On the Logarithmic Derivative of Zeta Functions for Compact, Odd-dimensional Hyperbolic Spaces

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Abstract: In this paper we investigate the Selberg zeta functions and the Ruelle zeta functions associated with locally homogeneous bundles over compact locally symmetric spaces of rank one. Our basic object will be a compact locally symmetric Riemannian manifold with negative sectional curvature. In particular, our research will be restricted to compact, odd-dimensional, real hyperbolic spaces. For this class of spaces, the Titchmarsh-Landau style approximate formulas for the logarithmic derivative of the aforementioned zeta functions are derived. As expected in this setting, the obtained formulas are given in terms of zeros of the attached Selberg zeta functions. Our results follow from the fact that these zeta functions can be represented as quotients of two entire functions of order not larger than the dimension of the underlying compact, odd-dimensional, locally symmetric space, and the application of suitably chosen Weyl asymptotic law. The obtained formulas can be further applied in the proof of the corresponding prime geodesic theorem.

Key–Words: Zeta functions of Selberg and Ruelle, logarithmic derivative, locally symmetric spaces, approximate formulas

1 Introduction and preliminaries

In this paper we derive approximate formulas for the logarithmic derivative of the zeta functions of Selberg and Ruelle described in [3].

As it is well known, such formulas are very well applied in proofs of prime geodesic theorems (in various settings), especially when it comes to the remainder improvement process.

Our object of research will be real hyperbolic spaces.

Let Y be a compact, d-dimensional (d odd, $d \geq 3$), locally symmetric Riemannian manifold with strictly negative sectional curvature.

$Y = \Gamma \setminus G \ / \ K = \Gamma \setminus X$, where G is a connected semi-simple Lie group of real rank one, K is a maximal compact subgroup of G, Γ is a discrete, co-compact, torsion-free subgroup of G, and X is the universal covering of Y.

In our case, X is a real hyperbolic space.

We require G to be linear in order to have the possibility of complexification.

Also, we shall assume that the metric on Y is normalized to be of sectional curvature -1.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$, and M the centralizer of $\mathfrak{a}$ in K.

The adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ determines the root system $\Phi(\mathfrak{g}, \mathfrak{a})$. By W we denote its Weyl group.

Fix a system of positive roots $\Phi^+(\mathfrak{g}, \mathfrak{a}) \subset \Phi(\mathfrak{g}, \mathfrak{a})$.

Define

$$\rho = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})} \dim(\mathfrak{n}_\alpha)\, \alpha,$$

where $\mathfrak{n}_\alpha$ is the root space that corresponds to $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})$.

Since $d \geq 3$ is odd, X is the real hyperbolic space $H\mathbb{R}^d$, where $K = Spin(d)$, $M = Spin(d-1)$ or $K = SO(d)$, $M = SO(d-1)$.

Moreover, $\rho = \frac{d-1}{2}$.

The root system $\Phi^+(\mathfrak{g}, \mathfrak{a})$ is of the form $\Phi^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha\}$ or $\Phi^+(\mathfrak{g}, \mathfrak{a}) = \left\{\frac{\alpha}{2}, \alpha\right\}$, where α is the long root.

Put $T_1 = |\alpha|$.

For $s \in \mathbb{C}$, $\mathrm{Re}(s) > \rho$ resp. $s \in \mathbb{C}$, $\mathrm{Re}(s) > 2\rho$, the Selberg zeta function $Z_{S,\chi}(s, \sigma)$ resp. the Ruelle zeta function $Z_{R,\chi}(s, \sigma)$ is defined by the infinite product given by Definition 3.2 resp. Definition 3.1 in [3, pp. 96-97].

Here σ and χ are some finite-dimensional unitary representations of M and Γ, respectively.

It is well known that the Ruelle zeta function can be represented as a product of the corresponding Selberg zeta functions (see, e.g., [6]).

Namely, there are sets

$$I_p = \left\{ (\tau, \lambda) \,:\, \tau \in \hat{M}, \lambda \in \mathbb{R} \right\}$$

such that

$$Z_{R,\chi}(s, \sigma) =$$
$$\prod_{p=0}^{d-1} \left(\prod_{(\tau,\lambda)\in I_p} Z_{S,\chi}(s + \rho - \lambda, \tau \otimes \sigma) \right)^{(-1)^p}.$$

Fix some $\sigma \in \hat{M}$ and $\chi \in \hat{\Gamma}$.

We simplify our notation by omitting σ and χ.

Hence,

$$Z_R(s) = \prod_{p=0}^{d-1} \left(\prod_{(\tau,\lambda)\in I_p} Z_S(s + \rho - \lambda, \tau) \right)^{(-1)^p}.$$

The Poincare duality

$$I_{d-1-p} = \{ (\tau, 2\rho - \lambda) \,:\, (\tau, \lambda) \in I_p \},$$

$p \in \left\{0, 1, ..., \frac{d-1}{2} - 1\right\}$, applied to the last equality gives us

$$Z_R(s)$$
$$= \prod_{p=0}^{\frac{d-1}{2}-1} \left(\prod_{(\tau,\lambda)\in I_p} Z_S(s + \rho - \lambda, \tau) \cdot \right.$$
$$\left. Z_S(s - \rho + \lambda, \tau) \right)^{(-1)^p} \cdot$$
$$\left(\prod_{(\tau,\lambda)\in I_{\frac{d-1}{2}}} Z_S(s + \rho - \lambda, \tau) \right)^{(-1)^{\frac{d-1}{2}}}.$$

Finally, reasoning as in [4, pp. 40-45], we obtain that

$$Z_R(s)$$
$$= \prod_{p=0}^{\frac{d-1}{2}-1} \left(Z_S\left(s + \frac{d-1}{2} - p, \sigma_p\right) \cdot \right.$$
$$\left. Z_S\left(s - \frac{d-1}{2} + p, \sigma_p\right) \right)^{(-1)^p} \cdot \quad (1)$$
$$\left(Z_S\left(s, \sigma_{\frac{d-1}{2}}\right) \right)^{(-1)^{\frac{d-1}{2}}}.$$

Here, σ_p, $p \in \left\{0, 1, ..., \frac{d-1}{2}\right\}$ is the p-th exterior power of the standard representation of $SO(d-1)$.

σ_p is irreducible unless $p = \frac{d-1}{2}$.

If $p = \frac{d-1}{2}$, then, there exists a splitting $\sigma_{\frac{d-1}{2}} = \sigma^+ \oplus \sigma^-$ into two irreducible components σ^+ and σ^-.

Since the metric on Y is normalized to be of sectional curvature -1, it follows that $T_1 = 1$ (see, [3, p. 150]).

By [3, p. 150, Prop. 5.5], the Selberg zeta function $Z_S(s, \sigma_p)$, $p \in \left\{0, 1, ..., \frac{d-1}{2}\right\}$, has the following singularities:

- a zero at $0 \neq s = \mathrm{i}\,\lambda \in \mathrm{i}\,\mathbb{R} \cup \left(-\frac{d-1-2p}{2}, \frac{d-1-2p}{2}\right)$ of order
$$\dim\Big\{\Delta_p\omega =$$
$$\Big(\lambda^2 + \Big(\frac{d-1-2p}{2}\Big)^2\Big)\omega,\ \delta\omega = 0\Big\},$$
- if $p \neq \frac{d-1}{2}$, a zero at $s = 0$ of order
$$2\dim\Big\{\Delta_p\omega = \Big(\frac{d-1-2p}{2}\Big)^2\omega,\ \delta\omega = 0\Big\},$$
- a singularity at $s = \frac{d-1-2p}{2}$ of order
$$\sum_{k=0}^{p} (-1)^{p-k}\, b_k,$$
- a singularity at $s = -\frac{d-1-2p}{2}$ of order
$$\sum_{k=0}^{p} (-1)^{p-k}\, b_k.$$

If $p = \frac{d-1}{2}$, the latter two singularities coincide, and the orders add up.

Here, Δ_p is the form Laplacian on Y, δ is the codifferential, and b_k is the k-th Betti number of Y.

2 Main Result

Theorem 1. *Let $\varepsilon > 0$ and $d - 1 \geq \eta > 0$.*

(a) Let $p \in \left\{0, 1, ..., \frac{d-1}{2}\right\}$.

(i) Suppose $t \gg 0$ is chosen so that $\frac{d-1}{2} - p + \mathrm{i}\,t$ is not a zero of $Z_S\left(s - \frac{d-1}{2} + p, \sigma_p\right)$. Then,

$$\frac{Z_S'\left(s - \frac{d-1}{2} + p, \sigma_p\right)}{Z_S\left(s - \frac{d-1}{2} + p, \sigma_p\right)}$$
$$= O\left(t^{d-1+\varepsilon}\right) + \sum_{|t - \gamma_{S,p}| \leq 1} \frac{1}{s - \rho_{S,p}}$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}-p\leq\sigma^1<\frac{1}{4}t+\frac{d-1}{2}-p$, *where* $\rho_{S,p}=\frac{d-1}{2}-p+\mathrm{i}\,\gamma_{S,p}$ *is a zero of* $Z_S\left(s-\frac{d-1}{2}+p,\sigma_p\right)$ *on the line* $\mathrm{Re}\,(s)=\frac{d-1}{2}-p$.

(ii) Suppose $t\gg 0$ *is chosen so that* $-\frac{d-1}{2}+p+\mathrm{i}\,t$ *is not a zero of* $Z_S\left(s+\frac{d-1}{2}-p,\sigma_p\right)$. *Then,*

$$\frac{Z_S'\left(s+\frac{d-1}{2}-p,\sigma_p\right)}{Z_S\left(s+\frac{d-1}{2}-p,\sigma_p\right)}=O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,p}\right|\leq 1}\frac{1}{s-\rho_{S,p}}$$

for $s=\sigma^1+\mathrm{i}\,t$, $-\frac{d-1}{2}+p\leq\sigma^1<\frac{1}{4}t-\frac{d-1}{2}+p$, *where* $\rho_{S,p}=-\frac{d-1}{2}+p+\mathrm{i}\,\gamma_{S,p}$ *is a zero of* $Z_S\left(s+\frac{d-1}{2}-p,\sigma_p\right)$ *on the line* $\mathrm{Re}\,(s)=-\frac{d-1}{2}+p$.

(iii) Suppose $t\gg 0$ *is chosen so that* $\frac{d-1}{2}-p+\mathrm{i}\,t$ *is not a zero of* $Z_S\left(s-\frac{d-1}{2}+p,\sigma_p\right)$. *Then,*

$$\frac{Z_S'\left(s-\frac{d-1}{2}+p,\sigma_p\right)}{Z_S\left(s-\frac{d-1}{2}+p,\sigma_p\right)}=O\left(\frac{1}{\eta}t^{d-1+\varepsilon}\right)$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}-p+\eta\leq\sigma^1<\frac{1}{4}t+\frac{d-1}{2}-p$.

(iv) Suppose $t\gg 0$ *is chosen so that* $-\frac{d-1}{2}+p+\mathrm{i}\,t$ *is not a zero of* $Z_S\left(s+\frac{d-1}{2}-p,\sigma_p\right)$. *Then,*

$$\frac{Z_S'\left(s+\frac{d-1}{2}-p,\sigma_p\right)}{Z_S\left(s+\frac{d-1}{2}-p,\sigma_p\right)}=O\left(\frac{1}{\eta}t^{d-1+\varepsilon}\right)$$

for $s=\sigma^1+\mathrm{i}\,t$, $-\frac{d-1}{2}+p+\eta\leq\sigma^1<\frac{1}{4}t-\frac{d-1}{2}+p$.

(b) Suppose $t\gg 0$ *is chosen so that* $\mathrm{i}\,t$ *is not a zero of* $Z_S\left(s,\sigma_p\right)$, $p\in\left\{0,1,...,\frac{d-1}{2}\right\}$. *Then,*

(i)

$$\frac{Z_R'(s)}{Z_R(s)}=O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,0}\right|\leq 1}\frac{1}{s-\rho_{S,0}}$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\leq\sigma^1<\frac{1}{4}t-\frac{d-1}{2}$.

(ii)

$$\frac{Z_R'(s)}{Z_R(s)}=O\left(\frac{1}{\eta}t^{d-1+\varepsilon}\right)$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}+\eta\leq\sigma^1<\frac{1}{4}t-\frac{d-1}{2}$.

Proof. (a) (i), (ii)

For the sake of simplicity, we shall consider the function

$$Z_S\left(s-\frac{d-1}{2},\sigma_0\right)$$

in the representation (1).

Let $r=\frac{1}{2}t$. We choose c such that

$$d-1<c<\frac{1}{8}t+\frac{d-1}{2},$$

and put $s_0=c+\mathrm{i}\,t$.

Since

$$\frac{1}{8}t+\frac{d-1}{2}-\frac{d-1}{2}=\frac{1}{8}t,$$

it follows immediately that the circles $|s-s_0|\leq\frac{1}{2}t$, $|s-s_0|\leq\frac{1}{4}t$ and $|s-s_0|\leq\frac{1}{8}t$ cross the line $\mathrm{Re}\,(s)=\frac{d-1}{2}$.

According to the singularity pattern given above, $Z_S\left(s-\frac{d-1}{2},\sigma_0\right)$ has no poles in the circle $|s-s_0|\leq\frac{1}{2}t$. This means that $Z_S\left(s-\frac{d-1}{2},\sigma_0\right)$ is regular in $|s-s_0|\leq\frac{1}{2}t$.

By [1, p. 306, Th. 2], there exist entire functions $Z_1(s)$, $Z_2(s)$ of order at most d, such that

$$Z_S(s,\sigma_0)=\frac{Z_1(s)}{Z_2(s)},$$

where the zeros of $Z_1(s)$ correspond to the zeros of $Z_S(s,\sigma_0)$, and the zeros of $Z_2(s)$ correspond to the poles of $Z_S(s,\sigma_0)$. The orders of the zeros of $Z_1(s)$ resp. $Z_2(s)$ equal the orders of the corresponding zeros resp. poles of $Z_S(s,\sigma_0)$.

According to the very definition of the order of a function, d is the infimum of numbers ω such that

$$|Z_1(s)|=O\left(e^{|s|^{\omega}}\right),$$

$s\to\infty$.

So, we have that

$$|Z_1(s)|=O\left(e^{|s|^{d+\varepsilon}}\right),$$

$s\to\infty$, where $\varepsilon>0$ is fixed at the beginning of Theorem.

Having in mind that $s \to \infty$ if and only if $s - \frac{d-1}{2} \to \infty$, we obtain that (substituting $s - \frac{d-1}{2}$ instead of s)

$$\left| Z_1 \left(s - \frac{d-1}{2} \right) \right| = O \left(e^{\left| s - \frac{d-1}{2} \right|^{d+\varepsilon}} \right),$$

$s - \frac{d-1}{2} \to \infty$ ($s \to \infty$).

Hence,

$$\left| Z_1 \left(s - \frac{d-1}{2} \right) \right| \leq Q_1 e^{\left| s - \frac{d-1}{2} \right|^{d+\varepsilon}},$$

$s \to \infty$, where Q_1 is a constant. Now,

$$\left| s - \frac{d-1}{2} \right| \leq |s| + \frac{d-1}{2} \leq |s| + Q_2 |s| = Q_3 |s|$$

for some constants Q_2 and Q_3. Namely, $|s|$ is arbitrarily large, and $\frac{d-1}{2}$ is a constant, so $\frac{d-1}{2} \leq Q_2 |s|$.

We obtain,

$$\left| Z_1 \left(s - \frac{d-1}{2} \right) \right| \leq Q_1 e^{Q_3^{d+\varepsilon} |s|^{d+\varepsilon}},$$

$s \to \infty$. Since $Q_1 \leq e^{Q_4 |s|^{d+\varepsilon}}$ for a constant Q_4, it follows that

$$\left| Z_1 \left(s - \frac{d-1}{2} \right) \right| \leq e^{Q_5 |s|^{d+\varepsilon}}, \tag{2}$$

$s \to \infty$, where Q_5 is some constant.

Let's pay our attention to the half-strip $c - \frac{1}{2}t \leq \sigma^1 \leq c + \frac{1}{2}t$, $t^1 \geq \alpha$, where $\alpha > 0$ is large, and $s = \sigma^1 + \mathrm{i}\, t^1$. We may assume that $\alpha \ll \frac{1}{2}t$.

So, notice that $\sigma^1 = \mathrm{Re}\,(s)$, $t^1 = \mathrm{Im}\,(s)$. For such $s = \sigma^1 + \mathrm{i}\, t^1$, (2) yields that (note that $|s|$ is large now)

$$\left| Z_1 \left(s - \frac{d-1}{2} \right) \right| \leq e^{Q_5 \left| \sigma^1 + \mathrm{i}\, t^1 \right|^{d+\varepsilon}}.$$

We have, $\left| \sigma^1 + \mathrm{i}\, t^1 \right| \leq \left| \sigma^1 \right| + t^1$.

Clearly, $\left| c - \frac{1}{2}t \right| \leq c + \frac{1}{2}t$.

This means that the lower bound of the half-strip $c - \frac{1}{2}t \leq \sigma^1 \leq c + \frac{1}{2}t$, $t^1 \geq \alpha$ is not larger than the upper bound (taken in absolute sense). Note that the absolute value of the upper bound is exactly $c + \frac{1}{2}t$. Namely,

$$d - 1 < c < \frac{1}{8}t + \frac{d-1}{2},$$

i.e., c is positive.

Consequently, it follows that $\left| \sigma^1 \right| \leq c + \frac{1}{2}t$.

Now,

$$\begin{aligned} \left| \sigma^1 + \mathrm{i}\, t^1 \right| &\leq \left| \sigma^1 \right| + t^1 \leq c + \frac{1}{2}t + t^1 \\ &< \frac{1}{8}t + \frac{d-1}{2} + \frac{1}{2}t + t^1 \\ &= Q_6 t + Q_7 + t^1. \end{aligned}$$

Clearly, $Q_7 \leq Q_8 t$ for some Q_8. Namely, t is large.

We have,

$$\left| \sigma^1 + \mathrm{i}\, t^1 \right| \leq Q_6 t + Q_8 t + t^1 = Q_9 t + t^1$$

for some constant Q_9.

Furthermore, Q_9, $1 \leq Q_{10}$ for a Q_{10} ($Q_{10} = \max(Q_9, 1)$ for example). Therefore,

$$\begin{aligned} \left| \sigma^1 + \mathrm{i}\, t^1 \right| &\leq Q_{10} t + Q_{10} t^1 = Q_{10} \left(t + t^1 \right) \\ &= Q_{10} \left(t + \mathrm{Im}\,(s) \right). \end{aligned}$$

We conclude,

$$\left| \sigma^1 + \mathrm{i}\, t^1 \right|^{d+\varepsilon} \leq Q_{10}^{d+\varepsilon} \left(t + \mathrm{Im}\,(s) \right)^{d+\varepsilon},$$

i.e.,

$$\left| Z_1 \left(s - \frac{d-1}{2} \right) \right| \leq e^{Q_5 Q_{10}^{d+\varepsilon} (t + \mathrm{Im}(s))^{d+\varepsilon}},$$

i.e.,

$$\left| Z_1 \left(s - \frac{d-1}{2} \right) \right| = e^{O\left((t + \mathrm{Im}(s))^{d+\varepsilon} \right)},$$

for $s = \sigma^1 + \mathrm{i}\, t^1$, $c - \frac{1}{2}t \leq \sigma^1 \leq c + \frac{1}{2}t$, $t^1 \geq \alpha$.

Reasoning in the same way, we obtain that

$$\left| Z_2 \left(s - \frac{d-1}{2} \right) \right| = e^{O\left((t + \mathrm{Im}(s))^{d+\varepsilon} \right)},$$

for $s = \sigma^1 + \mathrm{i}\, t^1$, $c - \frac{1}{2}t \leq \sigma^1 \leq c + \frac{1}{2}t$, $t^1 \geq \alpha$.

Therefore,

$$\begin{aligned} &\left| Z_S \left(s - \frac{d-1}{2}, \sigma_0 \right) \right| \\ &= \frac{\left| Z_1 \left(s - \frac{d-1}{2} \right) \right|}{\left| Z_2 \left(s - \frac{d-1}{2} \right) \right|} \\ &= e^{O\left((t + \mathrm{Im}(s))^{d+\varepsilon} \right) - O\left((t + \mathrm{Im}(s))^{d+\varepsilon} \right)} \\ &= e^{O\left((t + \mathrm{Im}(s))^{d+\varepsilon} \right)}, \end{aligned}$$

for $s = \sigma^1 + \mathrm{i}\, t^1$, $c - \frac{1}{2}t \leq \sigma^1 \leq c + \frac{1}{2}t$, $t^1 \geq \alpha$.

Hence,

$$\left| Z_S \left(s - \frac{d-1}{2}, \sigma_0 \right) \right| = e^{O\left((t + \mathrm{Im}(s))^{d+\varepsilon} \right)},$$

for $s=\sigma^1+\mathrm{i}\,t^1$, $|s-s_0|\le\frac{1}{2}t$.

In particular,

$$\left|Z_S\left(s_0-\frac{d-1}{2},\sigma_0\right)\right|=e^{O\left((t+t)^{d+\varepsilon}\right)}=e^{O\left(t^{d+\varepsilon}\right)}.$$

Note that $t^1\le\frac{3}{2}t$ for $s=\sigma^1+\mathrm{i}\,t^1$, $|s-s_0|\le\frac{1}{2}t$. Consequently,

$$\left|\frac{Z_S\left(s-\frac{d-1}{2},\sigma_0\right)}{Z_S\left(s_0-\frac{d-1}{2},\sigma_0\right)}\right|=e^{O\left(t^{d+\varepsilon}\right)}.$$

for $s=\sigma^1+\mathrm{i}\,t^1$, $|s-s_0|\le\frac{1}{2}t$.

Thus, there exists a constant C such that

$$\left|\frac{Z_S\left(s-\frac{d-1}{2},\sigma_0\right)}{Z_S\left(s_0-\frac{d-1}{2},\sigma_0\right)}\right|<e^{Ct^{d+\varepsilon}}$$

for $s=\sigma^1+\mathrm{i}\,t^1$, $|s-s_0|\le\frac{1}{2}t$.

Put $M=Ct^{d+\varepsilon}$.

Since $Z_S\left(s-\frac{d-1}{2},\sigma_0\right)$ is regular in the circle $|s-s_0|\le\frac{1}{2}t$, and

$$\left|\frac{Z_S\left(s-\frac{d-1}{2},\sigma_0\right)}{Z_S\left(s_0-\frac{d-1}{2},\sigma_0\right)}\right|<e^{M}$$

for $s=\sigma^1+\mathrm{i}\,t^1$, $|s-s_0|\le\frac{1}{2}t$, it follows by Lemma α in [11] that

$$\frac{Z_S'\left(s-\frac{d-1}{2},\sigma_0\right)}{Z_S\left(s-\frac{d-1}{2},\sigma_0\right)}=O\left(t^{d-1+\varepsilon}\right)+\sum_{\rho_{S,0}\in P}\frac{1}{s-\rho_{S,0}}$$

for $s=\sigma^1+\mathrm{i}\,t^1$, $|s-s_0|\le\frac{1}{8}t$, where P denotes the set of zeros of $Z_S\left(s-\frac{d-1}{2},\sigma_0\right)$ lying in the circle $|s-s_0|\le\frac{1}{4}t$.

Since $s=\sigma^1+\mathrm{i}\,t^1$, $|s-s_0|\le\frac{1}{8}t$, and the circle $|s-s_0|\le\frac{1}{8}t$ crosses the line $\mathrm{Re}\,(s)=\frac{d-1}{2}$, it follows that the last equality remains valid for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1\le c+\frac{1}{8}t$. Thus,

$$\frac{Z_S'\left(s-\frac{d-1}{2},\sigma_0\right)}{Z_S\left(s-\frac{d-1}{2},\sigma_0\right)}=O\left(t^{d-1+\varepsilon}\right)+\sum_{\rho_{S,0}\in P}\frac{1}{s-\rho_{S,0}} \qquad (3)$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1\le c+\frac{1}{8}t$.

As we already noted, $\rho_{S,0}=\frac{d-1}{2}+\mathrm{i}\,\gamma_{S,0}$.

Now,

$$|\rho_{S,0}-s_0|\le\frac{1}{4}t$$

if and only if

$$t-\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}\le\gamma_{S,0}\le t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}.$$

Since

$$d-1<c<\frac{1}{8}t+\frac{d-1}{2},$$

it follows that

$$\sqrt{\frac{1}{16}t^2-\left(\frac{d-1}{2}\right)^2}>\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}>\frac{\sqrt{3}}{8}t.$$

Thus, $t\gg 0$ implies that

$$\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}>1.$$

The equation (3) becomes,

$$\begin{aligned}&\frac{Z_S'\left(s-\frac{d-1}{2},\sigma_0\right)}{Z_S\left(s-\frac{d-1}{2},\sigma_0\right)}\\&=O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,0}\right|\le 1}\frac{1}{s-\rho_{S,0}}+\\&\sum_{t+1<\gamma_{S,0}\le t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}\frac{1}{s-\rho_{S,0}}+\\&\sum_{t-\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}\le\gamma_{S,0}<t-1}\frac{1}{s-\rho_{S,0}}\end{aligned}$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1\le c+\frac{1}{8}t$.

Note that $|s-\rho_{S,0}|\ge\gamma_{S,0}-t$ for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1\le c+\frac{1}{8}t$, where $\rho_{S,0}=\frac{d-1}{2}+\mathrm{i}\,\gamma_{S,0}\in P$ is such that

$$t+1<\gamma_{S,0}\le t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}.$$

Let $N_{S,0}\left(y\right)$ be the number of zeros $\rho_{S,0}=\frac{d-1}{2}+\mathrm{i}\,\gamma_{S,0}$ of $Z_S\left(s-\frac{d-1}{2},\sigma_0\right)$ on the interval $\frac{d-1}{2}+\mathrm{i}\,x$, $0<x\le y$.

As it is known (see, e.g., [5, p. 89, Th. 9.1.]),

$$N_{S,0}(y)=C_1y^d+O\left(y^{d-1}(\log y)^{-1}\right)$$

for some explicitly known constant C_1.

In this paper, however, the factor $(\log y)^{-1}$ does not improve the result. Therefore, we assume that

$$N_{S,0}(y)=C_1y^d+O\left(y^{d-1}\right).$$

Now, we estimate

$$\left|\sum_{t+1<\gamma_{S,0}\le t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}\frac{1}{s-\rho_{S,0}}\right|$$
$$\le\sum_{t+1<\gamma_{S,0}\le t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}\frac{1}{|s-\rho_{S,0}|}$$
$$\le\sum_{t+1<\gamma_{S,0}\le t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}\frac{1}{\gamma_{S,0}-t}$$
$$=\int_{t+1}^{t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}\frac{dN_{S,0}(y)}{y-t}$$
$$=\int_{t+1}^{t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}\frac{d\left(N_{S,0}(y)-C_1t^d\right)}{y-t}$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1\le c+\frac{1}{8}t$.

Integration by parts applied to the last integral gives us

$$\int_{t+1}^{t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}\frac{d\left(N_{S,0}(y)-C_1t^d\right)}{y-t}$$
$$=\frac{N_{S,0}(y)-C_1t^d}{y-t}\Bigg|_{t+1}^{t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}+$$
$$\int_{t+1}^{t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}\frac{N_{S,0}(y)-C_1t^d}{(y-t)^2}dy.$$

Note that

$$\frac{N_{S,0}(y)-C_1t^d}{y-t}\Bigg|_{t+1}^{t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}=$$

$$\frac{N_{S,0}\left(t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}\right)-C_1t^d}{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}$$
$$-N_{S,0}(t+1)+C_1t^d$$
$$=\frac{C_1\left(t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}\right)^d-C_1t^d}{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}$$
$$+\frac{O\left(\left(t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}\right)^{d-1}\right)}{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}$$
$$-C_1(t+1)^d+C_1t^d.$$

Obviously,

$$\frac{C_1\left(t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}\right)^d-C_1t^d}{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}$$
$$=O\left(t^{d-1}\right),$$
$$\frac{O\left(\left(t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}\right)^{d-1}\right)}{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}$$
$$=O\left(t^{d-2}\right),$$
$$-C_1(t+1)^d+C_1t^d$$
$$=O\left(t^{d-1}\right).$$

Therefore,

$$\int_{t+1}^{t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}\frac{d\left(N_{S,0}(y)-C_1t^d\right)}{y-t}$$
$$=O\left(t^{d-1}\right)+$$
$$\int_{t+1}^{t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}\frac{N_{S,0}(y)-C_1t^d}{(y-t)^2}dy.$$

Putting $y-t=v$,we obtain that

$$\int_{t+1}^{t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}}\frac{d\left(N_{S,0}(y)-C_1t^d\right)}{y-t}$$
$$=O\left(t^{d-1}\right)+$$

$$\int_1^{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} \frac{N_{S,0}(v+t)-C_1t^d}{v^2}dv$$
$$=O\left(t^{d-1}\right)+$$
$$\int_1^{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} \frac{C_1(v+t)^d-C_1t^d}{v^2}dv+$$
$$\int_1^{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} \frac{O\left((v+t)^{d-1}\right)}{v^2}dv.$$

Now,

$$\int_1^{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} \frac{C_1(v+t)^d-C_1t^d}{v^2}dv$$
$$=\int_1^{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} C_1v^{d-2}dv+$$
$$\int_1^{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} C_1'v^{d-3}tdv+...$$
$$+\int_1^{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} C_{d-3}'vt^{d-3}dv+$$
$$\int_1^{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} C_{d-2}'t^{d-2}dv+$$
$$\int_1^{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} C_{d-1}'\frac{t^{d-1}}{v}dv$$
$$=O\left(t^{d-1}\right)+O\left(t^{d-1}\right)+...+O\left(t^{d-1}\right)$$
$$+O\left(t^{d-1}\right)+O\left(t^{d-1}\log t\right)=O\left(t^{d-1}\log t\right)$$

for explicitly known constants C_1', C_2',..., C_{d-1}'.

Similarly,

$$\int_1^{\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} \frac{O\left((v+t)^{d-1}\right)}{v^2}dv$$
$$=O\left(t^{d-2}\log t\right).$$

Therefore,

$$\int_{t+1}^{t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} \frac{d\left(N_{S,0}(y)-C_1t^d\right)}{y-t}$$
$$=O\left(t^{d-1}\right)+O\left(t^{d-1}\log t\right)+O\left(t^{d-2}\log t\right)$$
$$=O\left(t^{d-1}\log t\right).$$

Consequently,

$$\sum_{t+1<\gamma_{S,0}\le t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}} \frac{1}{s-\rho_{S,0}}$$
$$=O\left(t^{d-1}\log t\right)$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1\le c+\frac{1}{8}t$.

Reasoning in the same way, we obtain that

$$\sum_{t-\sqrt{\frac{1}{16}t^2-\left(c-\frac{d-1}{2}\right)^2}\le\gamma_{S,0}<t+1} \frac{1}{s-\rho_{S,0}}$$
$$=O\left(t^{d-1}\log t\right)$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1\le c+\frac{1}{8}t$.

We end up with

$$\frac{Z_S'\left(s-\frac{d-1}{2},\sigma_0\right)}{Z_S\left(s-\frac{d-1}{2},\sigma_0\right)}$$
$$=O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,0}\right|\le1}\frac{1}{s-\rho_{S,0}}+$$
$$O\left(t^{d-1}\log t\right)+O\left(t^{d-1}\log t\right)$$
$$=O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,0}\right|\le1}\frac{1}{s-\rho_{S,0}}$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1\le c+\frac{1}{8}t$.

However, $c<\frac{1}{8}t+\frac{d-1}{2}$. Hence,

$$\frac{Z_S'\left(s-\frac{d-1}{2},\sigma_0\right)}{Z_S\left(s-\frac{d-1}{2},\sigma_0\right)}$$
$$=O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,0}\right|\le1}\frac{1}{s-\rho_{S,0}}$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1<\frac{1}{4}t+\frac{d-1}{2}$.

(iii), *(iv)*

Once again, for the sake of simplicity, we shall consider the function

$$Z_S\left(s-\frac{d-1}{2},\sigma_0\right)$$

in the representation (1).

By *(i)*,

$$\frac{Z_S'\left(s-\frac{d-1}{2},\sigma_0\right)}{Z_S\left(s-\frac{d-1}{2},\sigma_0\right)} = O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,0}\right|\le 1}\frac{1}{s-\rho_{S,0}}$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1<\frac{1}{4}t+\frac{d-1}{2}$.

Suppose that $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}+\eta\le\sigma^1<\frac{1}{4}t+\frac{d-1}{2}$.

We obtain,

$$\begin{aligned}&\left|\sum_{\left|t-\gamma_{S,0}\right|\le 1}\frac{1}{s-\rho_{S,0}}\right| \\ &\le\sum_{\left|t-\gamma_{S,0}\right|\le 1}\frac{1}{\left|s-\rho_{S,0}\right|}<\frac{1}{\eta}\sum_{\left|t-\gamma_{S,0}\right|\le 1}1 \\ &=O\left(\frac{1}{\eta}\left(N_{S,0}\left(t+1\right)-N_{S,0}\left(t-1\right)\right)\right) \\ &=O\Bigg(\frac{1}{\eta}\Big(C_1\left(t+1\right)^d+O\left(\left(t+1\right)^{d-1}\right) \\ &\quad -C_1\left(t-1\right)^d-O\left(\left(t-1\right)^{d-1}\right)\Big)\Bigg) \\ &=O\left(\frac{1}{\eta}t^{d-1}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{Z_S'\left(s-\frac{d-1}{2},\sigma_0\right)}{Z_S\left(s-\frac{d-1}{2},\sigma_0\right)} \\ &=O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,0}\right|\le 1}\frac{1}{s-\rho_{S,0}} \\ &=O\left(t^{d-1+\varepsilon}\right)+O\left(\frac{1}{\eta}t^{d-1}\right)=O\left(\frac{1}{\eta}t^{d-1+\varepsilon}\right)\end{aligned}$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}+\eta\le\sigma^1<\frac{1}{4}t+\frac{d-1}{2}$.

(b)*, *(i)

By (1),

$$\begin{aligned}\frac{Z_R'\left(s\right)}{Z_R\left(s\right)}=&\sum_{p=0}^{\frac{d-1}{2}-1}\left(-1\right)^p\frac{Z_S'\left(s+\frac{d-1}{2}-p,\sigma_p\right)}{Z_S\left(s+\frac{d-1}{2}-p,\sigma_p\right)}+ \\ &\sum_{p=0}^{\frac{d-1}{2}-1}\left(-1\right)^p\frac{Z_S'\left(s-\frac{d-1}{2}+p,\sigma_p\right)}{Z_S\left(s-\frac{d-1}{2}+p,\sigma_p\right)}+ \\ &\left(-1\right)^{\frac{d-1}{2}}\frac{Z_S'\left(s,\sigma_{\frac{d-1}{2}}\right)}{Z_S\left(s,\sigma_{\frac{d-1}{2}}\right)} \\ =&\sum_{p=0}^{\frac{d-1}{2}-1}\left(-1\right)^p\frac{Z_S'\left(s+\frac{d-1}{2}-p,\sigma_p\right)}{Z_S\left(s+\frac{d-1}{2}-p,\sigma_p\right)}+ \\ &\frac{Z_S'\left(s-\frac{d-1}{2},\sigma_0\right)}{Z_S\left(s-\frac{d-1}{2},\sigma_0\right)}+ \\ &\sum_{p=1}^{\frac{d-1}{2}-1}\left(-1\right)^p\frac{Z_S'\left(s-\frac{d-1}{2}+p,\sigma_p\right)}{Z_S\left(s-\frac{d-1}{2}+p,\sigma_p\right)}+ \\ &\left(-1\right)^{\frac{d-1}{2}}\frac{Z_S'\left(s,\sigma_{\frac{d-1}{2}}\right)}{Z_S\left(s,\sigma_{\frac{d-1}{2}}\right)}.\end{aligned}$$

Since $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1<\frac{1}{4}t-\frac{d-1}{2}$, it follows from *(a)* that

$$\begin{aligned}\frac{Z_R'\left(s\right)}{Z_R\left(s\right)}=&\sum_{p=0}^{\frac{d-1}{2}-1}\frac{1}{d-1-p}O\left(t^{d-1+\varepsilon}\right) \\ &+O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,0}\right|\le 1}\frac{1}{s-\rho_{S,0}} \\ &+\sum_{p=1}^{\frac{d-1}{2}-1}\frac{1}{p}O\left(t^{d-1+\varepsilon}\right)+\frac{1}{\frac{d-1}{2}}O\left(t^{d-1+\varepsilon}\right) \\ =&O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,0}\right|\le 1}\frac{1}{s-\rho_{S,0}}.\end{aligned}$$

Thus,

$$\frac{Z_R'\left(s\right)}{Z_R\left(s\right)}=O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,0}\right|\le 1}\frac{1}{s-\rho_{S,0}}$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1<\frac{1}{4}t-\frac{d-1}{2}$.

(ii)

By *(b)* *(i)*,

$$\frac{Z_R'\left(s\right)}{Z_R\left(s\right)}=O\left(t^{d-1+\varepsilon}\right)+\sum_{\left|t-\gamma_{S,0}\right|\le 1}\frac{1}{s-\rho_{S,0}}$$

for $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}\le\sigma^1<\frac{1}{4}t-\frac{d-1}{2}$.

Let $s=\sigma^1+\mathrm{i}\,t$, $\frac{d-1}{2}+\eta\le\sigma^1<\frac{1}{4}t-\frac{d-1}{2}$.

We deduce,

$$\begin{aligned}&\left|\sum_{\left|t-\gamma_{S,0}\right|\le 1}\frac{1}{s-\rho_{S,0}}\right| \\ &\le\sum_{\left|t-\gamma_{S,0}\right|\le 1}\frac{1}{\left|s-\rho_{S,0}\right|}<\frac{1}{\eta}\sum_{\left|t-\gamma_{S,0}\right|\le 1}1 \\ &=O\left(\frac{1}{\eta}t^{d-1}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{Z_R^{'}(s)}{Z_R(s)} &= O\left(t^{d-1+\varepsilon}\right) + \sum_{|t-\gamma_{S,0}|\le 1} \frac{1}{s-\rho_{S,0}} \\ &= O\left(t^{d-1+\varepsilon}\right) + O\left(\frac{1}{\eta}t^{d-1}\right) \\ &= O\left(\frac{1}{\eta}t^{d-1+\varepsilon}\right)\end{aligned}$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{d-1}{2} + \eta \le \sigma^1 < \frac{1}{4}t - \frac{d-1}{2}$.

This completes the proof. □

3 Remarks

Note that an analogue of Theorem 1 is derived in [2]. There, the authors derived approximate formulas for the logarithmic derivative of the Selberg and the Ruelle zeta functions over compact, even-dimensional, locally symmetric spaces of real rank one. Thus, the present paper represents a natural continuation of [2].

Approximate formulas of the form derived in this paper are very well applied in [7], [9], [8] and [10]. There, they are applied in the case of compact even-dimensional locally symmetric Riemannian manifolds of strictly negative sectional curvature, hyperbolic manifolds with cusps, and compact Riemann surfaces, respectively.

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