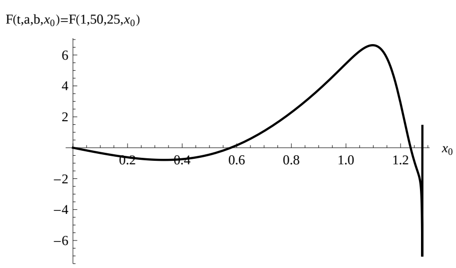

Figure 9: The graph of the function $F(t, a, b, x_0) = F(1, 50, 25, x_0)$ (defined in (38)).

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