Solution of Algebraic and Transcendental Equations using Fuzzified He's Iteration Formula in terms of Triangular Fuzzy Numbers

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Abstract: - In the literature, a lot of numerical methods are available for solving both algebraic and transcendental equations. The Newton-Raphson method is the most commonly used because of its simplicity and faster convergence. The intent of this paper is to fuzzify the generalized Newton Raphson type iterative scheme, known as He's iteration for solving the nonlinear algebraic and transcendental equations arising in fuzzy environment. Several examples are taken for depicting the efficiency of new fuzzified He's iterative scheme and its comparison table is given depicting the number of iterations required in Newton-Raphson, He's Iteration and Fuzzified He's iteration method.

Key-Words: - He's iteration, nonlinear equations, Newton-Raphson method, Fuzzified iterative scheme.

1 Introduction

In mathematics, most of the real life problems can be represented by means of nonlinear equation $T(x) = 0$, whether they arise in engineering, physics, biology, chemistry or any streams of science. Most of the nonlinear equation in their usual forms can be easily solvable by several methods available in the literature. But due to uncertainty in life or in real life situations, these nonlinear equations cannot describe the problems exactly what they are. Hence in general for any practical situations, if we collect data or information to measure something, then in most of the cases that data is not accurate and if on the basis of that data, we model the problem through some non linear equation then that nonlinear equation also not describes the situation exactly. Zadeh [1] provides a solution, known as fuzzy set theory to deal these kinds of situations. On the basis of this fuzzy theory, many researchers [2-10] studied the fuzzy nonlinear equations in which the parameters used in the equation are fuzzified. Standard numerical techniques dealing with usual nonlinear equation are not suitable in the case of fuzzy nonlinear equation. Therefore it is needed to develop new numerical schemes which deal these kinds of problems. There are many schemes that are available in numerical analysis by which it is very easy to solve nonlinear equations, such as bisection method, secant method [8], method of false position, fixed point iteration [4], Newton Raphson method [9-10], etc. To deal with fuzzy nonlinear equation, all these iterative methods are fuzzified by several authors [5-7]. Out of these iterative schemes N-R method is the most widely used method [2-3], because it provides faster convergence. Its iterative formula is given as:

$$x_{n+1} = x_n - \frac{T(x_n)}{T'(x_n)}.$$

Newton-Raphson method is also very sensitive to the initial guess and hence this method fails when functions derivative becomes zero. Thus to deal with this problems, many modifications [11-12] are proposed in the literature, out of which one is known as He's iteration [11] is also very much effective. Its iterative formula is given in the following form:

$$x_{n+1} = x_n - \frac{T(x_n)}{T'(x_n)+\alpha T(x_n)} \ ,$$

where α, is a free parameter, known as the control parameter, which can be adjusted according to the need, for the convergence of the iterative scheme.

The concept of fuzzy number was introduced by Dubois and Parde [13] which well incorporates the vagueness in the data. Since then a lot of work on the theory development on fuzzy numbers is done. Triangular fuzzy number [3], an extension on fuzzy number, gives better result as compared to fuzzy number. Triangular fuzzy numbers are used in many applications such as student assessment [14], collaborative filtering recommendation to measure users' similarity [15], etc.

Now in this work we have fuzzified He's iteration method (also known as modified Newton's method) and uses triangular fuzzy number to increase the convergence towards the solution by taking several examples.

2 Basic Concepts and Fuzzification of He's Iteration

In this section, we will define some basic definitions such as fuzzy sets, fuzzy number, triangular fuzzy number and its algebra.

Definition 2.1. (Fuzzy Sets) [1]: Let us consider a non-empty set *X*. A fuzzy set *A* defined on the elements of the set *X* having the membership value $\mu_A(x)$, defined as $A = \{< x, \mu_A(x) >: x \in X, \mu_A(x) \in [0,1]\}$.

Definition 2.2. (Fuzzy Number) [2]: A map $T_a: R \rightarrow [0, l]$ such that $T_a(x) = 1 \Leftrightarrow x = a$ and $lim(|x| \rightarrow \infty) T(x) = 0$ will be simply called a fuzzy number.

Definition 2.3. Triangular Fuzzy Number [3]: A fuzzy number A is called triangular, if the 3-tuples of real numbers $(a,\ b,\ c)$ with $a \le b \le c$ and having membership function are defined as follows:

$$\mu_A(x) = \begin{cases} \dfrac{x-a}{b-a}, & for\ \ a \le x \le b \\ \dfrac{c-x}{c-b}, & for\ b \le x \le c \\ 0 & for\ x < a\ and\ \ x > c \end{cases}$$

And set containing triangular fuzzy number is called triangular fuzzy set.

Let us suppose $A = [A_1\ B_1\ C_1]$ and $B = [A_2\ B_2\ C_2]$ are two triangular fuzzy numbers and either $T(A) > 0, T(B) < 0$ or $T(A) < 0, T(B) > 0$, then the root of fuzzy nonlinear equation exists within these two triangular fuzzy numbers. The membership functions of these two triangular fuzzy numbers *A* and *B* are defined as:

$$\mu_A(x) = \begin{cases} \dfrac{x-A_1}{B_1-A_1}, & for\ \ A_1 \le x \le B_1 \\ \dfrac{C_1-x}{C_1-B_1}, & for\ \ B_1 \le x \le C_1 \\ 0 & for\ x < A_1\ and\ \ x > C_1 \end{cases}$$

and

$$\mu_B(x) = \begin{cases} \dfrac{x-A_2}{B_2-B_1}, & for\ \ A_2 \le x \le B_2 \\ \dfrac{C_2-x}{C_2-B_2}, & for\ \ B_2 \le x \le C_2 \\ 0 & for\ x < A_2\ and\ \ x > C_2 \end{cases}$$

Definition 2.4. α –cut on Fuzzy set [3]: Let *A* be any fuzzy interval defined in *R* and $\alpha \in (0,1]$. The α -cut of *A* is the crisp set, denoted by $[A]^\alpha$ containing all the elements having some membership degree in *A* greater than or equal to α, i.e.

$$[A]^\alpha = \{x \in R | A(x) \ge \alpha\}.$$

The α −cuts with respect to triangular fuzzy numbers is defined as

$$[A]^\alpha = [A_1 + \alpha(B_1 - A_1), C_1 + \alpha(B_1 - C_1)\ ]$$

and

$$[B]^\alpha = [A_2 + \alpha(B_2 - A_2), C_2 + \alpha(B_2 - C_2)\ ]$$

Definition 2.5. Addition of two triangular fuzzy numbers [3]:

The addition of two triangular fuzzy numbers $A = [A_1\ \ B_1\ \ C_1]$ and $B = [A_2\ \ B_2\ \ C_2]$ are defined as:

$[A_1\ \ B_1\ \ C_1]\ +\ [A_2\ \ B_2\ \ C_2] = [A_1 + A_2\ \ B_1 + B_2\ \ C_1 + C_2]$.

Definition 2.6. Subtraction of two triangular fuzzy numbers [3]:

The subtraction of two triangular fuzzy numbers $A = [A_1\ \ B_1\ \ C_1]$ and $B = [A_2\ \ B_2\ \ C_2]$ are given as:

$[A_1\ \ B_1\ \ C_1]\ -\ [A_2\ \ B_2\ \ C_2] = [A_1 - C_2\ \ B_1 - B_2\ \ C_1 - A_2]$.

Definition 2.7. Multiplication of two triangular fuzzy numbers [3]:

The multiplication of two triangular fuzzy numbers $A = [A_1\ \ B_1\ \ C_1]$ and $B = [A_2\ \ B_2\ \ C_2]$ is:

$[A_1\ \ B_1\ \ C_1] \times [A_2\ \ B_2\ \ C_2] = [A_1A_2\ \ B_1B_2\ \ C_1C_2]$.

Definition 2.8. Division of two triangular fuzzy numbers [3]:

The division of two triangular fuzzy numbers $A = [A_1\ \ B_1\ \ C_1]$ and $B = [A_2\ \ B_2\ \ C_2]$ are defined as

$[A_1\ \ B_1\ \ C_1]\ /\ [A_2\ \ B_2\ \ C_2] = [A_1/C_2\ \ B_1/B_2\ \ C_1/A_2]$.

Definition 2.9. Scalar Multiplication of triangular fuzzy numbers [3]:

Let λ be any scalar and $A = [A_1\ \ B_1\ \ C_1]$ be any triangular fuzzy number. Then their scalar multiplication is defined as

$\lambda[A_1\ \ B_1\ \ C_1] = [\lambda A_1\ \ \lambda B_1\ \ \lambda C_1]$, where $\lambda > 0$

$\lambda[A_1\ \ B_1\ \ C_1] = [\lambda C_1\ \ \lambda B_1\ \ \lambda A_1]$, where $\lambda < 0$.

Definition 2.10. Intermediate Value Theorem

Let $I = [A, B]$ be any real valued interval and $T: I \to R$ be any continuous function defined on R. If w be any real number between $T(a) < 0$ and $T(b) > 0$, or $T(a) > 0$ and $T(b) < 0$, then there exists is at least one $c \in (a, b)$ such that $T(c) = w$.

3 Methodology

We start with some nonlinear equation $T(x) = 0$, which may be transcendental or algebraic. The intermediate value theorem (IVT) helps to find the interval of convergence, where function changes its sign, that means, if the value of any continuous function $T(x)$ at $x = A$ is negative and at $x = B$ positive or vice-versa, then the root lies between those two points A and B. In this research paper, we will use triangular fuzzy number for the solution of both non-linear algebraic and transcendental equations. Here He's iterative scheme is fuzzified and the results are compared with original Newton-Raphson method and defuzzified He's iteration method.

To start any iterative scheme, we have to choose some initial value so that the first initial value to the root of the fuzzy nonlinear equation $T(x) = 0$ is $x_0 = A$ or B.

According to He's iteration, we have

$$x_{n+1} = x_n - \frac{T(x_n)}{T'(x_n) + \alpha T(x_n)}.$$

In fuzzy environment, the above scheme reduces for $n = 0$, as

$$x_1 = x_0 - \frac{T(x_0)}{T'(x_0) + \alpha T(x_0)} = [x_1^{(1)}, x_1^{(2)}, x_1^{(3)}]$$

with membership function

$$\mu_{x_1}(x) = \begin{cases} \dfrac{x - x_1^{(1)}}{x_1^{(2)} - x_1^{(1)}} & for\ x_1^{(1)} \le x \le x_1^{(2)} \\ \dfrac{x - x_1^{(3)}}{x_1^{(2)} - x_1^{(3)}} & for\ x_1^{(2)} \le x \le x_1^{(3)} \\ 0 & for\ x < x_1^{(1)}\ and\ x > x_1^{(3)} \end{cases}$$

and $[x_1]^\alpha = \left[x_1^{(1)} + \alpha\left(x_1^{(2)} - x_1^{(1)}\right),\ x_1^{(3)} + \alpha\left(x_1^{(2)} - x_1^{(3)}\right)\right]$.

Similarly,

$$x_2 = x_1 - \frac{T(x_1)}{T'(x_1) + \alpha T(x_1)} = [x_2^{(1)}, x_2^{(2)}, x_2^{(3)}]$$

with membership function

$$\mu_{x_2}(x) = \begin{cases} \dfrac{x - x_2^{(1)}}{x_2^{(2)} - x_2^{(1)}} & for\ x_2^{(1)} \le x \le x_2^{(2)} \\ \dfrac{x - x_2^{(3)}}{x_2^{(2)} - x_2^{(3)}} & for\ x_2^{(2)} \le x \le x_2^{(3)} \\ 0 & for\ x < x_2^{(1)}\ and\ x > x_2^{(3)} \end{cases}$$

and$[x_2]^\alpha = \left[x_2^{(1)} + \alpha\left(x_2^{(2)} - x_2^{(1)}\right),\ x_2^{(3)} + \alpha\left(x_2^{(2)} - x_2^{(3)}\right)\right]$ and so on.

4 Numerical Implementations

We have taken following examples, whose solutions are given step by step to illustrate the effectiveness of this fuzzy iterative scheme.

Example 1: $T(x) = x^{10} - 1$

Solution Let $A = [0.49,\ 0.5,\ 0.51]$, $B = [1.59,\ 1.6,\ 1.61]$ be two fuzzy triangular numbers. We see that $T(A) = [-0.9992, -0.9990, -0.9988]$, which is negative and $T(B) = [102.2693, 108.9512,\ 116.0196]$, which is positive, hence the root lies between two fuzzy triangular numbers A and B using intermediate value theorem.

Now using fuzzified iteration formula for $\alpha = -2$, $x_{n+1} = x_n - \frac{T(x_n)}{T'(x_n) - 2T(x_n)}$,

$x_{n+1} = x_n - \frac{T(x_n)}{T'(x_n) - 2T(x_n)}$, $x_0 = [0.49, 0.5, 0.51]$

$$x_1 = x_0 - \frac{T(x_0)}{T'(x_0) - T(x_0)}$$

$x_1 = [0.49, 0.5, 0.51] -$

$$\frac{[-.9992, -.9990,\ -.9988]}{[.0163,\ .0195, .0233] - 2[-.9992, -.9990, -.9988]}$$

$x_1 = [0.986, 0.9952, 1.0042]$,

Crisp value = 0.9951.

The crisp value is calculated by taking the mean value of the triangular fuzzy number.

$$x_2 = x_1 - \frac{T(x_1)}{T'(x_1) - 2T(x_1)}$$

$x_2 = [0.986, 0.9952, 1.0042] -$

$$\frac{[-.1315,\ -.0470,\ .0428]}{[8.8083,\ 9.5762,\ 10.3844] - 2[-0.1315, -.0470,\ .0428]}$$

$x_2 = [0.9909, 1.0001, 1.0001]$, Crisp value = 1.0025

Example2: $T(x) = e^{sinx} - x$, $x_0 = 1.0$, $T'(x) = \cos x . e^{sinx} - 1$.

For $\alpha = -2$,

$$x_{n+1} = x_n - \frac{T(x_n)}{T'(x_n) - 2T(x_n)},$$

$x_0 = [0.99, 1.00, 1.01]$

$$x_1 = x_0 - \frac{T(x_0)}{T'(x_0) - 2T(x_0)}$$

$x_1 = [0.99, 1.00, 1.01] -$

$$\frac{[1.31718, 1.31978,\ 1.32225]}{[.26593,\ .25338,\ .24043] - 2[1.3171, 1.3197,\ 1.3222]}$$

$x_1 = [1.5459, 1.55309, 1.56022]$,

Crisp value = 1.55307.

$x_2 = [1.5459, 1.55309, 1.56022] -$

$$\frac{[1.17154,\ 1.16477,\ 1.15791]}{[-.93235, -.95188, -.9712] - 2[1.1715, 1.1647,\ 1.1579]}$$

$x_2 = [1.90238, 1.90805, 1.9137]$,

Crisp value = 1.90804

$x_3 = [1.90238, 1.90805, 1.9137] -$

$$\frac{[0.67179, 0.66134, 0.65085]}{[-1.83799, -1.8502, -1.8622] - 2[.6717, .6613,\ .6508]}$$

$x_3 = [2.10968, 2.11649, 2.12325]$,

Crisp value = 2.11647

$$x_4 = [2.10968, 2.11649, 2.12325] - \frac{[0.24942, 0.23434, 0.21930]}{[-2.2106, -2.2201, -2.229] - 2[.2494, .2343, \ .21930]}$$

$x_4 = [2.19246, 2.20364, 2.21467]$,

Crisp value = 2.20359.

$$x_5 = [2.19246, 2.20364, 2.21467] - \frac{[.06200, .03608, .01037]}{[-2.312, -2.324, -2.335] - 2[.06200, .0360, \ 0.0103]}$$

$x_5 = [2.1969, 2.21869, 2.23988]$,

Crisp value = 2.21849.

Example 3: $T(x) = x - \cos x$

Let $A = [-0.01, \ 0, \ 0.01]$, $B = [0.99, \ 1.0, \ 1.01]$,

$T(A) = [-1.00995, -1, -0.99995]$, which is negative and $T(B) = [0.44131, 0.45970, 0.47814]$, which is positive. Hence root lies between A and B. The root lies near B, so we choose $x_0 = B$.

For $\alpha = 1.0$, $x_{n+1} = x_n - \frac{T(x_n)}{T'(x_n)+T(x_n)}$,

$x_0 = [0.99, 1.0, 1.01]$

$$x_1 = x_0 - \frac{T(x_0)}{T'(x_0) + T(x_0)}$$

$$x_1 = [0.99, 1.0, 1.01] - \frac{[0.44131, \ 0.45970, \ 0.47814]}{[1.836, 1.841, 1.846] + [.4413, .4597, .4781]}$$

$x_1 = [0.78004, 0.80023, 0.82019]$,

Crisp value = 0.80015.

$$x_2 = [0.78004, 0.80023, 0.82019] - \frac{[0.0691, 0.1036, \ 0.1381]}{[1.7033, 1.7175, 1.7312] - [0.0691, 0.1036, \ 0.1381]}$$

$x_2 = [0.6918, 0.73986, 0.77859]$,

Crisp value = 0.73675.

$$x_3 = [0.6918, 0.73986, 0.77859] - \frac{[-.07830, -.00130, \ .06669]}{[1.637, 1.674, 1.702] + [.0783, -.00130, .06669]}$$

$x_3 = [0.64904, 0.74064, 0.82285]$,

Crisp value = 0.73751.

Example 4: $T(x) = \sin x$, $\alpha = -1$.

Let $A = [2.99, \ 3.0, \ 3.01]$, $B = [3.99, \ 4.0, \ 4.01]$, $T(A) = [0.15101, 0.14112, 0.13121]$, which is positive and

$T(B) = [-0.75023, -0.75680, -0.76330]$,

which is negative. Hence root lies between A and B. The root lies near A, so we choose $x_0 = A =$ [2.99, 3.0, 3.01].

For $\alpha = -1.0$,

$x_{n+1} = x_n - \frac{T(x_n)}{T'(x_n)-T(x_n)}$, $x_0 = [2.99, \ 3.0, \ 3.01]$

$$x_1 = x_0 - \frac{T(x_0)}{T'(x_0) - T(x_0)}$$

$$x_1 = [2.99, 3.0, 3.01] - \frac{[0.15101, 0.14112, \ 0.13121]}{[-0.9885, -0.9899, -0.9913] - [0.1510, 0.1411, \ 0.1312]}$$

$x_1 = [3.10718, 3.12476, 3.14219]$,

Crisp value = 3.12471.

$$x_2 = [3.10718, 3.12476, 3.14219] - \frac{[0.03441, 0.01683, -0.00060]}{[-0.9994, -0.9998, -1.0] - [0.0344, 0.0168, -0.00060]}$$

$x_2 = [3.10658, 3.14131, 3.17546]$,

Crisp value = 3.14112

Following Table 1 shows the comparison between Newton-Raphson method, He's iteration method and fuzzified He's iteration method.

Table 1: Comparison between the Newton Raphson Method, He's iteration method and the fuzzified He's Iteration Method

Newton Raphson Method			He' iteration Method			Fuzzy He's iteration Method		
Examples	n	x_n	α	n	x_n	α	n	x_n
Ex. 1	42	1.0	-1	8	1.0	-2	2	1.0025
Ex. 2	20	2.2191	-1	3	2.2191	-2	5	2.21849
Ex. 3	28	0.7391	1	9	0.7391	1	3	0.73751
Ex. 4	7	31.4159	-1	6	3.1416	-1	2	3.14112

4 Conclusion

In the present study we have solved algebraic and transcendental equations by fuzzyfying the He's iteration method using triangular fuzzy number, which is the modification of Newton-Raphson method. In this way we have provided a new fuzzified iterative method which can be used for solving various fuzzy nonlinear algebraic and transcendental equations. He's iteration scheme in a fuzzified form is not applied earlier as per our literature review. The future work will be devoted to develop new iterative schemes in a fuzzified form so that the convergence rate can be improved.

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