Recursive Residuals Partial Sums Method for Testing Model Validity in Modelling of Spatial Data

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Abstract: The main objective of this work is to model spatial observations using linear regression analysis defined on a compact experimental region. To check the validity of an assumed model, tests based on Kologorov-Smirnov and Cramér-von Mises functionals of the partial sums (CUSUM) of the recursive residuals of the observations are proposed. It is shown that the limit of the sequence of the CUSUM processes of the recursive residuals for triangular array of design points does not depend on the model. It is given by the set-indexed Brownian sheet when the model is true. The performance of the tests are also studied by deriving the non trivial limiting power functions of the tests when the model is not true. Their finite sample size behaviors are compared with those of the well-known asymptotic F test and are investigated by simulation. It is shown in this study that both Cramér-von Mises and F tests perform better than the Kolmogorov-Smirnov test. The application of the proposed method in a real data is also exhibited. The design under which the data has been collected is given by a regular lattice.

Key–Words: Recursive residual, Gaussian white noise, Brownian sheet, linear regression, Kolmogorov-Smirnov test, Cramér-von Mises test, F-test.

1 Introduction

Modelling spatial observations using linear regression has been studied extensively in various disciplines such as in economics, agriculture, geology, and other earth sciences. Testing the validity of an assumed model is important prior to the application of the model in prediction and other quantifications. Intensive researches have been conducted for developing such preliminary diagnostic method. The most common approach is based on the investigation of the residuals of the observations or variant of them, see [2, 20, 30, 28, 29] for references.

Motivated by the preceding works, [23] studied the set-indexed CUSUM of the ordinary least squares (OLS) residuals defined in [18, 31] for establishing asymptotic model validity test (model-check) in modelling of spatial observations. By generalizing the geometric approach proposed in [6, 7] which is different to that of [18, 31], [23] obtained the limit process as a complicated functional of the set-indexed Brownian sheet which depends not only on the design but also on the regression functions defining the assumed model, see also [5] for regression on a closed interval. By this reason the application of the method is restricted since it suffered from being unable to compute the critical values analytically.

We notice that the dependency of the limit process on the regression model is due to the fact that the OLS residuals are mutually correlated although the error terms are mutually independent. If a transformation can be defined in such a way that the resulting residuals are either uncorrelated or mutually independent, then the uniform central limit theorem in [1] and [19] guarantee that the limiting distribution of the CUSUM of such kind of residuals will not depend on the model, so that the test can be easily implemented in the practice. In the case of time series observations, the transformation are defined recursively which leads to the so called recursive residuals, see [8, 10, 12, 16, 21].

It is the purpose of the present paper to give an investigation to the application of CUSUM test based on the recursive residuals obtained from a sequence of triangular arrays of observations defined on a compact rectangle. CUSUM as well as MOSUM test based on the recursive residuals of time series observation have been well investigated in [8, 10, 12, 16, 21]. However there is no documentation available for CUSUM as well as MOSUM test based on the recursive residuals of observations obtained from a sequence of triangular arrays of design points. Due to the absence of order, more effort is needed for the derivation of the limit process in such case.

To see the problem in more detail, let us consider a spatial process $\{Y(\mathbf{t}) : \mathbf{t} := (t_1, \ldots, t_d)^\top \in \mathbf{D}\}$, satisfying a regression relationship defined by

$$Y(\mathbf{t}) = g(\mathbf{t}) + \varepsilon(\mathbf{t}),\ \mathbf{t} \in \mathbf{D}, \tag{1}$$

where g is the true, but unknown regression function defined on $\mathbf{D} := \Pi_{i=1}^d [a_i, b_i] \subset \mathcal{R}^d$, $\varepsilon(\mathbf{t})$ is the random error with $\mathbf{E}(\varepsilon(\mathbf{t})) = 0$ and $Var(\varepsilon(\mathbf{t})) = \sigma^2$, $0 < \sigma^2 < \infty$. In practice, d is mostly 2 or 3. Let $f_1, \ldots, f_p$ be linearly independent as functions in $L_2(P_0)$, where P_0 is the Lebesque measure on $\mathbf{D}$. Model check in regression analysis concerns with the problem of testing whether or not an assumed model

$$H_0 : Y(\mathbf{t}) = \sum_{i=1}^{p} \beta_i f_i(\mathbf{t}) + \varepsilon(\mathbf{t}),\ \mathbf{t} \in \mathbf{D}, \tag{2}$$

holds true, for some unknown constants $\beta_1, \ldots, \beta_p$. When the observations are normally distributed, it is well known that likelihood ratio test which coincides with F-test is usually applied, see [2, 20]. To simplify, we consider in the present work the case where $d = 2$. Let $\{Y_{j_1 j_2} : 1 \le j_1 \le n_1,\ 1 \le j_2 \le n_2\}$ be a triangular array of the observations of Model (2) obtained over a triangular array of design points $\{t_{j_1 j_2} : 1 \le j_1 \le n_1,\ 1 \le j_2 \le n_2\} \subset \mathbf{D}$ and let $\{\varepsilon_{j_1 j_2} : 1 \le j_1 \le n_1,\ 1 \le j_2 \le n_2\}$ be the corresponding triangular array of independent and identically distributed random errors with $\mathbf{E}(\varepsilon_{j_1,j_2}) = 0$ and $Var(\varepsilon_{j_1,j_2}) = \sigma^2$. Then for fixed $n_1 \ge 1$ and $n_2 \ge 1$, the observations under H_0 can be written as

$$\mathbf{Y} = \mathbf{X}\beta + \mathcal{E}, \tag{3}$$

where

$$\mathbf{Y} := (Y_{11}, \ldots, Y_{n_1 1}, \ldots, Y_{1 n_2}, \ldots, Y_{n_1 n_2})^\top$$
$$\mathbf{X} := (\mathbf{f}(t_{11}), \ldots, \mathbf{f}(t_{n_1 1}), \mathbf{f}(t_{1 n_2}), \ldots, \mathbf{f}(t_{n_1 n_2}))^\top$$
$$\mathcal{E} := (\varepsilon_{11}, \ldots, \varepsilon_{n_1 1}, \ldots, \varepsilon_{1 n_2}, \ldots, \varepsilon_{n_1 n_2})^\top$$
$$\beta = (\beta_1, \beta_2, \ldots, \beta_p)^\top,$$

thereby $\mathbf{f} = (f_1, \ldots, f_p)^\top$. The CUSUM process of the OLS residuals of Model (3) indexed by the family of convex subsets $\mathcal{A}$ of $\mathbf{D}$ is defined by

$$\mathbf{V}_{n_1 n_2}(\mathbf{R}_{n_1 n_2})(A) = \sum_{j_1=1}^{n_1} \sum_{j_1=1}^{n_2} \mathbf{1}_A(t_{j_1 j_2}) r_{j_1 j_2}, A \in \mathcal{A}$$

where

$$\mathbf{R}_{n_1 n_2} = \mathbf{Y} - \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{Y}.$$

Next by applying the invariance principle of [1, 19], [23, 18, 31] showed that

$$\frac{1}{\sigma \sqrt{n_1 n_2}} \mathbf{V}_{n_1 n_2}(\mathbf{R}_{n_1 n_2}) \Rightarrow Z^*_{P_0},$$

where $Z^*_{P_0}$ is a centered Gaussian process indexed by $\mathcal{A}$, defined by

$$Z^*_{P_0} = Z_{P_0} - \sum_{i=1}^{p} \left(\int_{\mathbf{D}}^{R} f_i(t,s) dZ_{P_0}(t,s) \right) h_{f_i},$$

where $h_{f_i}(A) := \int_A f_i(x,y) P_0(dx,dy)$ and Z_{P_0} is the set-indexed Brownian sheet (Gaussian white noise) defined in [1, 17, 19, 23, 31]. Thereby $\int^R$ stands for the Riemann-Stiltjes integral. It can be seen that the limit process $Z^*_{P_0}$ depends heavily on the regression function assumed under H_0. If the assumed model is zero model which is not so important in the practice, the limit is given by the standard set-indexed Brownian sheet Z_{P_0}. Furthermore, if under H_0 a constant model is assumed, we get for large n_1 and n_2 the standard set-indexed Gaussian pillow. The computation of the critical values of the Kolmogorov-Smirnov and Cramér-von Mises statistics $KS := \sup_{(t,s)\in\mathbf{D}} \left| Z^*_{P_0}(t,s) \right|$ and $CvM := \int_{\mathbf{D}} (Z^*_{P_0}(t,s))^2 P_0(dt,ds)$ become complicated task for higher order regression models. To the knowledge of the authors there are no literatures available for computing such critical values for the case other than Brownian sheet. A contrast situation has been found for the case of the standard Brownian motion $B(t)$ and Brownian bridge $B^\circ(t)$, for $t \in [0,1]$ in which it is well known that the critical values of $\sup_{t\in[0,1]} |B(t)|$ as well as $\sup_{t\in[0,1]} |B^\circ(t)|$ can be computed analytically by formulas (9.14) and (9.40) in [4], whereas that of $\int_{[0,1]} (B^\circ(t))^2 dt$ can be computed by using formulas (12) in [22], p. 147.

The rest of the present paper is organized as follows. In Section 2 we introduce the notion of recursive residual for spatial observation. The limit process of the sequence of set-indexed partial sums processes of the recursive residuals under H_0 as well as under H_1 are derived in Section 3. To be able to compare the performance of the tests we propose in Section 4 simulation study concerning the finite sample size behavior of the corresponding power function of the tests. The application of the test method is presented in Section 5. We close the paper with some conclusions and remarks for future research, see Section 6. Proofs are postponed to the appendix.

2 Spatial recursive residuals

We assume throughout the paper that the observations are collected according to an order defined by

$$Y_{11}, \ldots, Y_{n_1 1}, Y_{12}, \ldots, Y_{n_1 2}, \ldots, Y_{1 n_2}, \ldots, Y_{n_1 n_2}.$$

Let (j_1^*, j_2^*) be a fixed pair of integers such that $Y_{j_1^* j_2^*}$ becomes the first p-th observed response according to the preceding order. We define the following notations:

$$\mathbf{T}_{n_1n_2} := \{(j_1,j_2):\ 1\le j_1\le n_1,\ 1\le j_2\le n_2\}$$
$$\mathbf{T}_{n_1n_2-p} := \mathbf{T}_{n_1n_2-p+1}\setminus\{(j_1^*,j_2^*)\},$$

where

$$\mathbf{T}_{n_1n_2-p+1} = \{(j_1^*,j_2^*),(j_1^*+1,j_2^*),\cdots,(n_1,j_2^*), (1,j_2^*+1),\cdots,(n_1,j_2^*+1),\cdots,(n_1,n_2)\}.$$

Thus $\mathbf{T}_{n_1n_2},\mathbf{T}_{n_1n_2-p}$ and $\mathbf{T}_{n_1n_2-p+1}$ consist respectively of n_1n_2, n_1n_2-p and n_1n_2-p+1 ordered pairs. For every pair $(j_1,j_2)\in\mathbf{T}_{n_1n_2-p+1}$ we define the n_1n_2-p+1 regression models under H_0:

$$\mathbf{Y}^{(n_1,n_2)}_{j_1j_2} := \mathbf{X}^{(n_1,n_2)}_{j_1j_2}\beta+\mathcal{E}^{(n_1,n_2)}_{j_1j_2}, \qquad (4)$$

where

$$\begin{aligned}\mathbf{Y}^{(n_1,n_2)}_{j_1j_2} &:= (Y_{11},Y_{21},\ldots,Y_{j_1j_2})^\top,\\ \mathbf{X}^{(n_1,n_2)}_{(j_1j_2)} &:= (\mathbf{f}(t_{11}),\mathbf{f}(t_{21}),\ldots,\mathbf{f}(t_{j_1j_2}))^\top,\\ \mathcal{E}^{(n_1,n_2)}_{j_1j_2} &:= (\varepsilon_{11},\varepsilon_{21},\ldots,\varepsilon_{j_1j_2})^\top.\end{aligned}$$

Suppose the design $\{t_{j_1j_2}: 1\le j_1,j_2\le n\}$ is constructed in such away that $rank(\mathbf{X}^{(n_1,n_2)}_{(j_1j_2)})=p$, for all $(j_1,j_2)\in\mathbf{T}_{n_1n_2-p+1}$. Then the corresponding least squares estimator of β based on (4) is given by

$$\widehat{\beta}^{(n_1,n_2)}_{j_1j_2} = \left(\mathbf{X}^{(n_1,n_2)\top}_{j_1j_2}\mathbf{X}^{(n_1,n_2)}_{(j_1j_2)}\right)^{-1}\mathbf{X}^{(n_1,n_2)\top}_{j_1j_2}\mathbf{Y}^{(n_1,n_2)}_{(j_1j_2)}$$

for every $(j_1,j_2)\in\mathbf{T}_{n_1n_2-p+1}$.

Definition 1 *For $(j_1,j_2)\in\mathbf{T}_{n_1n_2-p}$, the n_1n_2-p recursive residuals of the spatial observations under H_0 are defined by*

$$w_{j_1j_2} := \frac{Y_{j_1j_2}-\mathbf{f}^\top(t_{j_1j_2})\widehat{\beta}^{(n_1,n_2)}_{j_1-1j_2}}{\sqrt{1+\mathbf{f}^\top(t_{j_1j_2})\left(\mathbf{X}^{(n_1,n_2)\top}_{j_1-1j_2}\mathbf{X}^{(n_1,n_2)}_{j_1-1j_2}\right)^{-1}\mathbf{f}(t_{j_1j_2})}},$$
$$for\ j_1\neq 1,\ j_2=j_2^*,j_2^*+1,\ldots,n_2,$$

and

$$w_{1j_2} := \frac{Y_{1j_2}-\mathbf{f}^\top(t_{1j_2})\widehat{\beta}^{(n_1,n_2)}_{n_1j_2-1}}{\sqrt{1+\mathbf{f}^\top(t_{1j_2})\left(\mathbf{X}^{(n_1,n_2)\top}_{n_1j_2-1}\mathbf{X}^{(n_1,n_2)}_{n_1j_2-1}\right)^{-1}\mathbf{f}(t_{1j_2})}},$$
$$j_1=1,\ j_2=j_2^*+1,\ldots,n_2.$$

Important properties of the recursive residuals under H_0 are summarized in the following proposition. The proof is given in the appendix.

Proposition 2 *For every $(j_1,j_2)\in\mathbf{T}_{n_1n_2-p}$, there exists a vector $\mathbf{a}_{j_1j_2}\in\mathcal{R}^{n_1n_2}$, such that $w_{j_1j_2}=\mathbf{a}^\top_{j_1j_2}\mathcal{E}_{n_1n_2}$, where for $j_2\in\{j_2^*,j_2^*+1,\ldots,n_2\}$, and $j_1\neq 1$,*

$$\mathbf{a}_{j_1j_2}\sqrt{d_{j_1j_2}} := \left(-\mathbf{f}^\top(t_{j_1j_2})\left(\mathbf{X}^{(n_1,n_2)\top}_{j_1-1j_2}\mathbf{X}^{(n_1,n_2)}_{j_1-1j_2}\right)^{-1}\mathbf{X}^{(n_1,n_2)\top}_{j_1-1j_2}, 1,0,\ldots,0\right)^\top,$$

with

$$d_{j_1j_2} = 1+\mathbf{f}^\top(t_{j_1j_2})(\mathbf{X}^{(n_1,n_2)\top}_{(j_1-1j_2)}\mathbf{X}^{(n_1,n_2)}_{(j_1-1j_2)})^{-1}\mathbf{f}(t_{j_1j_2}),$$

and for $j_1=1$ and $j_2\in\{j_2^*+1,\ldots,n_2\}$,

$$\mathbf{a}_{1j_2}\sqrt{d_{1j_2}} := \left(-\mathbf{f}^\top(t_{1j_2})\left(\mathbf{X}^{(n_1,n_2)\top}_{n_1j_2-1}\mathbf{X}^{(n_1,n_2)}_{n_1j_2-1}\right)^{-1}\mathbf{X}^{(n_1,n_2)\top}_{n_1j_2-1}, 1,0,\ldots,0\right)^\top$$

with

$$d_{1j_2} := 1+\mathbf{f}^\top(t_{1j_2})\left(\mathbf{X}^{(n_1,n_2)\top}_{n_1j_2-1}\mathbf{X}^{(n_1,n_2)}_{n_1j_2-1}\right)^{-1}\mathbf{f}(t_{1j_2}).$$

Furthermore, for any pairs $(j_1,j_2),(j_1',j_2')\in\mathbf{T}_{n_1n_2-p}$ with $j_1\neq j_1'$ or $j_2\neq j_2'$, it holds

$$\mathbf{a}^\top_{j_1j_2}\mathbf{a}_{j_1'j_2'} = \begin{cases}1 & ;\quad j_1=j_1'\ and\ j_2=j_2'\\ 0 & ;\quad j_1\neq j_1'\ or\ j_2\neq j_2'\end{cases}.$$

Proposition 2 says that $w_{j_1j_2}$ is a linear function of the vector of random errors $\mathcal{E}_{n_1n_2}$, for every $(j_1,j_2)\in\mathbf{T}_{n_1n_2-p}$ and they are mutually uncorrelated for at least $j_1\neq j_2$ or $j_1'\neq j_2'$, with $(j_1,j_2),(j_1',j_2')\in\mathbf{T}_{n_1n_2-p}$. Hence, if for $1\le j_1\le n_1$ and $1\le j_2\le n_2$, $\varepsilon_{j_1j_2}$ are independent and identically distributed (iid) $N(0,\sigma^2)$, then $w_{j_1j_2}$ are iid $N(0,\sigma^2)$, see also [8, 12, 21].

Let $\mathbf{W}_{n_1\times n_2} := (w_{j_1j_2})^{n_1,n_2}_{j_1=1,j_2=1}$, $n_1\ge 1$ and $n_2\ge 1$ be the sequence of the matrices of recursive residuals, where $w_{n(j_1,j_2)} := 0$, for $(j_1,j_2)\in\mathbf{T}_{n_1n_2}\setminus\mathbf{T}_{n_1n_2-p}$. Let $\mathcal{A}$ be the collection of convex subsets of $\mathbf{D} := [a_1,b_1]\times[a_2,b_2]$, such that $\mathcal{A}$ is totally bounded and have convergence integral entropy in the sense of Alexander and Pyke (1986). The set-indexed CUSUM process of the recursive residuals with respect to $\mathcal{A}$ is defined by

$$S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2})(A) := \sum_{(j_1,j_2)}\mathbf{1}_A(t_{j_1j_2})w_{j_1j_2},$$

where the sum is over all $(j_1,j_2)\in\mathbf{T}_{n_1n_2-p}$. It is noticed that $S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2})(\emptyset) := 0$ and

$S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2})(A) := 0$, whenever A is any element of $\mathcal{A}$ for which no design point $t_{j_1j_2}$ with the corresponding pair $(j_1, j_2) \in \mathbf{T}_{n_1n_2-p}$ are catched by A. To be able to sum the recursive residuals over all pairs $(j_1, j_2) \in \mathbf{T}_{n_1n_2}$, we set the value of $w_{j_1j_2}$ to zero, for $(j_1, j_2) \in \mathbf{T}_{n_1n_2} \setminus \mathbf{T}_{n_1n_2-p}$. So the CUSUM of the recursive residuals can be expressed as

$$S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2})(A) := \sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2} \mathbf{1}_A(t_{j_1j_2})w_{j_1j_2}.$$

In the sequel we use the partial sums operator of this kind, unless otherwise stated.

The Kolmogorov-Smirnov and Cramér-von Mises type statistics reasonable for testing H_0 are defined respectively by

$$\mathcal{KS}_{n_1\times n_2} := \frac{\sup_{A\in\mathcal{A}} |S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2})(A)|}{\sigma\sqrt{n_1n_2-p}}$$

$$\mathcal{CM}_{n_1\times n_2} := \frac{\sum_{A\in\mathcal{A}} (S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2})(A))^2}{(\sigma(n_1n_2-p))^2},$$

where H_0 will be rejected for large values of $\mathcal{KS}_{n_1\times n_2}$ or $\mathcal{CM}_{n_1\times n_2}$. The limiting distributions of $\mathcal{KS}_{n_1\times n_2}$ and $\mathcal{CM}_{n_1\times n_2}$ will be derived in the next section by generalizing the functional central limit theorem of Gaensler [13], Pyke [19] and Alexander and Pyke [1]. See also [14] for further reference in this topic.

3 The limit of the CUSUM processes

Theorem 3 below presents the limit process of $S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2})$ under H_0. By this theorem we can approximate the finite sample critical values of $\mathcal{KS}_{n_1\times n_2}$ and $\mathcal{CM}_{n_1\times n_2}$ by using their associated limiting distribution .

Theorem 3 *Let* $\left\{\mathbf{W}_{n_1\times n_2} = (w_{j_1j_2})_{j_1=1,j_2=1}^{n_1,n_2}\right\}$, $n_1 \geq 1$ *and* $n_2 \geq 1$ *be the sequence of the matrices of recursive residuals of Model 2 under normally distributed error terms and let the design is given by an equidistance design, defined by*

$$t_{j_1j_2} = \left(a_1 + (b_1 - a_1)\frac{j_1}{n_1}, a_2 + (b_2 - a_2)\frac{j_2}{n_2}\right),$$

for $1 \leq j_1 \leq n_1,\ 1 \leq j_2 \leq n_2$. *Then for* $n_1, n_2 \to \infty$, *it holds,*

$$\frac{1}{\sigma\sqrt{n_1n_2-p}} S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2}) \Rightarrow Z_{P_0}.$$

The asymptotic rejection region of the test using $\mathcal{KS}_{n_1\times n_2}$ and $\mathcal{CM}_{n_1\times n_2}$ are exhibited in the following corollary. The result is obtained by applying the well known continuous mapping theorem, see Theorem 27 in Billingsley [4].

Corollary 4 *Let* $\left\{\mathbf{W}_{n_1\times n_2} = (w_{j_1,j_2})_{j_1=1,j_2=1}^{n_1,n_2}\right\}$ *be the sequence of matrices of recursive residuals of Model 2, for* $1 \leq j_1 \leq n_1$ *and* $1 \leq j_2 \leq n_2$. *An asymptotic Kolmogorov-Smirnov test of size* α *will reject* H_0, *if and only if* $\mathcal{KS}_{n_1\times n_2} \geq \tilde{k}_{1-\alpha}$, *where* $\tilde{k}_{1-\alpha}$ *is a constant that satisfies*

$$\mathbf{P}\left\{\sup_{A\in\mathcal{A}} |Z_{P_0}(A)| \geq \tilde{k}_{1-\alpha}\right\} = \alpha.$$

Likewise, an asymptotic Cramér-von Mises test of size α *will reject* H_0, *if and only if* $\mathcal{CM}_{n_1\times n_2} \geq \tilde{t}_{1-\alpha}$, *where* $\tilde{t}_{1-\alpha}$ *is a constant that satisfies*

$$\mathbf{P}\left\{\int_{\mathbf{D}} Z_{P_0}^2(A)dA \geq \tilde{t}_{1-\alpha}\right\} = \alpha.$$

Remark 5 *If the family of half-open rectangles* $\{(a_1, t_1] \times (a_2, t_2] : a_1 \leq t_1 \leq b_1,\ a_2 \leq t_2 \leq b_2\}$ *is considered as the index set, then the critical regions of the test can be constructed based on the statistic* $KS := \sup_{(t,s)\in\mathbf{D}} |B_2(t,s)|$ *and* $CvM := \int_{\mathbf{D}} B_2^2(t,s)dtds$, *where* B_2 *is the Brownian sheet which is frequently called two parameters Brownian motion. The critical values of* KS *have been already computed by simulation, whereas those of* CvM *can be computed by applying the approach in [31]. The sufficient conditions of Theorem 3 and Corollary 4 are fulfilled by our data which consists of the measurement of the rate of growth of corn plants observed over* 21×16 *lattice points, see Section 5. In the application* σ *can be estimated by any consistent estimator.*

As suggested by the $0-1$ decision rule in the statistical decision theory (cf. [9], p. 468), an optimal test is a test that maximizes the power under the alternative. Two or more asymptotic tests of size α can be compared by investigating their power functions. The best test is the one that has the greatest power when the alternative is true. For our test problem, we consider general localized model

$$Y(\mathbf{t}) = \frac{1}{\sqrt{n_1n_2-p}}g(\mathbf{t}) + \varepsilon(\mathbf{t}),\ \mathbf{t} \in \mathbf{D} \qquad (5)$$

in order to get the limiting power function. When H_0 is true, Theorem 3 guarantees that the limiting power function coincides with the size of the test.

Theorem 6 *Let* $g : \mathbf{D} \to \mathcal{R}$ *be of bounded variation and let the localized model (5) be observed over the array of the equidistance design. Let* $\mathbf{W}_{n_1\times n_2}^1 := (w_{j_1j_2}^1)_{j_1=1,j_2=1}^{n_1,n_2}$ *be the triangular array of the recursive residuals associated with Model (5). Then, as* $n_1, n_2 \to \infty$, *it holds*

$$\frac{1}{\sigma\sqrt{n_1n_2-p}} S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2}^1) \Rightarrow \frac{1}{\sigma}h_g + Z_{P_0},$$

where

$$h_g(A) := \int_A g(u,v)P_0(du,dv)$$
$$-\int_A \mathbf{f}^\top(u,v)\mathbf{G}^{-1}(u,v)(\mathbf{f}g)(u,v)P_0(du,dv),$$

with

$$\mathbf{G}(u,v) := \int_{B_{u,v}} \mathbf{f}(x,y)\mathbf{f}^\top(x,y)P_0(dx,dy)$$
$$(\mathbf{f}g)(u,v) := \int_{B_{u,v}} (f_i(x,y)g(x,y))_{i=1}^p P_0(dx,dy).$$

The set $B_{u,v}$ is determined by the variable $(u,v) \in A$.

By applying the continuous mapping theorem, the asymptotic power function of the $KS_{n_1\times n_2}$ and $CM_{n_1\times n_2}$ tests can be expressed as follows.

Corollary 7 *Suppose that for testing the hypothesis*

$$H_0 : g \in \mathbf{V} \; against \; H_1 : g \notin \mathbf{V},$$

the localized model (5) is observed under the equidistance design, where $\mathbf{V} := [f_1,\ldots,f_p] \subset L_2(P_0)$. Asymptotic power function of the size α Kolmogorov-Smirnov test is given by

$$\lim_{n_1,n_2\to\infty} \Upsilon_{\mathcal{KS}_{n_1\times n_2}}(g)$$
$$= \mathbf{P}\left\{\sup_{A\in\mathcal{A}}\left|\frac{1}{\sigma}h_g(A) + Z_{P_0}(A)\right| \geq \tilde{k}_{1-\alpha}\right\}.$$

Similarly, the asymptotic power function of the Cramér-von Mises test of size α is given by

$$\lim_{n_1,n_2\to\infty} \Upsilon_{\mathcal{CM}_{n_1\times n_2}}(g)$$
$$= \mathbf{P}\left\{\int_{\mathbf{D}}\left(\frac{1}{\sigma}h_g(A) + Z_{P_0}(A)\right)^2 \geq \tilde{t}_{1-\alpha}\right\}.$$

The finite sample size behavior of the tests will be investigated by simulation in the next section. The power of the $\mathcal{KS}_{n_1\times n_2}$ and $\mathcal{CM}_{n_1\times n_2}$ tests are compared with that of the F test studied in Arnold [2]. When H_0 is true, the trend term $\frac{1}{\sigma}h_g$ vanishes uniformly, so that the power attains the size of the test, that is $\Upsilon_{\mathcal{KS}_{n_1\times n_2}}(g) = \alpha = \Upsilon_{\mathcal{CM}_{n_1\times n_2}}(g)$, for g varies in $\mathbf{V}$. To this end under H_0 we observe the parametric model $Y(\mathbf{t}) = \mathbf{f}^\top(\mathbf{t})\beta + \varepsilon(\mathbf{t})$, $\mathbf{t} \in \mathbf{D}$, for some unknown vector of parameters β. Hence, we get

$$(\mathbf{f}g)(u,v) = (\mathbf{f}\mathbf{f}^\top\beta)(u,v) = \mathbf{G}(u,v)\beta.$$

This leads us to the following result

$$h_g(A) = \int_A \beta^\top\mathbf{f}(u,v)P_0(du,dv)$$
$$-\int_A \mathbf{f}^\top(u,v)\mathbf{G}^{-1}(u,v)\mathbf{G}(u,v)\beta P_0(du,dv)$$
$$= \int_A \beta^\top\mathbf{f}(u,v)P_0(du,dv)$$
$$-\int_A \beta^\top\mathbf{f}(u,v)P_0(du,dv) = 0.$$

4 Simulation

The purpose of this section is to investigate the finite sample size behavior of the $\mathcal{KS}_{n_1\times n_2}$, $\mathcal{CM}_{n_1\times n_2}$ and F tests by simulating the performance of their corresponding power functions. The computer program for the simulation is written using R version 3.3.3. Two scenarios are considered. In the first scenario we test the hypotheses

$$H_0 : Y(t,s) = \beta_0 f_0(t,s) + \varepsilon(t,s)$$

against the alternative

$$H_1 : Y(t,s) = \sum_{i=0}^{2}\beta_i f_i(t,s) + \varepsilon(t,s),$$

whereas in the second one we test for

$$H_0 : Y(t,s) = \sum_{i=0}^{2}\beta_i f_i(t,s) + \varepsilon(t,s),$$

against the alternative

$$H_1 : Y(t,s) = \sum_{i=0}^{5}\beta_i f_i(t,s) + \varepsilon(t,s),$$

for $(t,s) \in [0,1]\times[0,1]$, where β_i, $i = 0,1,2,3,4,5$ are unknown constants and $f_0(t,s) = 1$, $f_1(t,s) = t$, $f_2(t,s) = s$, $f_3(t,s) = t^2$, $f_4(t,s) = s^2$ and $f_5(t,s) = ts$ are real valued regression functions. In the simulation the error terms are generated independently from centered normally distributed random numbers. However, in the computation of the test statistics we assume that the variance σ^2 is unknown and it is estimated by $\widehat{\sigma}_n^2$ defined in [2]. The experimental design is given by $n\times n$ regular lattice on $\mathbf{I}$, defined by

$$\Xi_{n\times n} := \{(\ell/n, k/n) : \; 1 \leq \ell, k \leq n\}, \; n \geq 1.$$

For computational reason we restrict the index set to the family $\mathcal{R}_{\mathbf{I}} := \{[0,t]\times[0,s] : \; 0 \leq t,s \leq 1\}$.

Hence the test statistics reduce to

$$\mathcal{KS}_{n\times n} = \frac{1}{\sigma\sqrt{n^2-p}} \max_{1\leq \ell,k\leq n} \left| \sum_{j=1}^{\ell}\sum_{i=1}^{k} w_{ij} \right|$$

$$\mathcal{CM}_{n\times n} = \sum_{\ell=1}^{n}\sum_{k=1}^{n} \left(\frac{1}{\sigma(n^2-p)} \sum_{j=1}^{\ell}\sum_{i=1}^{k} w_{ij} \right)^2$$

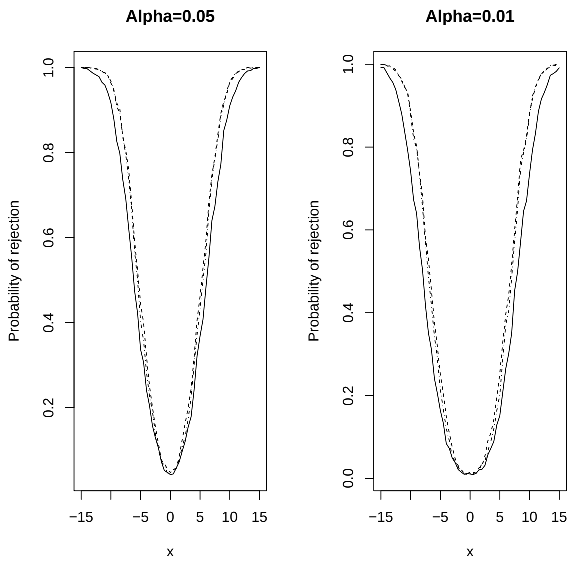

Figure 1: The graphs of the empirical power function of the $\mathcal{KS}_{n_1\times n_2}$ (solid line), $\mathcal{CM}_{n_1\times n_2}$ (dashed) and the F (dot-dashed) tests for constant model. Sample size $= 75 \times 75$ and the replication number $= 1500$.

The observations in the first case are generated based on the localized model

$$Y_{\ell k} = \frac{1}{\sqrt{n^2-1}}\left(4 + x\frac{l}{n} + x\frac{k}{n}\right) + \varepsilon_{\ell k},$$

for $1 \leq \ell, k \leq n,\ n \geq 1$, where x varies in the closed interval $[-15, 15]$. Hence, the observations are clearly from the model specified under H_0, when x attains 0. The error terms are generated from the standard normal distribution. The simulation results is exhibited in Figure 1. They represent the graphs of the approximated power functions of the tests for $\alpha = 0.05$ and $\alpha = 0.01$, respectively. The curves are drawn by joining the points $(x, P(x))$ by straight lines for several selected values of x and the corresponding values of $P(x)$, where $P(x)$ is the probability of rejection of H_0 when the true model is $g(\ell, k) = 4 + x\frac{l}{n} + x\frac{k}{n}$. It can be seen that in the case of constant model, $\mathcal{CM}_{n\times n}$ test is as good as F test, since they have the same empirical power. In contrast to $\mathcal{CM}_{n\times n}$ and $F_{n\times n}$ tests, the $\mathcal{KS}_{n\times n}$ test performs slightly lower power. However, when $x = 0$ the power of all tests achieve the values of α as they should be. In general the power of the tests increases as the model moves away from H_0.

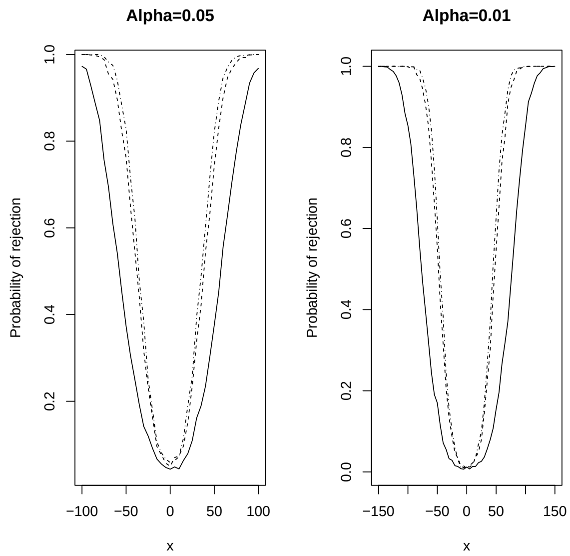

Figure 2: The graphs of the empirical power function of the $\mathcal{KS}_{n_1\times n_2}$ (solid line), $\mathcal{CM}_{n_1\times n_2}$ (dashed) and the F (dot-dashed) tests for first-order model. Sample size $= 75 \times 75$ and replication number $= 1500$, where $\varepsilon(\ell/n, k/n) \sim N(0, 2)$.

Figure 2 represents the empirical power function for testing the hypotheses defined in the second scenario. The observations for this simulation are generated based on the localized model

$$Y_{\ell k} = \frac{3 + \frac{3l}{n} - \frac{4k}{n} + x(\frac{l}{n})^2 + x(\frac{k}{n})^2 + x\frac{lk}{n^2}}{\sqrt{n^2-3}} + \varepsilon_{\ell k},$$

for $1 \leq \ell, k \leq n,\ n \geq 1$, where x varies in $[-100, 100]$ for $\alpha = 0.05$ and in $[-150, 150]$ for $\alpha = 0.01$. The model shows that when $x = 0$, the observations are from H_0, otherwise they are from H_1. In both situations we generate the random errors from the distribution $N(0, 2)$. Based on Figure 2, it can be seen that both $\mathcal{CM}_{n\times n}$ and F test in the second example show the same performance since they have almost the same power. As in the first scenario, the power of the $\mathcal{KS}_{n\times n}$ test is much lower than those of the $\mathcal{CM}_{n\times n}$ and F_n. This means that $\mathcal{CM}_n$ and F_n tests have better ability in detecting the alternative. When the value of the parameter x is set to zero, then all tests attain the probability of rejection α.

5 Application

In this section we study the application of the proposed method in the field of agriculture. We consider a real data provided by [27] consisting of the measurement of the rate of growth of corn plants observed over 21×16 lattice points running from South to North and from West to East, see Figure 3. The dimension of the lattice is $0.75m \times 0.75m$ spreads over the rectangular experimental region of dimension $[0, 12m] \times [0, 15.75m]$. Our aim is to build a regression model representing the functional relationship between the coordinate of the position of the plants on the experimental region and the rate of growth measured in cm/day. All plants have been given the same treatment until they grow up. By this model we can further access the fertility distribution of the land where the corn are planted. If a constant model is significance, then it can be stated that the land has a uniform fertility level.

Figure 3: The state of the 3 weeks old corn plants.

Preliminary diagnostic using the Kolmogorov-Smirnov goodness-of-fit test for testing the normality of the population results in the following quantities $ks = 0.0439$, and $p - value = 0.5$. A large $p - value$ indicates that there is a strong evidence that a normal distribution model is plausible for describing the probability distribution model for the rate of growth of the corn plants. Hence our test method is applicable to the data.

Furthermore, the three dimensional drop-line scatter plot of the rate of growth with respect to the coordinate $(x, y) \in [0, 12] \times [0, 15.75]$ is exhibited in Figure 4 showing a conjecture that a first-order model with positive slopes along both x and y axis seems to be reasonable for representing the model.

In the first step we test the hypothesis

$$H_0 : constant\ model\ holds\ true,$$

based on the CUSUM process of the recursive as

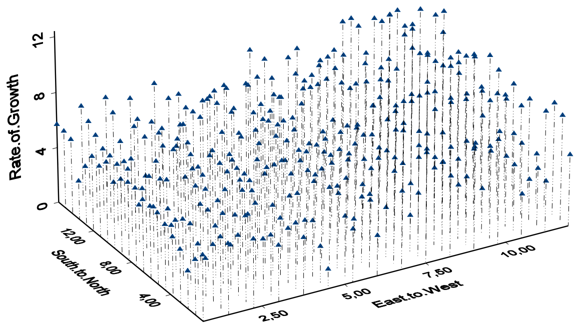

Figure 4: The drop line scatter plot of the rate of growth of corn plants.

well as ordinary residuals. The computation results of all statistics corresponding with their p-values are presented in Table 1. The quantities denoted by $\mathcal{OKS}_{21\times16}$ and $\mathcal{OCM}_{21\times16}$ stand respectively for the Kolmogorov-Smirnov and Cramér-von Mises type statistics based on CUSUM process of the ordinary least squares residuals, defined in [18, 31]. The extremely small p-values of all proposed tests lead us to the conclusion of the rejection of H_0 for all tests. This means that constant model does not fit well to the sample independent to which test is used. Next we consider the hypothesis:

$$H_0 : first - order\ model\ is\ a\ valid\ model.$$

To this conjecture, the p-values of all tests show relatively large values. Only F test rejects H_0 for all α larger than 26.60%. However this value is too risky for rejecting H_0 when H_0 is actually true. Therefore we insist on the conclusion that first-order polynomial is plausible to the rate of growth model.

Table 1: The critical values and the approximated p-values of the $\mathcal{KS}_{n_1\times n_2}$, $\mathcal{CM}_{n_1\times n_2}$ and F tests for the rate of growth of corn plants.

	Constant		First-order	
Test Statistic	Data	p-value	Data	p-value
$\mathcal{KS}_{21\times16}$	6.00	0.00	1.50	0.99
$\mathcal{CM}_{21\times16}$	6.05	0.00	0.20	1.00
$\mathcal{OKS}_{21\times16}$	3.48	0.00	0.79	0.45
$\mathcal{OCM}_{21\times16}$	3.30	0.00	0.08	0.36
$F_{21\times16}$	40.35	0.00	1.32	0.27

The least squares estimate for the parameters of the model gives the fitted model as follows:

$$\widehat{y}(t,s) = 0.24424 + 0.01328t + 0.00504s,$$

for $(t,s) \in [0,12] \times [0,15.75]$. The graph of the fitted model is presented in Figure 5.

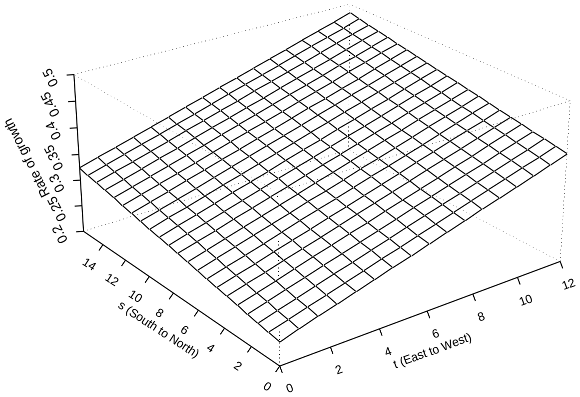

Figure 5: The graph of the fitted model of the rate of growth of corn plants.

There are several interpretations can be given to this fitted model. First, this model can help the practitioner in predicting the rate of growth of corn plan in every coordinate point on the experimental region. Second, by the positive values of the slope, the fertility level of the land increases as the position moves away from the origin $(0,0)$. In other world, the fertility level is not uniformly distributed over the region.

6 Conclusion

The validity of an assumed linear regression model for spatial observations can be tested asymptotically based on the partial sums of the recursive residuals. Traditionally, recursive residual approach was used for times series design, in this paper we extend this method for triangular array design points. In contrast to testing based on the partial sums of the least squares residuals, the most important advantage of testing based on the partial sums processes of the recursive residuals is that the limit process does not depend on the assumed model. It is given by the set-indexed Brownian sheet, so that the critical value of the test statistics can be computed analytically. The simulation results show that among the three types of tests, the Kolmogorv-Smirnov type has the lowest power. However all tests perform the similar behaviors when H_0 is true.

In the future, we extend the study to the problem of model validity check for multivariate observations based on the moving sums as well as the partial sums of the recursive residuals when the vector of observations does not follow normal distribution model and the design is not really a regular lattice. Survey for the partial sums of multivariate ordinary least squares residuals can be found in [25, 24, 26]. To the knowledge of the authors the test procedure under such a design strategy has been not yet investigated in the literatures. It is also interesting to extend the consideration to the case of spatial regression with second-order stationary observations.

Appendix

Proof of Proposition 2: Without loss of generality we consider the case of $j_1 < j_1'$ and $j_2 = j_2'$. Other cases can be handled analogously. Then there exists a positive integer k, such that $j_1' = j_1 + k \le n_1$. By the definition we get

$$\begin{aligned}
&\mathbf{a}_{j_1j_2}^{\top}\mathbf{a}_{j_1'j_2'} = \frac{1}{\sqrt{d_{j_1j_2}d_{j_1'j_2'}}}\\
&(-\mathbf{f}^{\top}(t_{j_1j_2})\left(\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}\right)^{-1}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top},\\
&1,0,\ldots,0,0,0,\ldots,0)\\
&\times(-\mathbf{f}^{\top}(t_{j_1'j_2'})\left(\mathbf{X}_{j_1'-1j_2'}^{(n_1,n_2)\top}\mathbf{X}_{j_1'-1j_2'}^{(n_1,n_2)}\right)^{-1}\mathbf{X}_{j_1'-1j_2'}^{(n_1,n_2)\top},\\
&1,0,\ldots,0,\ldots,0)^{\top}\\
&= \frac{1}{\sqrt{d_{j_1j_2}d_{j_1+kj_2}}}\\
&\times(\mathbf{f}^{\top}(t_{j_1j_2})\left(\mathbf{X}_{j_1+k-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1+k-1j_2}^{(n_1,n_2)}\right)^{-1}\mathbf{f}(t_{j_1+kj_2})\\
&-\mathbf{f}^{\top}(t_{j_1+kj_2})\left(\mathbf{X}_{j_1+k-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1+k-1j_2}^{(n_1,n_2)}\right)^{-1}\mathbf{f}(t_{j_1j_2})\\
&+0+\cdots+0) = 0.
\end{aligned}$$

In the case where $j_1 = j_1'$ and $j_2 = j_2'$, we have

$$\begin{aligned}
&\frac{\mathbf{a}_{j_1,j_2}^{\top}\mathbf{a}_{j_1,j_2}}{d_{j_1j_2}}\\
= \quad &(\mathbf{f}^{\top}(t_{j_1j_2})\left(\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}\right)^{-1}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top},\\
&1,0,\ldots,0)\\
&\times(\mathbf{f}^{\top}(t_{j_1j_2})\left(\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}\right)^{-1}\mathbf{X}_{(j_1-1j_2)}^{(n_1,n_2)\top},\\
&1,0,\ldots,0)^{\top}\\
&= \mathbf{f}^{\top}(t_{j_1j_2})\left(\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}\right)^{-1}\mathbf{f}(t_{j_1j_2})+1\\
&= 1.
\end{aligned}$$

The proof of the proposition is established.

Proof of Theorem 3: First we show that the finite dimensional distribution of the sequence

$$V_{n_1n_2} := \frac{1}{\sigma\sqrt{n_1n_2-p}}\sum_{\ell=1}^{m}\delta_\ell S_{n_1n_2-p}(\mathbf{W}_{n\times n})(A_\ell)$$

for $m \geq 1$, $\delta_1,\ldots,\delta_m$ be arbitrary m constants and $A_1,\ldots,A_m$ be arbitrary convex subsets of $\mathbf{D}$, converges to that of Z_{P_0}. Then, we have

$$\begin{aligned}
&Var\left(V_{n_1n_2}\right) = \mathbf{E}\left(V_{n_1n_2}\right)^2 \\
&= \sum_{\ell=1}^{m}\sum_{k=1}^{m}\frac{\delta_\ell\delta_k}{n_1n_2-p}\sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2}\sum_{j_1'=1}^{n_1}\sum_{j_2'=1}^{n_2}\mathbf{1}_{A_\ell}(t_{j_1j_2}) \\
&\times \mathbf{a}_{j_1j_2}^\top\mathbf{a}_{j_1'j_2'}\mathbf{1}_{A_k}(t_{j_1'j_2'}) \\
&= \sum_{\ell=1}^{m}\sum_{k=1}^{m}\frac{\delta_\ell\delta_k}{n_1n_2-p}\sum_{(j_1,j_2)\in\mathbf{T}_{n_1n_2-p}}\mathbf{1}_{A_\ell\cap A_k}(t_{j_1j_2}) \\
&= \sum_{\ell=1}^{m}\sum_{k=1}^{m}\frac{\delta_\ell\delta_k}{n_1n_2-p}\sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2}\mathbf{1}_{A_\ell\cap A_k}(t_{j_1j_2}) + o(1).
\end{aligned}$$

Since p is fixed and $p \ll n_1n_2$, then the term converges to zero as $n_1, n_2 \to \infty$. We notice that for $n_1 \geq 1$ and $n_2 \geq 1$, the equidistance design corresponds to a discrete probability measure $P_{n_1\times n_2}$ on $(\mathbf{D}, \mathcal{B}(\mathbf{D}))$, defined by

$$P_{n_1\times n_2}(A) := \frac{1}{n_1n_2}\sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2}\mathbf{1}_A(\mathbf{t}_{j_1j_2}),\ A \in \mathcal{B}(\mathbf{D}),$$

having the property $P_{n_1\times n_2} \Rightarrow P_0$, for $n_1, n_2 \to \infty$. Hence, it holds

$$\begin{aligned}
&\lim_{n_1,n_2\to\infty} Var\left(\sum_{\ell=1}^{m}\delta_\ell S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2})(A_\ell)\right) \\
&= \lim_{n_1,n_2\to\infty}\sum_{\ell=1}^{m}\sum_{k=1}^{m}\delta_\ell\delta_k\frac{n_1n_2}{n_1n_2-p} \\
&\times \int_{A_\ell\cap A_k} P_{n_1\times n_2}(dx,dy) \\
&= \sum_{\ell=1}^{m}\sum_{k=1}^{m}\delta_\ell\delta_k P_0(A_\ell\cap A_k).
\end{aligned}$$

The last expression is the variance of $\sum_{\ell=1}^{m}\delta_\ell Z_{P_0}(A_\ell)$.

Lindeberg condition is fulfilled by the sequence $V_{n_1n_2}$. To this end we write $V_{n_1n_2}$ as

$$V_{n_1n_2} = \sum_{(j_1,j_2)\in\mathbf{T}_{n_1n_2-p}}\frac{1}{\sigma\sqrt{n_1n_2-p}}\gamma_{j_1j_2}w_{j_1j_2},$$

where $\gamma_{j_1j_2} := \sum_{\ell=1}^{m}\delta_\ell\mathbf{1}_{A_\ell}(t_{j_1,j_2})$. For $\epsilon \in (0,1)$, let $\mathcal{L}_{n_1n_2}(\epsilon)$ be a sequence of constants defined by

$$\begin{aligned}
&\mathcal{L}_{n_1n_2}(\epsilon) := \\
&\sum_{(j_1,j_2)}\mathbf{E}\left(\left(\frac{\gamma_{j_1j_2}w_{j_1j_2}}{\sigma\sqrt{n_1n_2-p}}\right)^2\mathbf{1}_{\left\{\left|\frac{\gamma_{j_1j_2}w_{j_1j_2}}{\sigma\sqrt{n_1n_2-p}}\right|\geq\epsilon\right\}}\right),
\end{aligned}$$

where the sum is taken over $\mathbf{T}_{n_1n_2-p}$. We show $\lim_{n_1,n_2\to\infty}\mathcal{L}_{n_1n_2}(\epsilon) = 0$, for every $\epsilon \in (0,1)$. For that let $M := \max\{\delta_\ell : 1 \leq \ell \leq m\}$. Then for every $(j_1,j_2) \in \mathbf{T}_{n_1n_2-p}$, $\gamma_{j_1j_2}^2 \leq m^2M^2$ and $|\gamma_{j_1j_2}| \leq m|M|$. Hence, for every $\epsilon \in (0,1)$, it holds

$$\begin{aligned}
&\mathcal{L}_{n_1,n_2}(\epsilon) \leq \sum_{(j_1,j_2)}\frac{(mM)^2}{\sigma^2(n_1n_2-p)} \\
&\mathbf{E}\left(w_{j_1j_2}^2\mathbf{1}_{\left\{|w_{j_1j_2}|\geq\frac{\epsilon\sigma\sqrt{n_1n_2-p}}{m|M|}\right\}}\right) \\
&\leq \frac{(mM)^2}{\sigma^2}\mathbf{E}\left(w_{11}^2\mathbf{1}_{\left\{|w_{11}|\geq\frac{\epsilon\sigma\sqrt{n_1n_2-p}}{m|M|}\right\}}\right).
\end{aligned}$$

Hence, by the bounded convergence theorem (cf. Corollary 2.3.13 in [3]) and by recalling the fact that the set $\{w_{j_1j_2} : (j_1,j_2) \in \mathbf{T}_{n_1n_2-p}\}$ are independent and identically distributed, we get

$$0 \leq \lim_{n_1,n_2\to\infty}\mathcal{L}_{n_1,n_2}(\epsilon) \leq 0.$$

Thus by Lindeberg-Feller central limit theorem ([3], p. 345), it can be concluded that the finite dimensional distributions of V_{n_1,n_2} converges to those of Z_{P_0}. The tightness of the process can be shown by applying the approach proposed in [1] and [19], finishing the proof.

Proof of Theorem 6: For $(j_1,j_2) \in \mathbf{T}_{n_1n_2-p}$, let $w_{j_1j_2}^1$ be the recursive residuals associated to the localized model (5). Then we get

$$\begin{aligned}
w_{j_1j_2}^1 &= \frac{(Y_{j_1j_2} - \mathbf{f}^\top(t_{j_1j_2})\widehat{\beta}_{j_1-1j_2}^{(n_1,n_2)})}{\sqrt{d_{j_1j_2}}} \\
&= \frac{\frac{g(t_{j_1j_2})}{\sqrt{n_1n_2-p}} + \varepsilon_{j_1j_2}}{\sqrt{d_{j_1j_2}}} - \mathbf{f}^\top(t_{j_1j_2}) \times \\
&\frac{\left(\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}\right)^{-1}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{Y}_{j_1-1j_2}^{(n_1,n_2)}}{\sqrt{d_{j_1j_2}}} \\
&= \frac{g(t_{j_1j_2})}{\sqrt{d_{j_1j_2}(n_1n_2-p)}} - \mathbf{f}^\top(t_{j_1j_2}) \times \\
&\frac{\left(\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}\right)^{-1}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}\mathbf{g}_{j_1-1j_2}^{(n_1,n_2)}}{\sqrt{d_{j_1j_2}(n_1n_2-p)}} + w_{j_1j_2},
\end{aligned}$$

where the last term is the recursive residual under H_0. Hence, by considering the linearity of the partial sums operator we get for every $A \in \mathcal{A}$,

$$\frac{1}{\sigma\sqrt{n_1n_2-p}}S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2}^1)(A)$$

$$
\begin{aligned}
&= \sum_{(j_1,j_2)\in \mathbf{T}_{n_1n_2-p}} \mathbf{1}_A(t_{j_1j_2}) \frac{g(t_{j_1j_2})}{\sigma(n_1n_2-p)\sqrt{d_{j_1j_2}}} \\
&- \sum_{(j_1,j_2)\in \mathbf{T}_{n_1n_2-p}} \mathbf{1}_A(t_{j_1j_2}) \mathbf{f}^\top(t_{j_1j_2}) \\
&\times \frac{\left(\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}\right)^{-1}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{g}_{j_1-1j_2}^{(n-1,n_2)}}{\sigma(n_1n_2-p)\sqrt{d_{j_1j_2}}} \\
&+ \frac{1}{\sigma\sqrt{n_1n_2-p}} S_{n_1n_2-p}(\mathbf{W}_{n_1\times n_2})(A).
\end{aligned}
$$

The first term on the right-hand side of the last equation can be re-written as

$$
\frac{1}{\sigma}\frac{n_1n_2}{(n_1n_2-p)}\left(\frac{1}{n_1n_2}\sum_{(j_1,j_2)} \mathbf{1}_A(t_{j_1j_2})\frac{g(t_{j_1j_2})}{\sqrt{d_{j_1j_2}}}\right),
$$

where the sum is over $\mathbf{T}_{n_1n_2-p}$. Furthermore, for every $(j_1,j_2)\in \mathbf{T}_{n^2-p}$, the constant d_{j_1,j_2} can be written by

$$
\begin{aligned}
&d_{j_1j_2} = 1 \\
&+ \frac{\mathbf{f}^\top(t_{j_1j_2})}{\sqrt{n_1n_2}}\left(\frac{\mathbf{X}_{(j_1-1j_2)}^{(n_1,n_2)}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}}{n_1n_2}\right)^{-1}\frac{\mathbf{f}(t_{j_1j_2})}{\sqrt{n_1n_2}},
\end{aligned}
$$

where for $(j_1,j_2) \in \mathbf{T}_{n^2-p}$ the assumption that $rank(\mathbf{X}^n_{(j_1-1,j_2)}) = p$ guarantees the existence of the invers

$$
\left(\frac{\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}}{n_1n_2}\right)^{-1}.
$$

Hence by the well known decomposition theorem (cf. Harville [15], p.), there exists a matrix $\mathbf{A}$, such that

$$
\left(\frac{\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}}{n_1n_2}\right)^{-1} = \mathbf{A}\mathbf{A}^\top.
$$

By applying Cauchy-Schwarz inequality (cf. Conway [11]) to the absolute value of the Euclidean inner product, we get

$$
|d_{j_1j_2}-1| \le \frac{1}{n_1n_2}\|\mathbf{f}^*(tj_1j_2)\|^2 < \infty
$$

for some vector $\mathbf{f}^*(t_{j_1j_2}) = \mathbf{A}^\top\mathbf{f}(t_{j_1j_2}) \in \mathcal{R}^p$, where $\|\cdot\|$ is the usual Euclidean norm on $\mathcal{R}^p$. Thus, for every $(j_1,j_2)\in \mathbf{T}_{n_1n_2-p}$, d_{j_1,j_2} converges to one, as $n_1,n_2\to\infty$. Since g has bounded variation on $\mathbf{D}$ and $P_{n_1\times n_2} \Rightarrow P_0$, then the fact that $p \ll n_1n_2$ implies

$$
\begin{aligned}
&\lim_{n_1,n_2\to\infty} \sum_{(j_1,j_2)\in \mathbf{T}_{n_1n_2-p}} \frac{\mathbf{1}_A(t_{j_1j_2})g(t_{j_1j_2})}{\sigma(n_1n_2-p)\sqrt{d_{j_1j_2}}} \\
&= \lim_{n_1,n_2\to\infty} \int_A \frac{(n_1n_2)g(x,y)P_{n_1\times n_2}(dx,dy)}{\sigma(n_1n_2-p)} \\
&= \frac{1}{\sigma}\int_A g(x,y)P_0(dx,dy).
\end{aligned}
$$

For the second term we have

$$
\begin{aligned}
&\sum_{(j_1,j_2)\in \mathbf{T}_{n_1n_2-p}} \mathbf{1}_A(t_{j_1j_2})\mathbf{f}^\top(t_{j_1j_2}) \\
&\times \frac{\left(\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}\right)^{-1}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}\mathbf{g}_{j_1-1j_2}^{(n_1,n_2)}}{\sigma(n_1n_2-p)\sqrt{d_{j_1j_2}}} \\
&= \frac{1}{n^2}\sum_{(j_1,j_2)\in \mathbf{T}_{n_1n_2-p}} \frac{(n_1n_2)\mathbf{1}_A(t_{j_1j_2})\mathbf{f}^\top(t_{j_1j_2})}{\sigma(n_1n_2-p)\sqrt{d_{j_1j_2}}} \\
&\times \left(\frac{\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)\top}\mathbf{X}_{j_1-1j_2}^{(n_1,n_2)}}{n_1n_2}\right)^{-1}\frac{\mathbf{X}_{j_1-1j_2}^{(n-1,n_2)\top}\mathbf{g}_{j_1-1j_2}^{(n_1,n_2)}}{n_1n_2} \\
&= \frac{n_1n_2}{\sigma(n_1n_2-p)}\int_A \frac{\mathbf{f}^\top(u,v)}{\sqrt{d_{j_1j_2}}} \times \\
&\left(\int_{B_{u,v}} (f_k(x,y)f_\ell(x,y))_{k,\ell=1}^{p,p}\, dP_{n_1\times n_2}\right)^{-1} \\
&\times \left(\int_{B_{u,v}} f_k(x,y)g(x,y)dP_{n_1\times n_2}\right)_{k=1}^{p} dP_{n_1\times n_2}.
\end{aligned}
$$

Hence, by applying the similar argument as in the preceding result, we get

$$
\begin{aligned}
&\lim_{n_1,n_2\to\infty} \frac{n_1n_2}{\sigma(n_1n_2-p)}\int_A \frac{\mathbf{f}^\top(u,v)}{\sqrt{d_{j_1j_2}}} \\
&\times \left(\int_{B_{u,v}} (f_k(x,y)f_\ell(x,y))_{k,\ell=1}^{p,p}\, dP_{n_1\times n_2}\right)^{-1} \\
&\times \left(\int_{B_{u,v}} f_k(x,y)g(x,y)dP_{n_1\times n_2}\right)_{k=1}^{p} dP_{n_1\times n_2} \\
&= \int_A \mathbf{f}^\top(u,v)\mathbf{G}^{-1}(u,v)(\mathbf{f}g)(u,v)P_0(du,dv),
\end{aligned}
$$

finishing the proof.

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References:

[1] K.S Alexander and R. Pyke, A uniform central limit theorem for set-indexed partial-sum processes with finite variance, *Ann. Probab.* 14, 1986, pp. 582–597.

[2] S.F. Arnold, *The Theory of Linear Models and Multivarite Analysis*, John Wiley & Sons, Inc., New York 1981.

[3] K.B. Athreya and S.N. Lahiri, *Measure Theory and Probability Theory,* Springer Science+Business Media, LLC., New York 2006.

[4] P. Billingsley, *Convergence of Probability Measures (2nd ed.),* John Wiley & Sons, New York 1999.

[5] W. Bischoff, A functional central limit theorem for regression models, *Ann. Stat.* 6, 1998, pp. 1398–1410.

[6] W. Bischoff, The structure of residual partial sums limit processes of linear regression models, *Theory of Stochastic Processes*, 2, 2002, pp. 23–28.

[7] W. Bischoff and W. Somayasa, The limit of the partial sums process of spatial least squares residuals, *J. Multivariate Analysis,* 100, 2009, pp. 2167–2177.

[8] R.L. Brown, J. Durbin and J.M. Evans, Teqhniques for testing the regression relationships over time, *J. R. Statist. Soc. B* 37(2), 1975, pp. 149–192.

[9] G. Casella and R.L. Berger, *Statistical Inference,* Wadsworth, Inc., California 1990.

[10] C.S.J. Chu, K. Hornik and C.M. Kuan, MOSUM test for parameter constancy, *Biometrika* 82, 1995, pp. 603–617.

[11] J.B. Conway, *A Course in Functional Analysis (2nd ed.),* Springer–Verlag, Heidelberg 1990.

[12] J.M. Dufour, Recursive stability analysis of linear regression relationships, *J. Econometrics* 19, 1982, pp. 31–76.

[13] P. Gaenssler, On recent development in the theory of set-indexed processes (A unified approach to empirical and partial-sum processes) in *Asymptotic Statistics*, Springer Verlag, Berlin 1993.

[14] P. Gaensler, P. Molnér and D. Rost, On Continuity and strict increase of the CDF for the supfunctional of a Gaussian process with applications to Statistics, *Result. Math.* 51, 2007, pp. 51-60.

[15] D.A. Harville, *Matrix Algebra from a Statistician's Perspective*, Springer Verlag, New York 1997.

[16] W. Kramér, W. Ploberger and R. Alt, Testing for structural change in dynamic models, *Econometrica* 56, 1988, pp. 1355–1369.

[17] M. Lifshits, *Lectures on Gaussian Processes,* Springer Briefs in Mathematics, Springer–Verlag, Berlin 2012.

[18] I.B. MacNeill and V.K. Jandhyala, *Change-point methods for spatial data, Multivariate Environmental Statistics eds. by G.P. Patil and C.R. Rao,* Elsevier Science Publishers B.V., 1993, pp. 298–306.

[19] R. Pyke, A uniform central limit theorem for partial sum processes indexed by sets, *Ann. Probab.* 79, 1983, pp. 219–240.

[20] G.A.F. Seber and A.J. Lee, *Linear Regression Analysis (2nd ed.),* John Wiley & Sons, New Jersey 2003.

[21] P.K. Sen, Invarianc principle for recursive residuals, *Ann. Stat.* 10(1), 1982, pp. 307–312.

[22] G.R Shorack and J.A Wellner, *Empirical Process with Application to Statistics,* John Wiley & Sons, New York-Chichester-Brisbane-Toronto-Singapore 1986.

[23] W. Somayasa, Ruslan, E. Cahyono, L.O. Engkoimani, Cheking adequateness of spatial regressions using set-indexed partial sums technique, *Fareast J. Math. Sci.* 96(8), 2015, pp. 933–966.

[24] W. Somayasa and G.N.A. Wibawa, Asymptotic model-check for multivariate spatial regression with correlated responses *Fareast J. Math. Sci.* 98(5), 2015, pp. 613–939.

[25] W. Somayasa, G.N.A. Wibawa, L. Hamimu and L.O. Ngkoimani, Asymptotic theory in model diagnostic for general multivariate spatial regression, *Int. J. Mathematics and Mathematical Sciences* 2016, pp. 1–16.

[26] W. Somayasa and H. Budiman, Testing the mean im multivariate regression using set-indexed Gaussian white noise, *Statistics and Its Interface* 11, 2018, pp. 61–77.

[27] W. Somayasa and G.N.A. Wibawa, *Spatial Modelling and Goodness-of-Fit-Using Moving Window Approach (Research Report),* Halu Oleo University, Kendari 2018.

[28] W. Stute, Nonparametric model checks for regression, *Ann. Statist.*, 25, 1997, pp. 613–641.

[29] W. Stute, W.G. Manteiga and M.P. Quindimil, Bootstrap approximations in model checks for regression, *J. Amer. Statist. Assoc.* 93(441), 1998, pp. 141149.

[30] W. Stute, W.L Xu and L.X. Zhu, Model diagnostic for parametric regression in high-dimensional space, *Biometrika,* 95(2), 2008, pp. 451467.

[31] L. Xie and I.B. MacNeill, Spatial residual processes and boundary detection, *South African Statist. J.* 4, 2006, pp. 33–53.