On Finite Order Nearness in Soft Set Theory

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Abstract: - Soft set theory is a useful mathematical tool to deal with uncertainty in a parametric manner. Near sets have been used as a tool to study extensions of topological spaces. The present paper introduces and studies nearness of finite order, S_n-merotopy, in soft set theory. An S_m-merotopic space (U,ζ_m,E) is introduced for a given S_n-merotopic space (U,ζ,E), where m and n are integers with the restriction that $2\leq m\leq n$. For $m\leq n$ an S_n-merotopy ζ_* from a given S_n-merotopy ζ having the property that $\zeta=(\zeta_*)_m$ is constructed and the largest S_n-merotopy having such property is derived. For an S_n-merotopic space (U,ζ), every maximal $\zeta-$compatible family is a maximal $\zeta-$clan.

Key-Words: - Soft set; Grill operator; Soft Čech closure operator; Proximity spaces; Merotopic spaces.

1 Introduction

Molodtsov [14] initiated a completely new idea of soft sets as an innovative approach to deal with uncertainties and gave the basic results along with its application in various directions. After the inception of the concept of fuzzy sets [20], understanding the concept of uncertainty has got a paradigm shift. Soft set theory was further investigated by Maji, Roy and Biswas [12] and they gave the practical application of soft set in its ability to solve decision making problems.

The combination of fuzzy set and soft sets is studied by Maji, Biswas and Roy [13]. In comparison to fuzzy set, soft set theory can be seen to be free from inability of parameterization. Feng et al. [7] gave the concept of soft rough set which extended Pawlak's rough set model using soft sets and the notion of soft group is given in 2007 by Aktas & Cagman [1]. Feng et al. [6] defined soft semi rings. That a fuzzy topological space is a special case of a soft topological space is shown in [21], however the converse need not be true. Classical mathematical approaches are sometimes insufficient to provide effective or applicable models. Soft set theory has been the most recent topic in many areas like algebra, topology, etc. Babita and Sunil [2, 3] defined soft set relation function and ordering on soft sets and also investigated compatible soft set functions. The lattice structures of soft sets are constructed by Keyun Qin and Hong [10].

The main purpose of the present paper is to introduce and study the notion of S_n-merotopic spaces in soft set theory. We define nearness of finite order in soft set theory which includes basic soft proximities [17]. The concept of nearness is given by Herrlich [8] in 1974. Chattopadhyay and Njåstad [5] gave the concept of Riesz merotopic spaces by using a weaker form of the condition (N5) given by Herrlich. Bentley [4] studied binary nearness spaces and a complete theory of nearness of two sets (proximity) is studied in [15]. Nearness of finite order leads to contiguity as a limiting case in soft set theory. We first define nearness of finite order which is an obvious generalization of soft proximities. Following Ward [19] the concept of F_n-merotopic space in fuzzy setting is given in [11]. Generalization of the concept of nearness of two sets and contiguity may be seen in [11,16, 18]. An S_n-merotopic space is a soft Čech closure spaces and S_n-merotopy is maximal clan generated. For the integers m and n with $2\leq m\leq n$ and a given S_n-merotopic space (U,ζ,E), an S_m-merotopic space (U,ζ_m,E) is introduced. We obtained an S_n-merotopy ζ_* from a given S_m-merotopy ζ such that $\zeta=(\zeta_*)_m$, where $m\leq n$. And hence the largest S_n-merotopy exhibiting such property is constructed. For a given soft Čech closure space (U,c) an S_n-merotopy ζ^c is obtained. Under a suitable condition, it is proved that an S_n-merotopy ζ is a subset of ζ^c.

2 Preliminaries

2.1 Definition

[14] Let U be an initial universe and $P(U)$ be the power set of U. Let E be the set of parameters. Then

a pair (F, E) is called a soft set over U, F is a mapping given by $F : E \rightarrow P(U)$.

2.1.1 Remark
We use the symbol $S(U, E)$ for the collection of all soft set over U. For brevity, the symbol "$\sqcup(\sqcap)$" will be used for union (intersection) of two soft sets, the symbol "$\Subset$" will be used for "does not belong to" throughout the paper.

2.2 Definition
For all $e \in E$.
If $F(e) = \emptyset$ then the soft set F is called null soft set, denoted by Φ. F is called absolute soft set, if $F(e) = U$, denoted by $\widetilde{U}$. If $F(e) \subseteq G(e)$ then F is a soft subset of G, denoted by $F \sqsubseteq G$. If $F \sqsubseteq G$ and $G \sqsubseteq F$, then the two soft sets F and G are said to be equal, denoted by $F = G$.The compliment of a soft set F is defined by $F^c(e) = U - F(e)$ and denoted by F^c. Here F^c is a map from the set of parameters to power set of U. The union K of two soft set F and G is defined by, $K(e) = F(e) \cup G(e)$, denoted by $F \sqcup G$. The intersection H of two soft set F and G is defined by $H(e) = F(e) \cap G(e)$, denoted by $F \sqcap G$.

2.3 Definition [21] The soft set (F, E) over U is called a soft point if there exists $x \in U$ and $e \in E$ such that $P(e) = \{x\}$ and $P(e^{'}) = \emptyset$ for every $e^{'} \in E\backslash\{e\}$. A soft point is denoted by x^e. Family of all soft set over U is denoted by $SP(U)$. If $x \in F(e)$, then a soft point x^e is said to a soft set F, denoted by $x^e \widetilde{\in} F$. Two soft point $x_1^{e_1}, x_2^{e_2}$ is said to be equal if $e_1 = e_2$ and $x_1 = x_2$, unequal if $e_1 \neq e_2$ and $x_1 \neq x_2$.

2.4 Definition
A sub-collection G of $S(U, E)$ is called soft grill on U if G satisfies: $\Phi \notin G$; $B \in G$ and $B \sqsubseteq C$, then $C \in G$; $B \sqcup C \in G$ implies that $B \in G$ or $C \in G$. The collection of all soft grill on U will be denoted by $\Gamma(S, U)$.

2.5 Definition [9]
A relation δ which is binary relation on $S(U, E)$ is called a soft proximity on U if for any $A, B, C \in S(U, E)$, the following conditions are satisfied: (i) $\Phi \overline{\delta} B$ (ii) if $B \sqcap C \neq \Phi$ then $B\delta C$, (iii) if $B\delta C$ then $C\delta B$, (iv) $B\delta\,(C \sqcup D)$ if and only if $B\delta C$ or $B\delta D$.
A triple (U, δ, E) is a proximity space consisting of set U, a set of parameter E and a proximity relation on $S(U, E)$. If $(A, B) \in (S, E)$, are δ −related then we shall write $(A, B) \in \delta$, otherwise $(A, B) \notin \delta$.

2.6 Definition
[19] Let U be set and n is an integer such that $n \geq 2$. Let $\bar{A}, \bar{B} \in P^2(U)$ and $\bar{\eta} \subseteq P^2(U)$**.** Then $(U, \bar{\eta})$ is called an $n -$nearness space, if the following axioms are satisfied:

a) $\bar{A} \in \bar{\eta}$ implies that $|\bar{A}| \leq n$,
b) for all $A \subseteq U, \{A\} \in \bar{\eta}$ iff $A \neq \Phi$,
c) $|\bar{A}| \leq n$ and $\bar{B}$ is such that $A \in \bar{A}$ implies that there exists $B \in \bar{B}, B \subseteq A$ then $B \in \bar{\eta}$ implies that $A \in \bar{\eta}$,
d) If A, B be a subset of U, $\bar{C} \subseteq P(U)$ and $|\bar{C}| \leq n - 1$ then $\bar{C} \cup \{A \cup B\}) \in \bar{\eta} \Rightarrow \bar{C} \cup \{A\}) \in \bar{\eta}$ or $\bar{C} \cup \{B\}) \in \bar{\eta}$.

3 $S_n -$ Merotopic Space

3.1 Definition
Let $\zeta \subseteq P(S(U, E))$ over U. Then ζ is called an $S_n -$ merotopy on U if the following conditions are satisfied for any integer $n \geq 2$:

a) $\mathcal{A} \in \zeta$ implies $|\mathcal{A}| \leq n$,
b) $\{A\} \in \zeta$ iff $A \neq \Phi$,
c) If $\mathcal{B} \in \zeta$ and $\mathcal{A}$ corefines $\mathcal{B}$ (i.e., for each $A \in \mathcal{A} \exists B \in \mathcal{B}$ such that $A \supseteq B$) with $|\mathcal{A}| \leq n$ then $\mathcal{A} \in \zeta$,
d) If $(\mathcal{C} \cup \{A \sqcup B\}) \in \zeta$ then $(\mathcal{C} \cup \{A\}) \in \zeta$ or $(\mathcal{C} \cup \{B\}) \in \zeta$, where $A, B \in S(U, E), \mathcal{C} \in P(S(U, E))$ with $|\mathcal{C}| \leq n - 1$,

A triplet (U, ζ, E) is said to be soft $S_n -$ merotopic spaces over U.

3.1.1 Remark
Arbitrary union of $S_n -$ merotopies is an $S_n -$ merotopy. For $n = 2$ an $S_n -$ merotopy on U is a soft proximity on U.

3.2 Definition
$sec\,\mathcal{A} = \{B \in S(U, E) : A \sqcap B \neq \Phi \text{ for all } A \in \mathcal{A}\}$

3.2.1 Remark
If $\mathcal{A}$ corefines $\mathcal{B}$ then $sec\,\mathcal{B} \subseteq sec\,\mathcal{A}$.

3.2.2 Example
Let $\zeta = \{\mathcal{A} \in P(S(U, E))\} : |\mathcal{A}| \leq n$ and $\Phi \neq \mathcal{A}\}$ is a soft $S_n -$ merotopy on U. The triple (U, ζ, E) is the largest soft $S_n -$ merotopy.

3.2.3 Example
Let $\zeta = \{\mathcal{A} \subseteq S(U, E) : |\mathcal{A}| \leq n, \mathcal{C} \cap secA \neq \Phi\} \cup \{A \subset S(U, E) : |A| \leq n$ and $\sqcap \mathcal{A} \neq \Phi\}$ is a soft $S_n -$ merotopy on U.

3.2.4 Example

Let $\zeta = \{\mathcal{A} \in P(S(U,E))\}: |\mathcal{A}| \leq n$ and $\sqcap\mathcal{A} \neq \Phi\}$is a soft $S_n -$ merotopy on U. The triple (U,ζ,E) is the smallest soft $S_n -$ merotopy.

3.2.5 Example

Let there be a three machines in the universe set U. $U = (m_1, m_2, m_3)$ and $A = (ergonomics\,(e), cost\,(c))$. The soft set describe the "efficiency of machines". Let G be a collection of those soft sets for which $G_i(c) = (m_1, m_2, m_3)$ for all i and $G = \{G_1,\ G_2,\ G_3,\ G_4, G_5, G_6, U\}$ *i.e.* the cost of all machines are same defined by;

$G_1 = \{(e, m_1), (c, \{m_1, m_2, m_3\})\}$
$G_2 = \{(e, m_2), (c, \{m_1, m_2, m_3\})\}$
$G_3 = \{(e, m_3), (c, \{m_1, m_2, m_3\})\}$
$G_4 = \{(e, \{m_1, m_2\}), (c, \{m_1, m_2, m_3\})\}$
$G_5 = \{(e, \{m_1, m_3\}), (c, \{m_1, m_2, m_3\})\}$
$G_6 = \{(e, \{m_2, m_3\}), (c, \{m_1, m_2, m_3\})\}$

Then G is a soft grill over U.

3.3 Proposition

If p, q are positive integers such that $pq \leq n$ and ζ be an $S_n -$ merotopy on U, then $\{A_r \sqcup B_s: 1 \leq r \leq p,\ 1 \leq s \leq q\} \in \zeta$ iff $\{A_r: 1 \leq r \leq p\} \in \zeta$ or $\{B_s: 1 \leq s \leq q\} \in \zeta$.

Proof. Let $\{A_r: 1 \leq r \leq p\} \in \zeta$ or $\{B_s: 1 \leq s \leq q\} \in \zeta$. Then using 3.1 (c), we get $\{A_r \sqcup B_s: 1 \leq r \leq p,\ 1 \leq s \leq q\} \in \zeta$.

Conversely, let $\{A_r \sqcup B_s: 1 \leq r \leq p,\ 1 \leq s \leq q\} \in \zeta$. Then again using 3.1 (d), we get $\{A_l, B_m: 1 \leq l \leq p,\ 1 \leq m \leq q\} \in \zeta$. Now, by using 3.1 (c), $\{A_r: 1 \leq r \leq p\} \in \zeta$or $\{B_s: 1 \leq s \leq q\} \in \zeta$.

3.4 Proposition

If ζ is an $S_n -$ merotopy on U, then $c_\zeta(B) = \sqcup\{x_e \tilde{\in} \tilde{X} : (x_e, B) \in \zeta\}$, $B \in S(U,E)$ is a soft Čech closure operator on U.

3.5 Definition

(a) If $2 \leq m \leq n$ and ζ is a given $S_n -$ merotopy on U, then ζ_m, the restriction m of ζ, is defined by $\mathcal{A} \in \zeta_m$ iff $\mathcal{A} \in \zeta$ and $|\mathcal{A}| \leq m$. (b) Let (U,ζ,E) be an $S_n -$ mereotopic space and $\mathcal{A} \in \mathrm{P}((S,E))$. If $\mathcal{A} \subseteq \mathcal{B}$ and $|\mathcal{B}| \leq n$ implies $\mathcal{B} \in \zeta$, then $\mathcal{A}$ is said to be $\zeta -$compatible. An $\zeta -$ compatible soft grill is called an $\zeta -$ clan.

3.6 Proposition

If ζ is an $S_n -$ merotopy and $(2 \leq m \leq n)$, then ζ_m is an $S_m -$ merotopy.

3.7 Proposition

Let (U,ζ) be an $S_n -$ merotopic space. Then every maximal $\zeta -$compatible family is a maximal $\zeta -$ clan.

Proof. Let (U,ζ) be an $S_n -$merotopic space. Suppose that $\mathcal{A} \in \mathrm{P}((S,E))$ be a maximal $\zeta -$ compatible family, we will now show that $\mathcal{A}$ is a grill. For this note that $\Phi \notin \mathcal{A}$. Let $A \sqsubseteq B$ and $A \in \mathcal{A}$. Then $\{B\} \cup \mathcal{A}$ corefines $\mathcal{A}$ and so $\{B\} \cup \mathcal{A}$ is a $\zeta -$compatible family. But, since $\mathcal{A}$ is a maximal $\zeta -$compatible family, $B \in \mathcal{A}$. Let $A \sqcup B \in \mathcal{A}$. Further it is our claim that either $\{A\} \cup \mathcal{A}$ or $\{B\} \cup \mathcal{A}$ is a $\zeta -$compatible family. Let $\alpha = \{\mathcal{B} \in P((S,E)): \mathcal{B} \subseteq \mathcal{A}\}$and suppose that the claim be not true. Then there exist $\mathcal{B}_1$ and $\mathcal{B}_2$ in α such that $\{A\} \cup \mathcal{B}_1 \notin \zeta$ and $\{B\} \cup \mathcal{B}_2 \notin \zeta$, it implies that $(\{A\} \cup \mathcal{B}_1) \sqcup (\{B\} \cup \mathcal{B}_2) \notin \zeta$. Since $(\{A\} \cup \mathcal{B}_1) \sqcup (\{B\} \cup \mathcal{B}_2) \notin \zeta$ corefines $(\{A \sqcup B\} \cup \mathcal{B}_1 \cup \mathcal{B}_2)$, we have $(\{A \sqcup B\} \cup \mathcal{B}_1 \cup \mathcal{B}_2) \notin \zeta$. This contradicts that $\mathcal{A}$ is a $\zeta -$compatible family. Hence our claim is true and so by the maximality of $\mathcal{A}$, either $A \in \mathcal{A}$ or $B \in \mathcal{A}$.

3.8 Proposition

Let (U,ζ) be an $S_n -$merotopic space. Then every $\zeta -$compatible soft grill is a subset of a maximal $\zeta -$compatible soft grill.

Proof. By Zorn's lemma the proof follows.

3.9 Definition

A family $\zeta \subseteq \mathrm{P}((S,E))$ is said to be *clan* (*soft*) generated if ζ satisfies: $\mathcal{A} \in \zeta$ implies that there exists a $\zeta -$ clan G on U such that $\mathcal{A} \subseteq G$.

3.10 Theorem

Every $S_n -$merotopy ζ is maximal clan (soft) generated.

Proof. Since every ζ-compatible family is a subset of a maximal ζ-clan (proposition 3.6, 3.7), the result follows.

3.11 Theorem

(a) Let c be a soft Čech closure operator on U. Define ζ^c by $\mathcal{A} \in \zeta^c$ iff $|\mathcal{A}| \leq n$ and $\sqcap\{c(A): A \in \mathcal{A}\} = \Phi, \mathcal{A} \in P((S,E))$. Then ζ^c is an $S_n -$ merotopy on U. Moreover, $(\zeta_m)^c = (\zeta^c)_m$, for $2 \leq m \leq n$. (b) Let (U,ζ,E) be an $S_n -$merotopic space and c be a soft Čech closure operator on U such that for every $\zeta -$ clan $\mathcal{B}$, $\sqcap\{c(A): A \in \mathcal{B}\} \neq \Phi$. Then $\zeta \subseteq \zeta^c$.

Proof. For the proof of (ii), let $\mathcal{A} \in \zeta$ using Theorem 3.10, $\mathcal{A} \subseteq \mathcal{C}$, where $\mathcal{C}$ is a maximal $\zeta -$ clan. Then $\Phi \neq \sqcap\{c(A): A \in \mathcal{C}\} \sqsubseteq \sqcap\{c(A): A \in \mathcal{A}\}$.

3.11.1 Example

Let (U,ζ) be an $S_m -$merotopic space, where $n \geq m$ and n, m are integers. Define ζ_* by $\mathcal{A} \in \zeta_*$ iff there exists $\mathcal{B} \in \zeta$ such that $\mathcal{A}$ corefines $\mathcal{B}$ with $|\mathcal{A}| \leq n$. Then ζ_* is an $S_n -$merotopy on U. ζ is a restriction of ζ_*. It is sufficient to prove 3.1 (c) and (d). For the proof 3.1 (c), let $|\mathcal{A}| \leq n$ and $\mathcal{A}$ corefines $\mathcal{B}$ and $\mathcal{B} \in \zeta_*$. Then there exist $\mathcal{C} \in \zeta$ such that $\mathcal{B}$ corefines $\mathcal{C}$ implies that $\mathcal{B} \in \zeta$. Further to prove 3.1(d), let $\mathcal{C} \in P(S,E))$ such that $|\mathcal{C}| \leq n-1$ and $(\mathcal{C} \cup \{A \sqcup B\}) \in \zeta_*$. Then there exists $B \in \mathcal{B}$, $\mathcal{B} \in \zeta$ such that $\mathcal{C} \cup \{A \sqcup B\}$ corefines $\mathcal{B}$. Then $\mathcal{C} \cup \{A \sqcup B\} \in \zeta$ and so $(\mathcal{C} \cup \{A\}) \in \zeta$ or $(\mathcal{C} \cup \{B\}) \in \zeta$.

3.12 Proposition

Let ζ be an $S_m -$merotopy on U and $n \geq m$ where n and m are integers. Then the largest $S_n -$merotopy on U, whose restriction is ζ, is given by $\mathcal{B} \in \zeta^*$ iff $\mathcal{B} \subseteq \mathcal{C}, \mathcal{C}$ is an $\zeta -$clan and $|\mathcal{B}| \leq n$. Moreover, every $\zeta -$clan is an $\zeta^* -$clan.

Proof. It is sufficient to prove 3.1(d). For any $\zeta -$clan $\mathcal{C}$, let $\mathcal{B} \cup \{B_1\} \Subset \mathcal{C}$ and $\mathcal{B} \cup \{B_2\} \Subset \mathcal{C}$, $\mathcal{B} \subseteq \mathrm{P}((S,E))$. Then there exist $B' \in \mathcal{B} \cup \{B_1\}$ and $B'' \in \mathcal{B} \cup \{B_2\}$ such that $B' \notin \mathcal{C}$ and $B'' \notin \mathcal{C}$. Suppose that $B' \in \mathcal{B}$. Then $\mathcal{B} \cup \{B_1 \sqcup B_2\} \Subset \mathcal{C}$. However if $B' = B_1$ and $B'' = B_2$ then $B_1 \sqcup B_2 \in \mathcal{B} \cup \{B_1 \sqcup B_2\}$ and $B_1 \sqcup B_2 \Subset \mathcal{C}$. This gives that $\mathcal{B} \cup \{B_1 \sqcup B_2\} \Subset \mathcal{C}$.

4 Conclusion

Near structure plays a crucial role, specifically in the extension problem of topological spaces. In the present study we define nearness of finite order in soft set theory and supported the concept introduced with some basic examples. The concept of merotopic space is generalized in soft setting which is a useful generalization using the theory of grills. Soft nearness of finite order is shown to be maximal clan generated. The notion of near set is a useful tool for theoretical studies in image processing also Soft set being the most recent concept, has a wide scope of applicability in different branches of study and the present work gives an initial platform to the same.

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