Integrating the Gaussian through differentiable topological manifolds

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Abstract: - We introduce a new method for solving differential equations through differentiable manifolds. The Gaussian integral is used as an illustrative example, simply because it has been declared in many texts as unsolvable through other mathematical procedures. Our argument is that the notion of whether an integral could be un-integrable, or a differential equation unsolvable, depends on the space one is working in.

Key-Words: - Differential equations, Fibre bundles, Quotient spaces, Equivalent classes

1 Introduction

Our approach to the Gaussian is hinged on the modification of the method of variation parameters. This is achieved through differentiable topological manifolds. In its current state, as treated in metric spaces, the method is rigged with contradictions. For example, some quantities that begin as constants transform into functions without any logical connection offered, and none can be deduced.

Such inconsistencies clear automatically when inspected through differentiable topological manifolds. In metric spaces, if there is no distance between two points, then the points are identical; like when two billiard balls overlap. For our method to work, we extend the notion of the zero distance to two points that are in each other's immediate neighbourhood; this allows for the two balls to step out of each other, and simply touch.

The Gaussian integral, or simply the Gaussian,

$$\int^{z} \frac{1}{\sqrt{(2\,\pi)}}\, e^{-\frac{s^2}{2}}\, d\, s, \tag{1}$$

is said to be un-integrable because a function $F(z)$, called the anti-derivative, cannot be found, such that

$$\frac{d\,F}{d\,z} = \frac{1}{\sqrt{(2\,\pi)}}\, e^{-\frac{z^2}{2}}, \tag{2}$$

This anti-derivative notion derives from the fundamental theorem of calculus, credited to Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716). This was before Leonhard Euler (1707-1783) appeared on the scene, and introduced topology, though today topology is credited to the likes of Johann Benedict Listing (1808 - 1882), for coining the term *topology* and largely to Felix Hausdorff (1868 - 1942), of Hausdorff topology.

The extension of topology to the study of differential equations, is an ongoing process, as evident from [1], [2], [3], [4] and [5]. Our contribution is to infinitely differentiable solutions, in particular those requiring the fundamental theorem be presented in the form

$$V_{f,z} = \frac{1}{\sqrt{(2\,\pi)}}\, e^{-\frac{z^2}{2}}, \tag{3}$$

for some mapping f, with the velocity operator $V_{f,z} \epsilon\, TP$, where

$$TP = \{V_{f,z}\ | f\colon \mathbb{R} \to X\}, \tag{4}$$

a tangent vector space, with $\mathbb{R}$ being the set of real numbers.

In brief, what we are proposing is that integrals that are un-integrable on the metric space, may not necessarily be so on differentiable topological spaces. Quotient spaces arise naturally in higher derivatives. As such, we will address

$$\frac{d^2\,F}{d\,z^2} = -\frac{z}{\sqrt{(2\,\pi)}}\,e^{-\frac{z^2}{2}}, \qquad (5)$$

or simply

$$\frac{d^2\,\psi}{d\,x^2} = -x\,\frac{d\,\psi}{d\,x}\,, \qquad (6)$$

after replacing $F(z)$ with $\psi(x)$.

Section 2 is on our theoretical basis of our proposed integration procedure. The theory is built from variation of parameters method, discussed briefly in subsection 2.1. Some integrals may not evaluate in elementary spaces because of the exclusion the function f pointed out in (3). This is discussed further in subsection 2.2. A connection to quotient spaces is shown in subsection 2.3.

The theory is tested with an application on the Gaussian integral in section 3.

2 The theoretical basis

2.1 The variation of parameters method

Consider the differential equation

$$a\,\frac{d^2\,y}{d\,x^2} + b\,\frac{d\,y}{d\,x} + c\,y = f(x), \qquad (7)$$

where a,b and c are constants.

To solve this equation through the method of variation of parameters, one has to first determine the fundamental solution $\{y_1, y_2\}$, which is the solution of the homogeneous case, resulting from setting

$$f(x) = 0\,, \qquad (8)$$

The complementary solution is then

$$y_C = C_1 y_1 + C_2 y_2\,, \qquad (9)$$

where C_1 and C_2 are constants. To get particular solution, the result is expressed as

$$y_P = v_1 y_1 + v_2 y_2\,, \qquad (10)$$

where the functions v_1 and v_2 have taken the places of the constants.

What we take from here to the next subsection and beyond, is the two assumptions which we have highlighted with the bullets •. The zero $f(x) = 0$ observed here, in our cases are multiple zeros, eventually leading to (38) and (53). The constants C_1, C_2 and the functions v_1,v_2 lead to (16) and (17) which generate (39) and (47).

Differentiable Topological Manifolds

A topological space $M = (X, J_X)$ is a set X with a topology J_X. For it to be a differentiable topological manifold, or simply a differentiable manifold $DM = (X, J_X, A)$, an atlas A is required in addition.

To build a differentiable manifold, we start with a metrizable topological space $M = (X, J_X)$. That is, a set X with the topology J_X, with the properties that

- it is Hausdorff, meaning that any two points x_p and x_q in this space can be isolated in their own open set $U_P = \{x_p\}$ and $U_q = \{x_q\}$, and these sets can never intersect. That is, $U_p \cap U_q = \emptyset$.
- It is second countable.
- It has para-compactness.

These three properties ensures that the space is metrizable. We also require a local or simply Euclidean space IR^N, with the usual topology J_{IR^N}.

Next is a homeomorphism f from (X, J_X) to $(\mathbb{R}^N, J_{\mathbb{R}^N})$. That is, the mapping f has an inverse f^{-1}. It is bi-continuous, onto, and one to one.

We now consider the topologies $(U_p, J_X|_{U_p})$ and $(U_q, J_X|_{U_q})$. That is, the topology of X restricted to U_p and U_q. A mapping ψ_p, if it exists, then maps the space $(U_p, J_X|_{U_p})$ into the Euclidean space $(\mathbb{R}^N, J_{\mathbb{R}^N}|_{\psi_p(U_p)})$. Similarly, ψ_q maps $(U_q, J_X|_{U_q})$ into the Euclidean space $(\mathbb{R}^N, J_{\mathbb{R}^N}|_{\psi_q(U_q)})$.

If this this mappings are homeomorphisms, then the set A, with

$$A = \{(U_p, \psi_p), (U_q, \psi_q)\} \tag{11}$$

is called an atlas, with ψ_p, ψ_q called coordinates.

Our interest is in one of the charts mapping equivalence classes. Therefore,

$$A = \{([U_p], [\psi_p]), (U_q, \psi_q)\} \tag{12}$$

Similarly, for manifolds in derivatives of ψ, we get the atlases

$$A^{(i)} = \left\{\left([U_p], [\psi_p^{(i)}]\right), \left(U_q, \psi_q^{(i)}\right)\right\}. \tag{13}$$

2.2 Transmission mappings

The mapping from $(\mathbb{R}, J_{\mathbb{R}}|_{\mathbb{R}\psi([U_P])})$ to $(\mathbb{R}, J_{\mathbb{R}}|_{\mathbb{R}\psi(U_q)})$, having stepped down from $\mathbb{R}^N$ to $\mathbb{R}$. It is given by

$$\psi_p(\psi_q^{-1}(\psi_q([U_p]))), \tag{14}$$

and it is called a transition mapping. Its inverse is

$$\psi_q(\psi_p^{-1}(\psi_p(U_q))). \tag{15}$$

We are interested in case where $[U_p]$ and U_q overlap, so that there is a point x in the neighbourhood of both p and q such that

$$[\psi[x]] = \psi(x). \tag{16}$$

The transmission mappings in derivative spaces lead to

$$\frac{d^n[\psi[x]]}{dx^n} = \frac{d^n\psi(x)}{dx^n}, \tag{17}$$

for $n = 1,2,3,\cdots$.

2.3 Tangent spaces

As indicated earlier, tangent spaces assist in establishing a function f, that allows for results to be projected onto the metric space. A tangent space is a set

$$\mathrm{TP} = \{V_{\gamma,P} \mid \gamma : \mathbb{R} \to X\}, \tag{18}$$

such that

$$V_{\gamma,P} f = (f \circ \gamma^{-1})[\gamma(\tau_0)], \tag{19}$$

where $f \in C^\infty(X)$, $V_{\gamma,P} : C^\infty(\mathrm{M}) \to \mathbb{R}$, $\gamma(\tau_0) = P$.

The tangent space TP has the basis vectors $\{\partial X^i\}$. Any vector then can be represented in terms of it, so that

$$X = \xi^i \frac{\partial}{\partial X^i}\bigg|_P. \tag{20}$$

That is $X \in \mathrm{T}_P X = \mathrm{T}_P \mathrm{M}$.

2.4 Cotangent spaces

A tangent space is a vector space, and where there is one there should also be a co-vector space, hence the cotangent space. It is the set of all maps in the tangent space to $\mathbb{R}$. That is,

$$\omega : \mathrm{T}_p X \to \mathbb{R}, \tag{21}$$

with ω being an element of the cotangent space. The symbol $(df)_p$ represents a co-vector acting on mapping f at P. A cotangent space, therefore, is

$$T P^* = \{(df)_p | f \in C^\infty(X)\}, \quad (22)$$

and it is a vectors space, and is the dual of TP.

The basis vectors of a cotangent space requires that

$$(d\omega^j)_p \left(\frac{\partial}{\partial x^i}\right)|_p = \delta_i^j, \quad (23)$$

so that the basis of $T P^*$ is

$$\left\{\frac{\partial}{\partial x^i}\right\}|_p. \quad (24)$$

Therefore an element ω of $T P^*$ can be written

$$\omega = \omega_i(dx^i)|_p. \quad (25)$$

At any point of a differentiable manifold (X, J_X, A), with a multiple points P, Q, R, S P,Q there exists tangent spaces T_PX, T_QX, T_RX, T_SX, which are the tangent bundles. This can be extended to cotangent bundles $T_{P^*}X, T_{Q^*}X, T_{R^*}X, T_{S^*}X$.

2.5 Quotient spaces

Consider the general ordinary differential equation

$$f = f\left[x, \psi, \frac{d\psi}{dx}, \frac{d^2\psi}{dx^2}, \frac{d^3\psi}{dx^3}, \cdots\right] \quad (26)$$

with

$$\psi : X \to Y. \quad (27)$$

A set

$$S = \{x_0, x_1, x_2, \cdots\} \subset X, \quad (28)$$

such that

$$x_i = P(x_j) = x_j + 2\pi k_s \quad (29)$$

where k_s is an integer, is called an equivalence class. This leads to a Quotient space $\mathbb{R} / \sim$. It is the set of all equivalent classes in $\mathbb{R}$, and is given by

$$\mathbb{R} / \sim = \{[x_0],[x_1],[x_2],\cdots\}. \quad (30)$$

It is a differentiable topological space. In our study, the image of this set, is also an equivalence class

$\{[\psi(x_0)],[\psi(x_1)],[\psi(x_2)],\cdots\}$, (31) as such there is a homomorphism, and it extends to the derivative spaces

$\{[\psi^{(i)}(x_0)],[\psi^{(i)}(x_1)],[\psi^{(i)}(x_2)],\cdots\}$, (32) for $i = 1,2,3,\cdots$.

3 Integrating the Gaussian (6)

We begin with the trivial integration of (5). The first integration gives

$$\int \frac{d^2F}{dx^2}dx = -\int x\frac{dF}{dx}dx - D_1. \quad (33)$$

That is,

$$\frac{dF}{dx} = -\int x\frac{dF}{dx}dx - D_1, \quad (34)$$

where D_1 is a constant of integration. The second integration:

$$\begin{aligned} \int F'dx &= -\int\left(\int xF'(x)dx\right)dx \\ & n - D_1x - D_2, \end{aligned} \quad (35)$$

where D_2 is also a constant of integration. That is,

$$F = -\int\left(\int xF'(x)dx\right)dx - D_1x - D_2, \quad (36)$$

or

$$\psi + \int\left(\int x\psi'dx\right)dx + D_1x + D_2 = 0. \quad (37)$$

We turn to quotient spaces to resolve the remaining integral, by generating equivalence classes, guided by the theory developed in the previous section.

3.1 The equivalence classes

From expression (6), it is clear that ψ is smooth, i.e., infinitely differentiable. Note again that ψ and $\frac{d^2\psi}{dx^2}$ share the same infinite zeroes, and that this set of zeroes constitutes an equivalence class, in both x and ψ, and its derivatives. Hence,

$$\frac{[\psi]''}{[\psi]} = \frac{[\psi]^{(3)}}{[\psi]'}, \tag{38}$$

which has the solution

$$[\psi] = \frac{[a]\sin(i[\omega]([x]+[\phi]))}{i[\omega]}. \tag{39}$$

To evaluate determine a and ω through (37) and (54) we note that

$$\frac{d[\psi]}{d[x]} = \frac{d\psi}{dx}. \tag{40}$$

Hence,

$$[\psi] + \int\left(\int [x][\psi]' d[x]\right) d[x] + D_1[x] = -D_2, \tag{41}$$

so that

$$\frac{[a]\sin(i[\omega]([x]+[\phi]))}{i[\omega]} + \int\left(\int [x]\left(\frac{[a]\sin(i[\omega]([x]+[\phi]))}{i[\omega]}\right) d[x]\right) d[x] + D_1[x] + D_2 = 0. \tag{42}$$

It gives

$$\frac{[a]\sin(i[\omega]([x]+[\phi]))}{i[\omega]} - \frac{2[a]\cos(i[\omega]([x]+[\phi]))}{(i[\omega])^4} - \frac{i[\omega][a]\sin(i[\omega]([x]+[\phi]))}{(i[\omega])^4} + D_1[x] + D_2 = 0, \tag{43}$$

so that

$$[a] = \frac{D_1[x]+D_2}{\frac{2\cos(i[\omega]\theta)+i[\omega]\sin(i[\omega]\theta)-(i[\omega])^3\sin(i[\omega]\theta)}{(i[\omega])^4}}, \tag{44}$$

with $\theta = [x] + [\varphi]$.

Differentiating (41) with respect to $[x]$ gives

$$\frac{d[\psi]}{d[x]} + \int [x]\frac{d[\psi]}{d[x]} d[x] + D_1 = 0. \tag{45}$$

Substituting

$$\frac{d[\psi]}{d[x]} = [a]\cos(i[\omega]([x]+[\phi])), \tag{46}$$

transforms it into the differential

$$[a]\cos(i[\omega]\theta)d[x] + d([a]\cos(i[\omega]\theta)) - d(D_1). \tag{47}$$

Hence,

$$[a]\cos(i[\omega]\theta) + \left(\frac{[a]}{i[\omega]}\right)\frac{\sin(i[\omega]\theta) - i[\omega][x]\cos(i[\omega]\theta)}{(i[\omega])^2} + D_1 = 0. \tag{48}$$

Solving for a and ω leads to

$$\omega = \sqrt{S_1 - S_2}, \tag{49}$$

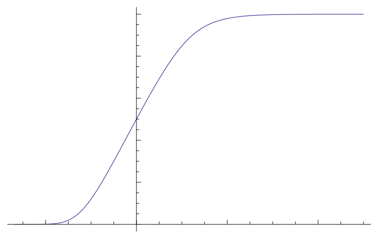

Figure 1: A plot of the Gaussian (1) obtained through Numerical techniques.

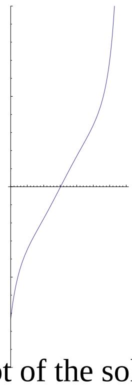

Figure 2: A plot of the solution in (55).

with

$$S_1 = \frac{D_2 ix}{2(-iD_2 - iD_1 x)} + \frac{D_1 ix^2}{2(-iD_2 - iD_1 x)} \quad (50)$$

and

$$S_2 = \frac{\sqrt{\chi_1}}{2(-iD_2 - iD_1 x)}, \quad (51)$$

where

$$\chi_1 = 8iD_1(-iD_2 - iD_1 x) + \left(-D_2 ix - D_1 ix^2\right)^2,$$

so that

$$a = \frac{1}{2}(D_2 + D_1 x)(S_1 - S_2). \quad (52)$$

The easier expression follows from

$$\frac{[\psi]^{(3)}}{[\psi']} = \frac{[\psi]^{(4)}}{[\psi]''}, \quad (53)$$

which has the solution

$$[\psi] = -\frac{[a]\cos(i[\omega]([x] + [\phi]))}{[\omega]^2} + F_0, \quad (54)$$

with a constant F_0. It leads to

$$\begin{aligned} \psi &= \\ &- \frac{4e^{-2\pi x^2}\sqrt{\pi}x \sec\left[\sqrt{\pi}x\sqrt{\chi_2}\right]}{-2e^{-\pi x^2} + 4e^{-\pi x^2}\pi x^2} \\ &+ \frac{2e^{-2\pi x^2}\sqrt{\pi}x}{-2e^{-\pi x^2} + 4e^{-\pi x^2}\pi x^2}, \end{aligned} \quad (55)$$

where $\chi_2 = e^{\pi x^2}\left(-2e^{-\pi x^2} + 4e^{-\pi x^2}\pi x^2\right)$, and is plotted in Figure 2, and compares favourably with the numerically result in Figure 1.

4 Discussion and conclusion

This paper was on a new method for solving differential equations through quadrature, and it was tested successfully on the Gaussian integral, posed as a differential equations. We have addressed this integral before, using a technique we based on Sophus Lie (1842-1899)[6]'s symmetry group theoretical methods in [7], and later on [8]. The work we did here is an improvement, in that in our previous study we missed the function *f*, which displaces Figure 2 to Figure 1.

Similarly infinitely differentiable equation like (6) arise in a number of our models. For example, in fluid mechanics through similarity analyses of equations such as the Euler and the Navier-Stokes equations, and are still unsolved.

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