Behavior of Continuous Two-contours System

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Abstract: - A deterministic continuous dynamical system is considered. This system contains two contours. The length of the ith contour equals c_i, i=1,2. There is a moving segment (cluster) on each contour. The length of the cluster, located on the ith contour, equals l_i, i=1,2. If a cluster moves without delays, then the velocity of the cluster is equal to 1. There is a common point (node) of the contours. Clusters cannot cross the node simultaneously, and therefore delays of clusters occur. A set of repeating system states is called a spectral cycle. Spectral cycles and values of average velocities of clusters have been found. The system belongs to a class of contour networks. This class of dynamical systems has been introduced and studied by A.P. Buslaev.

Key-Words: - Dynamical system, Contour networks, Self-organization, Average velocity of particles

1 Introduction

In [1] (Nagel, Schreckenberg, 1992), a transport model has been introduced. This model is a cellular automaton. In the model, particles move on an infinite or closed sequence of cells in accordance with given rules. In general case, Nagel-Schreckenberg model and its versions are too complicated for analytic research and were studied by simulation.

Analytic results for a simple version of Nagel-Schreckenberg model have been obtained in [2] (Belitzky, Ferrary, 2005) (a preprint of this paper has been published in 1999). These results have been obtained under assumption that, at any step, each particle moves onto a cell forward if the cell ahead is vacant and does not move if this cell is occupied. It is noted in [2] that the model is equivalent to the elementary cellular automaton 184 (CA 184) in classification of Wolfram, [3]. Results, similar to results of [2], have been obtained independently in [4] (Blank, 2000). In accordance with results of [2], [4], all particles move after some moment at every time for any initial state if the density of particles (the number of particles divided by the number of cells) is not more than 1/2, then the average velocity of particles (the average number of transitions of a particle per time unit) equals $(1-\rho)/\rho$, where is the density. In [5] (Gray, Griffeath, 2001), analytical results have been obtained for somewhat more general traffic model. In this model, a particle moves from the cell i to a vacant cell $i+1$ behead of particle with probability depending on states of cells $i-1$, $i+2$ (cells are numbered in the direction of movement). In [5], the behavior of particles has been studied for some particular cases, and, in the general case, the formula for velocity has not obtained. In [6] (Kanai, 2008), a formula has been obtained for a stochastic version of the traffic model. In this system, at every step, each particle moves onto a cell forward if the cell ahead is vacant. Some generalizations of results of [3–5] have been obtained in [7] (Blank, 2010). In general case, the system state space, studied in [7], is continuous. In a particular case, the system is equivalent to the discrete system that is considered in [7].

A two-dimensional traffic model with a toroidal supporter (BML traffic model) has been introduced in [8] (Biham, Middelton, Levin, 1992). In this model, particle move in accordance with a rule, similar to the rule CA 184. Conditions of self-organization (every particle moves after some moment) and collapse (no particle moves after some moment) tained for BML model have been obtained in [9] (D'Souza, 2005), [10] (Angel, Holroyd, Martin, [11] (Austin, Benjamini, 2006),

In paper [12] (Bugaev, Buslaev, Kozlov, Yashina, 2011), the concept of a cluster traffic model with cluster movement has been introduced. In the discrete version of the cluster model, each contour contains a given number of cells. There are clusters of particles on each contour. Particles of each cluster occupy neighboring cells. All particles of each cluster move simultaneously in accordance with the rule of the cellular automaton 240. Clusters of the same contours can merge. Clusters can be delayed at nodes. In the continuous version of the model a cluster is a segment moving on the contour with constant velocity in a given directionr.

The concept of a contour network has been introduced in [13] (Buslaev contour networks). The supporter of a contour network is a system of contours with a network structure. Particles (clusters) move on contours in accordance with some rules. Some limitations are imposed on the system. These limitations allow us to study the system analytically.

In [14] (Buslaev, Fomina, Tatashev, Yashina, 2018) the concept of spectrum of a contour system has been introduced for a deterministic dynamical system with a infinite set of states. In such system, a sequence of states repeated periodically from some moment. This sequence of states is called a spectral cycle. The system, considered in [14], is a closed chain of contours. Particles move on each contour in accordance with the rule of the cellular automaton 240 (CA 240). There is one cluster on each contour. The spectrum of the system is a set of spectral cycles and corresponding values of clusters velocities.

In [15] (Buslaev, Tatashev, 2017) and [16] (Buslaev, Tatashev, 2018), a discrete two-contours system was considered. In this system, particles move on contours in accordance with the rule of CA 184 or CA 240. In [15], the following generalization was also considered. The supporter of the system contains N contours. There is one common point of the contours. In [15, 16] theorems have been proved for different versions of movement rules. In [15, 16], mainly, systems with contours of the same length were considered. For a system, containing contours of different lengths, in [15] conditions of self-organization (system resulting in a state of free movement from any initial state) have been obtained.

In this paper, a pair of contours is studied such that the lengths of the contours are different. We study a discrete and a continuous version of the system. There is a moving cluster on each contour. There exists a common point of contours (node). Delays occurs at the node. We have been found spectral cycles and obtained formulas for velocities of clusters.

The initial state of the system is given. This state should be admissible.

We study the spectrum of two-contours system with contours of different lengths. is equal to c_i, $i = 1,2$. There is a moving cluster on each contour. The length of the cluster, moving on the contour i, is equal to l_i, $i = 1,2$. There is a common point of the contours. This point is called the node. More than one cluster cannot cross the node simultaneously. A cluster stops if it comes to the node at time such that at this time the other cluster crosses the node. If clusters come to the node simultaneously, then only the cluster 1, located on the left, moves (left-priority rule). A set of states such that these states are repeated periodically is called a spectral cycle. We say that the system has the property of self-organization if the system results in the state of free movement over a finite time. In this paper, we have obtained conditions of self-organization. We have proved that, if the condition of self-organization does not hold, then, depending on c_1, c_2, l_1, l_2, there are one or two spectral cycles. Formulas for average velocities of clusters have been obtained.

2 System description

We consider a discrete dynamical system containing two *contours 1 and 2*, figure 1. The length of the contour i is equal to c_i, $i = 1,2$. There is a moving segment (*cluster* i) on the contour i ($i = 1,2$). The contours have a common point (*node*). At any time, each cluster moves in the direction of movement. The cluster i passes the distance c_i per c_i time units (the velocity equals 1) if there is no delay, $i = 1,2$. The length of the cluster i is equal to $0 < l_i < c_i$, $i = 1,2$. The coordinate system $[0, c_i]$ is introduced on the contour i, $i = 1,2$. The direction of the coordinate axis is reverse to the direction of movement by modulo c_i, $i = 1,2$. The coordinate of the node is equal to 0 for both contours. The state of the system is the vector $(\alpha_1(t), \alpha_2(t))$, where $\alpha_i(t)$ is the coordinate of the leading point of the cluster i, $i = 1,2$. We say that *the cluster i covers the node* at time t if $c_i - l_i < \alpha_i(t) < 1$, $i = 1,2$. We say that *the cluster i is at the node* at time t if $\alpha_i(t) = 0$, $i = 1,2$. *Admissible* states of the system are only the states such that no more than cluster covers the node. If at time t, the cluster i, is at the node, and the other cluster covers the node, then a delay of the cluster i occurs, $i = 1,2$, figure 2. If both clusters are at the node, i.e., the system is in the state (0,0), then a *conflict* occurs, figure 3. In the case of a conflict, only one cluster moves in accordance with a *conflict resolution rule*. We suppose that the conflict resolution rule is the following. If a conflict occurs at the initial time $t = 0$, or a conflict occurs at time $t = t_0$ and there are no delays at the time interval $[0, t)$, then, at time $t = t_0$, the cluster 1 moves. Assume that a conflict occurs at the time $t = t_0$, i.e., $\alpha_1(t_0) = \alpha_2(t_0) = 0$. Suppose the latest delay in the time interval $[0, t_0)$ occurs at time

$t = t_1$, and, at time t_1, the cluster i_0, $i_0 = 1{,}2$, does not move. Then, at time t_0, the same cluster i_0 does not move and the other cluster moves. The *initial* state of the system is given. This state should be admissible.

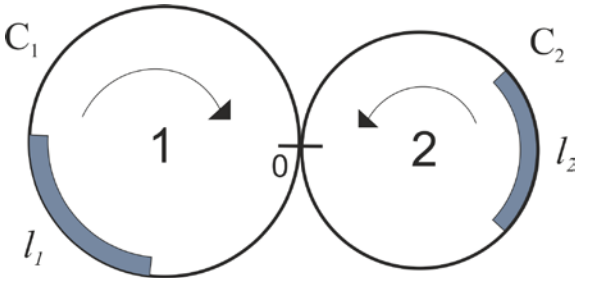

Figure 1: Two-contours system

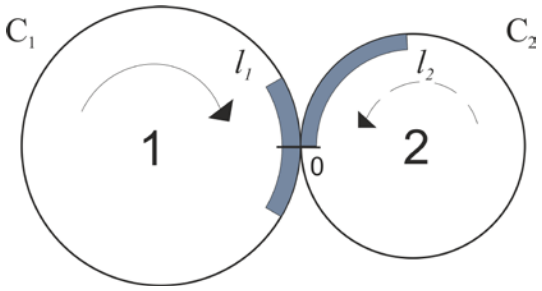

Figure 2: Delay of the cluster 2

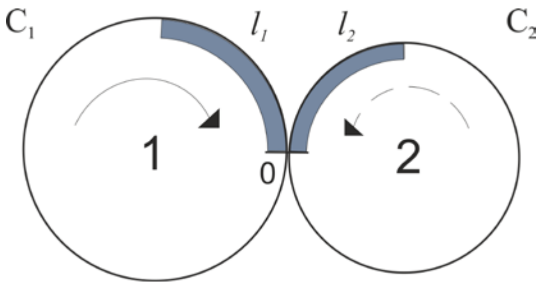

Figure 3: Conflict

3 Concepts of spectral cycle, velocity, free movement, and self-organization

A spectral cycle is a cyclic trajectory in the system state space. Suppose T is the *period* of the cycle; H_i is the distance that the cluster i passes on a spectral cycle for the period, $i = 1{,}2$. The number $v_i = H_i/T$ is called the *average velocity* of the cluster i on the spectral cycle, $i = 1{,}2$. The system is *in a state of free movement* at moment t_0 if both clusters move at any moment $t \geq t_0$. If the system results in a state of free movement, then the velocity of clusters is equal to 1.

The property of the system to result in a state of free movement over a finite time from any initial state is called *self-organization.*

4 Information from the theory of linear Diophantine equations

Let us consider the equation

$$ax + by + c = 0, \qquad (1)$$

where a and b are integer numbers not equal to 0, and c is an integer number.

There exist integer solutions of the system if and only if the greatest divisor of numbers a and b divides the number c, [20].

Let $x = x_0$, $y = y_0$ be solutions of (1). Then all solutions of (1) are

$$x = x_0 - bt, \qquad y = y_0 + at, \qquad t = \pm 1, \pm 2.$$

5 Conditions of self-organization

We shall prove theorems, which give conditions of free movement.

Lemma 1 *At time t_0, a delay of the cluster i begins or continues, or a conflict occurs, if and only if*

$$\alpha_i(t_0) = 0, \qquad (2)$$

$$0 \leq \alpha_j(t_0) < l_j,\ j \neq i. \qquad (3)$$

Proof: If (2), (3) holds then the cluster i is at the node, and the cluster j covers the node or this cluster is at the node. Therefore the cluster i does not move or a conflict of two clusters occurs.

On the other hand, if, at time t_0, the cluster j covers the node, then $0 \leq \alpha_j(t_0) < l_j$, and, if the cluster j is at the node, then $\alpha_j = 0$. In both these cases, (3) holds. Besides, if the cluster i is at the node, then (2) holds. Lemma 1 has been proved.

We have proved the following theorems, which give conditions of free movement.

Theorem 2 *If $\frac{c_2}{c_1}$ is an irrational number, then the system does not result in a state of free movement from any initial state.*

Proof: Since the value of $\frac{c_2}{c_1}$ is irrational, we have that for any $\varepsilon > 0$ and t_0 there exists $t \geq t_0$ such that for any $t \geq t_0$ the inequality $|\alpha_2(t) - \alpha_1(t)| < \varepsilon$ holds. From this and Lemma 1, Theorem 2 follows.

Theorem 3 *Suppose $\frac{c_2}{c_1}$ is a rational number, and $d = GCD(c_1, c_2)$ is the greatest common divisor of c_1 and c_2, i.e., d is the greatest number such that there exist natural numbers k_1, k_2 satisfying the condition $k_1 d = c_1$, $k_2 d = c_2$; then the following is true. If the condition*

$l_1 + l_2 \le d,$ (4)
holds, then the system results in the state of free movement over a infinite time from any initial state (self-organization). If the condition
$l_1 + l_2 > d,$ (5)
holds, then the system does not result in the state of free movement over a infinite time from any initial state.

Proof: Denote by k_i and m_i values c_i/d, and l_i/d respectively, $i = 1,2$. Assume that at time t_0 the system is in the state $(\alpha_1(t_0), \alpha_2(t_0))$. If non-negative integer numbers x, y, and a real number b satisfying the condition,
$k_1 x - k_2 y = \frac{\alpha_2(t_0) - \alpha_1(t_0) + b}{d}, \ -l_1 < b < l_2,$ (6)
then, either, at time $t_0 + k_1 x + \alpha_1$, a delay of the cluster 1 begins, or, at time $t = k_2 x + \alpha_2$, a delay of cluster 2 begins. Indeed, if no delays occur in the time interval $[t_0, t_0) + k_1 x + \alpha_1(t_0)$ and $b \ge 0$, then the cluster 1 is at the node, and $0 \le \alpha_2(t_0) = c_2 - b < l_2$. Therefore, at time $[t_0 + k_1 x + \alpha_1(t_0)$, a delay of the cluster 1 begins or a conflict occurs. If no delays occur in the time interval $[t_0 + k_1 x + \alpha_1(t_0) + |b|$ and $b < 0$, then the cluster 2 is at the node, and $0 \le \alpha_1(t_0) = c_1 - |b|$. Therefore, at time $[t_0 + k_1 x + \alpha_1(t_0) + |b|$, a delay of the cluster 2. Since d is the greatest common divisor of l_1 and l_2, then k_1, k_2 are co-prime numbers. Therefore, in accordance with theory of linear Diophantine equations, there exists a solution of equation (6) for any natural number in the right side of (6). If (5) holds, then there exists a value of b $(-l_1 < b < l_2)$ such that the right side of (6) is integer. Thus the inequality (4) is a necessary condition of self-organization.
Assume that (4) holds. If at time t_0, a delay of the cluster 1 ends, then, at time t_0, the system is in the state $(0, c_2 - l_2)$, and, if, at time t_1, a new delay begins, then
$k_1 x - k_2 y = \frac{c_2 - l_2 + b}{d},$ (7)
where b is a non-negative integer number such that $-l_1 < b < l_2$. Since d is a divisor of c_2, and (4) holds, then the right part of (7) cannot be integer for $-l_1 < b < l_2$.
If at time t_0, a delay of the cluster 2 ends. Then, at time t_0, the system is in the state $(c_1 - l_1, 0)$, and, if, at time t_1, a new delay begins, then
$k_2 y - k_1 x = \frac{c_1 - l_1 + b}{d},$ (8)
where b is a non-negative integer number such that $-l_2 < b < l_1$. Since d is a divisor of c_1, and (4) holds, then the right part of (8) cannot be integer for $-l_2 < b < l_1$. Thus (4) is a sufficient condition of self-organization.

6 Optional parameters

Assume the condition of self-organization (Theorems 2, 3) does not hold.
Let us introduce optional parameters g_1, g_2, b_1, b_2 and describe a way to calculate these parameters.
We shall see that, if the condition of self-organization does not hold, there is one spectral cycle or there are two spectral cycles or there are two spectral cycles depending on values g_1, g_2, b_1, b_2. *The average velocities of clusters depend on these parameters.*
Assume that A is the set of system states such that one cluster does not move in the state; A_i is the set of system states such that the cluster i does not move, $i = 1,2$. In accordance with Lemma 1, the set A_1 contains states $(0, \alpha_2)$, $c_2 - l_2 < \alpha_2 < 1$. The set A_2 contains states $\alpha_1, 0)$, $c_1 - l_1 < \alpha_1 < 1$. We have $A = A_1 \cup A_2 \cup (0,0)$.

Lemma 4 *If (5) holds, then there exists a moment such the system is at the state* $(0, c_2 - l_2)$ *or at the state* $(c_1 - l_1, 0)$.

Proof: Since the condition (5) holds, the system results in the states set A from any initial state over a finite time, and comes out a finite time, and comes out of the set A through the state $(0, c_2 - l_2)$ or $(c_1 - l_1, 0)$.

Lemma 5 *Let non-negative integer numbers* x, y *satisfy the condition*
$c_2 - l_2 < c_1 x - c_2 y < c_2.$ (9)
Suppose the system is in the state $(0, c_2 - l_2)$ at time t_0, and no delays occur in the time interval $[t_0, t_0 + c_1 x)$; then, at the time $t_0 + c_1 x$, a delay of the cluster 1 begins.

Proof If the condition of Lemma 5 holds, then, at time $t_0 + c_1 x$, the system is in the state $(0, c_2 - b_1)$, where $0 < b_1 = c_1 x - c_2 y - c_2 + l_2 < l_2$. From this, Lemma 5 follows.

Lemma 6 *Assume that non-negative integer numbers* x, y *satisfy the condition*
$c_2 - l_2 - l_1 < c_1 x - c_2 y < c_2 - l_2.$ (10)
Suppose the system is at the state $(0, c_2 - l_2)$ at time t_0, and no delays occur in the time interval $[t_0, t_0 + c_1 x + |b_1|)$, where $-l_1 < b_1 = c_1 x - c_2 y - c_2 + l_2 < 0$, then, at the time $t_0 + c_1 x + |b_1|$, a delay of the cluster 2 begins.

Proof If the condition of Lemma 6 holds, then, at time $t_0 + c_1 x + |b_1|$, the system is in the state $(c_1 - |b_1|, 0)$, $0 < b_1 < l_2$. Therefore, at time t_0, the cluster 2 is at the node, and the cluster 1 covers the node. From this, Lemma 6 follows.

Lemma 7 *Assume that non-negative integer numbers x, y satisfy the condition*

$$c_1x - c_2y = c_2 - l_2. \quad (11)$$

Suppose the system is at the state $(0, c_2 - l_2)$ at time t_0, and no delays occur in the time interval $[t_0, t_0 + c_1x)$; then, at the time $t_0 + c_1x_1$, a conflict occurs.

Proof If the condition of Lemma 7 holds, then, at time $t_0 + c_1x$, the system is in the state $(0,0)$. Therefore, at time t_0, the system is in the state $(0,0)$. Therefore, at time $t_0 + c_1x$, both clusters are at the node. Thus a conflict occurs.

Lemma 8 *Assume that the following holds. The system is in the state $(0, c_2 - l_2)$ at time t_0. There are no delays occur in time interval $[t_0, t_1)$. Suppose, at time t_1, the cluster 2 moves through the node, and a delay of the cluster 1 begins; then $t_1 = t_0 + c_1x_1^+$, and, at time t_0, the system is in the state $(0, c_2 - b_1^+$, where x_1^+ is the minimum non-negative integer number x such that there exists a non-negative integer number $y = y_1^+$ satisfying the condition*

$$c_2 - l_2 < c_1x_1^+ - c_2y < c_2, \quad (12)$$

and

$$b_1^+ = c_1x_1^+ - c_2y_1^+ - c_2 + l_2. \quad (13)$$

Proof. If conditions of the the lemma holds, then $\alpha_1(t_1) = 0$ and $c_2 - l_2 < \alpha_2(t_1) < 1$. From this, taking into account that both clusters move at any moment belonging time interval $[t_0, t_1)$, we get Lemma 8.

Suppose

$$g_1^+ = c_1x_1^+. \quad (14)$$

If there exist no non-negative integer numbers x, y satisfying (12), then we assume that $g_1^+ = \infty$.

Lemma 9 *Let the following hold. The system is in the state $(0, c_2 - l_2)$ at time t_0. There are no delays occur in time interval $[t_0, t_1)$. Suppose, at time t_1, the cluster 1 moves through the node, and a delay of the cluster 2 begins; then $t_1 = t_0 + c_1x_1^- + |b_1^-|$, and, at time t_1, the system is in the state $(c_1 - |b_1^-|, 0)$ where x_1^- is the minimum non-negative integer value of x such that there exists a non-negative integer number $y = y_1^-$ satisfying the condition*

$$c_2 - l_2 - l_1 < c_1x_1^- - c_2y_1^- < c_2 - l_2, \quad (15)$$

and

$$b_1^- = c_1x_1^- - c_2y_1^- - c_2 + l_2. \quad (16)$$

Proof. If conditions of the the lemma holds, then $c_1 - l_1 < \alpha_1(t_1) < 1$ and $\alpha_2(t_0) = 1$. From this, taking into account that both clusters move at any moment belonging time interval $[t_0, t_1)$, we get Lemma 9.

Suppose

$$g_1^- = c_1x_1^- + |b_1^-|. \quad (17)$$

If there exist no non-negative integer numbers x, y satisfying (15), then we assume that $g_1^- = \infty$.

Lemma 10 *Assume that the following holds. The system is in the state $(0, c_2 - l_2)$ at time t_0. There are no delays occur in time interval $[t_0, t_1)$. Suppose, at time t_1, the system is in the state $(0,0)$; then $t_1 = t_0 + c_1x_1^0$, where x_1^0 is the minimum non-negative integer number x such that there exists a non-negative integer number $y = y_1^0$ satisfying the condition*

$$c_1x_1^0 - c_2y_1^0 = c_2 - l_2. \quad (18)$$

Proof. Taking into account that, at time t_1, the system is in the state $(0,0)$ and both clusters move at any moment belonging time interval $[t_0, t_1)$, we get Lemma 8. Assume that

$$g_1^0 = c_1x_1^0. \quad (19)$$

If there exist no non-negative integer numbers x, y satisfying (18), then we assume that $g_1^0 = \infty$.

Lemma 11 *Let non-negative integer numbers x, y satisfy the condition*

$$c_1 - l_1 < c_2y - c_1x < c_1. \quad (20)$$

Suppose the system is at the state $(c_1 - l_1, 0)$ at time t_0, and no delays occur in the time interval $[t_0, t_0 + c_2y)$; then, at time $t_0 + c_2y$, a delay of the cluster 2 occurs.

Proof If the condition of the lemma holds, then, at time $t_0 + c_2y$, the system is in the state $(0, c_1 - b_2)$, where $0 < b_2 = c_2y - c_1x - c_1 + l_1 < l_1$. From this, Lemma 11 follows.

Lemma 12 *Assume that non-negative integer numbers x, y satisfy the condition*

$$c_1 - l_1 - l_2 < c_2y - c_1x < c_1 - l_1. \quad (21)$$

Suppose the system is at the state $(c_1 - l_1, 0)$ at time t_0, and no delays occur in the time interval $[t_0, t_0 + c_2y + |b_2|)$, where $b_2 = c_2y - c_1x - c_1 + l_1$; then, at the time $t_0 + c_1x + |b_2|$, a delay of the cluster 1 begins.

Proof If the condition of Lemma 12 holds, then, at time $t_0 + c_2y + |b_2|$, the system is in the state $(0, c_2 - |b_2|)$, $-l_2 < b_2 < 0$. Therefore, at time t_0, the cluster 1 is at the node, and the cluster 2 covers the node. From this, Lemma 12 follows.

Lemma 13 *Assume that non-negative integer numbers x, y satisfy the condition*

$$c_2y - c_1x = c_1 - l_1. \quad (22)$$

Suppose the system is at the state $(c_1 - l_1, 0)$ at time t_0, and no delays occur in the time interval $[t_0, t_0 + c_2 y)$; then, at the time $t_0 + c_2 y$, a conflict occurs.

Proof If the condition of Lemma 13 holds, then, at time $t_0 + c_2 y$, the system is in the state (0,0). Therefore, at time t_0, the system is in the state (0,0). Therefore, at time $t_0 + c_2 y$, both clusters are at the node. Thus a conflict occurs.

Lemma 14 *Assume that the following holds. The system is in the state $(c_1 - l_1, 0)$ at time t_0. There are no delays occur in time interval $[t_0, t_1)$. Suppose, at time t_1, the cluster 1 moves through the node, and a delay of the cluster 2 begins; then $t_1 = t_0 + c_2 y_0$, and, at time t_1, the system is in the state $(c_1 - b_2^+, 0)$, where y_2^+ is the minimum non-negative integer number y^+ such that there exists a non-negative integer number $x = x_2^+$ satisfying the condition*

$$c_1 - l_1 < c_2 y_2^+ - c_1 x_2^+ < c_1, \tag{23}$$

and

$$b_2^+ = c_2 y_2^+ c_1 x_2^+ - c_1 + l_1. \tag{24}$$

Proof. If conditions of the lemma holds, then $c_1 - l_1 < \alpha_1(t_1) < 1$ and $\alpha_2(t_1) = 0$. From this, taking into account that both clusters move at any moment belonging time interval $[t_0, t_1)$, we get Lemma 14. Suppose

$$g_2^+ = c_2 y_2^+. \tag{25}$$

If there exist no non-negative integer numbers x, y satisfying (23), then we assume that $g_2^+ = \infty$.

Lemma 15 *Let the following hold. The system is in the state $(c_1 - l_1, 0)$ at time t_0. There are no delays occur in time interval $[t_0, t_1)$. Suppose, at time t_1, the cluster 2 moves through the node, and a delay of the cluster 1 begins; then $t_1 = c_2 x_2^- + |b_2^-|$, and, at time t_1, the system is in the state $(c_1 - |b_2^-|, 0)$ where y_2^- is the minimum non-negative integer value of y such that there exists a non-negative integer number $x = x_2^-$ satisfying the condition*

$$c_1 - l_1 - l_2 < c_2 y_2^- c_1 x_2^- < c_1 - l_1, \tag{26}$$

and

$$b_2^- = c_2 y_2^- - c_1 x_2^- c_1 + l_1. \tag{27}$$

Proof. If conditions of the the lemma holds, then $c_1 - l_1 < \alpha_1(t_0) < 1$ and $\alpha_2(t_0) = 0$. From this, taking into account that both clusters move at any moment belonging time interval $[t_0, t_1)$, we get Lemma 15.

Assume that

$$g_2^- = c_2 y_0 + |b_2^-|. \tag{28}$$

If there exist no non-negative integer numbers x, y satisfying (26), then we assume that $g_2^- = \infty$.

Lemma 16 *Assume that the following holds. The system is in the state $(c_1 - l_1, 0)$ at time t_0. There are no delays occur in time interval $[t_0, t_1)$. Suppose, at time t_1, the system is in the state (0,0); then $t_0 = c_1 x_0^0$, and, at time t_0, the system is in the state $(c_1 - b_2^+, 0)$ where x_0^0 is the minimum non-negative integer number x such that there exists a non-negative integer number $y = y_0^0$ satisfying the condition*

$$c_2 y_2^+ - c_1 x_2^- = c_1 - l_1. \tag{29}$$

The proof of Lemma 16 is the same as the proof of Lemma 10. Assume that

$$g_2^0 = c_2 y_2^0. \tag{30}$$

If there exist no non-negative integer numbers x, y satisfying (29), then we assume that $g_2^0 = \infty$. Suppose

$$g_1 = \min g_1^+, g_1^-, g_1^0. \tag{31}$$

$$g_2 = \min g_2^+, g_2^-, g_2^0. \tag{32}$$

Denote by b_1 the value b_1^+, b_1^-, or b_1^0 if $g_1 = g_1^+$, $g_2 = g_2^-$ or $g_2 = g_2^0$ respectively. Denote by b_2 the value b_2^+, b_2^-, or b_2^0 if $g_2 = g_2^+$, $g_2 = g_2^-$ or $g_2 = g_2^0$ respectively.

7 The behavior of system in the case of no self-organization

We shall prove theorems about spectral cycles and average velocities of particles.

Theorem 17 *Suppose the condition of self-organization (4) does not hold and inequalities $b_1 \geq 0$, $b_2 < 0$ hold; then there exist a unique spectral cycle, and this cycle contains the state $(0, c_2 - l_2)$. The period of the cycle is equal to $g_1 + l_2 - b_1$. Average velocities of clusters are equal to*

$$v_1 = \frac{g_1}{g_1 + l_2 - b_1}, \ v_2 = 1. \tag{33}$$

Proof: Since the condition of self-organization does not hold, the system does not result in the state of free movement. Hence the system results in a state, belonging the set A_1, over a finite time, and, after this, in the state $(0, c_2 - l_2)$, or the system results in a state, belonging the set A_2, over a finite time, and, after this, in the state $(c_1 - l_1, 0)$. In accordance with Lemmas 5, 7, from the state $(0, c_2 - l_2)$, the system results again in a state, belonging to the set $A_1 \cup (0,0)$. In accordance Lemma 12, from the state $(c_1 - l_1, 0)$, the system results in a state, belonging to the set A_1. Therefore there is a unique spectral cycle, and this spectral cycle contains the state $(0, c_2 - l_2)$. On the spectral cycle, the system is in the states, not belonging to the set A (both clusters move in these states), during

g_1 time units and the system is in states, belonging to the set A_1 (only the cluster 2 moves in these states), during $l_2 - b_1$ time units. From this, Theorem 16 follows.

Example 1. Assume that $c_1 = 3,\ l_1 = 2,\ c_2 = 5,$ $l_2 = 3$. The greatest common divisor of c_1 and c_2 is equal to $d = 1$, and therefore the inequality (5) holds. Hence the condition of self-organization does not hold.

Let us find the values g_1^+, b_1^+.

Assume that $x = 0$. Then there exists no non-negative integer value of y satisfying (9).

Assume that $x = 1$. Then the number $y = y_1^+ = 0$ satisfies (9). Therefore, in accordance with (13), (14), $x_1^+ = 1, y_1^+ = 0,$

$$g_1 += 3,\ b_1^+ = 1. \quad (34)$$

Let us find the value $b_1^-, g_1 -$.

If $x = 0$ or $x = 1$, then there exists no non-negative integer value y satisfying (10).

If $x = 2$, then the number $y = 1$ satisfies (10). We have $x_1^- = 2$, $y_1^- = 1$, and, in accordance with (16), (17), $x_1^- = 2, y_1^- = 1,$

$$b_1^- = -1,\ g_1^- = 7. \quad (35)$$

Let us find the value g_1^0.

If $x = 0,\ x = 1,\ x = 2,$ or $x = 3$, then there exists no non-negative integer number y satisfying the equation (11). If $x = x_1^0 = 4$, then the number $y = y_1^0 = 2$ satisfies (11), and, in accordance with (19),

$$g_1^0 = c_1 c_1^0 = 12. \quad (36)$$

In accordance with (31),

$$g_1 = g_1^+ = 3,\ b_1 = b_1^+ = 1. \quad (37)$$

Let us find g_2^+ and b_2^+.

Assume that $y = 0$. Then there exists no non-negative integer value of y satisfying (20).

Assume that $y = 1$. Then the number $x = x_2^+ = 1$ satisfies (20). Therefore, $x_2^+ = 1,\ y_2^+ = 1$, and, in accordance with (24), (25),

$$b_2^+ = 1,\ g_2 += 5. \quad (38)$$

Let us find the value b_2^-, g_2^-.

Suppose $y = 0$; then the number $x = x_2^+ = 0$ satisfies (21). We have $x_2^- = 0,\ y_2^- = 0$, and, in accordance with (27), (28),

$$b_2^- = -1,\ g_2^- = 1. \quad (39)$$

Let us find the value g_2^0, b_2^0.

If $y = 0$ or $y = 1$, then there exists no non-negative integer number y satisfying the equation (29). If $y = y_2^0 = 2$, then the number $x = x_2^0 = 3$ satisfies (29), and, in accordance with (30),

$$g_2^0 = 10. \quad (40)$$

In accordance with (32), (38)–(40), we have

$$g_2 = 1,\ b_2 = -1. \quad (41)$$

In accordance with (37), (41), the conditions of Theorem 16 hold. In accordance with (33), (34), (37),

$$v_1 = \frac{3}{5}, \qquad v_2 = 1.$$

Theorem 18 *Suppose the condition of self-organization (3) does not hold and inequalities* $b_1 < 0,\ b_2 \geq 0$; *then there exists a unique spectral cycle, and this cycle contains the state* $(c_1 - l_1, 0)$. *The period of this spectral cycle equals* $g_2 + l_1 - b_2$. *Velocities of clusters are equal to*

$$v_1 = 1, \qquad v_2 = \frac{g_2}{g_2 + l_1 - b_2}.$$

We get Theorem 18 from Theorem 17 if we renumber the contours.

Theorem 19 *Suppose inequalities* $b_1 < 0,\ b_2 < 0$ *hold. Then there a unique spectral cycle. The spectral contains the states* $(0, c_2 - l_2)$ *and* $(c_1 - l_1, 0)$. *The period of the spectral cycle equals* $g_1 + g_2 + l_1 + l_2 - |b_1| - |b_2|$. *Velocities of clusters are equal to*

$$v_1 = 1 - \frac{l_2 - |b_2|}{g_1 + g_2 + l_1 + l_2 - |b_1| - |b_2|}, \quad (42)$$

$$v_2 = 1 - \frac{l_1 - |b_1|}{g_1 + g_2 + l_1 + l_2 - |b_1| - |b_2|}, \quad (43)$$

Proof. Since the condition of self-organization does not hold, the system does not result in the state of free movement. Hence the system results in a state, belonging the set A_1, over a finite time, and, after this, in the state $(0, c_2 - l_2)$, or the system results in a state, belonging the set A_2, over a finite time, and, after this, in the state $(c_1 - l_1, 0)$. In accordance Lemma 6, from the state $(0, c_2 - l_2)$, over a finite time, the system results in a state belonging to A_1. In accordance Lemmas 10, from the state $(c_1 - l_1, 0)$, over a finite time, the system results in a state belonging to A_2. Therefore there is a unique spectral cycle, and this spectral cycle contains the states $(0, c_2 - l_2)$ and $(c_1 - l_1, 0)$. On the spectral cycle, the system is in the states, not belonging to the set A (both clusters move in these states), during $g_1 + g_2$ time units, the system is in states, belonging to the set A_1 (only the cluster 2 moves in these states), during $l_2 - |b_2|$ time units, and the system is in states, belonging to the belonging to the set A_2 (only the cluster 1 moves in these states), during $l_1 - |b_1|$ time units. From this, Theorem 18 follows.

Example 2. Assume that $c_1 = \sqrt{2},\ l_1 = 1,\ c_2 = 1,\ l_2 = \frac{1}{2}$. The value c_2/c_1 is irrational, and therefore, in accordance with

Theorem 2, the condition of self-organization, does not hold.
Let us find the values g_1^+, b_1^+.
Assume that $x = 0$ or $x = 0$ or $x = 1$. Then there exists no non-negative integer value of y satisfying (9).
Assume that $x = 2$. Then the number $y = y_1^+ = 1$ satisfies (9). Therefore, $x_1^+ = 2,\ y_1^+ = 2$ and, in accordance with (13), (14),

$$g_1^+ = 2\sqrt{2},\ b_1^+ = 2\sqrt{2} - \frac{5}{2}. \quad (44)$$

Let us find the value b_1^-, g_1^-.
If $x = x_1^- = 0$, then the number $y = y_1^- = 0$ satisfies (10). Therefore, $x_1^+ = 0,\ y_1^+ = 0$ and, in accordance with (16), (17),

$$g_1^- = \frac{1}{2},\ b_1^- = -\frac{1}{2}. \quad (45)$$

There are no integer numbers x, y satisfying (11). Therefore, we have

$$g_1^0 = \infty. \quad (46)$$

In accordance with (32), (44)–(46), we have

$$g_1 = g_1^- = \frac{1}{2},\ b_1 = b_1^- = -\frac{1}{2}. \quad (47)$$

Let us find g_2^+ and b_2^+.
Assume that $y = 0$. Then there exists no non-negative integer value of y satisfying (20).
Assume that $y = 1$. Then the number $x = x_2^+ = 0$ satisfies (20). Therefore, $x_2^+ = 0,\ y_2^+ = 1$, and, in accordance with (24), (25),

$$b_2^+ = 2 - \sqrt{2},\ g_2{+}{=}\ 1. \quad (48)$$

Let us find the value b_2^-, g_2^-.
The values $x = x_2^+ = 0$ and $y = y_2^+ = 0$ satisfy (21). Therefore, in accordance with (27), (28),

$$b_2^- = -\sqrt{2} + 1,\ g_2^- = \sqrt{2} - 1.$$

There are no integer numbers x, y satisfying (11). Therefore, we have

$$g_2^0 = \infty. \quad (49)$$

In accordance with (32), (48)–(50), we have

$$g_2 = g_2^- = \frac{1}{2},\ b_2 = b_2^- = -\frac{1}{2}. \quad (50)$$

In accordance with (47), (51), the conditions of Theorem 16 hold. In accordance with (39), (47), (51),

$$v_1 = \frac{2}{3}, \qquad v_2 = \frac{2\sqrt{2}}{3}.$$

Theorem 20 *Suppose inequalities $b_1 \geq 0,\ b_2 \geq 0$ hold. Then there are two spectral cycles. One of these cycles contains the state $(0, c_2 - l_2)$. The period of this cycle equals $g_1 + l_2 - b_1$. On this cycle, velocities of clusters are equal to*

$$v_1 = \frac{g_1}{g_1 + l_2 - b_1}, \qquad v_2 = 1.$$

The other cycle contains the state $(c_1 - l_1, 0)$. The period of this cycle equals $g_2 + l_1 - b_2$. On this cycle, velocities of clusters are equal to

$$v_1 = 1, v_2 = \frac{g_2}{g_2 + l_1 - b_2}.$$

Proof. Depending on the initial state, the system results in a state of the set A_1, and after this, the system results in the state $(0, c_2 - l_2)$, or a state, belonging to the set A_2, over a finite time, and, after this, in the state $(c_1 - l_1, 0)$. If the system is in the state of the set A_2, then, returning to the set A, the system results in states belonging $A_2 \cup (0; 0)$. In this case, over a finite time, the system will not be in states belonging A_1, and the system will be only in the state belonging to the spectral cycle such that this spectral cycle contains the state $(0, c_2 - l_2)$. In this case, on the spectral cycle, the system is in the states, not belonging to the set A (both clusters move in these states), during g_1 time units and the system is in states, belonging to the set A_1 (only the cluster 2 moves in these states), during $l_2 - b_1$ time units, and therefore,

$$v_1 = \frac{g_1}{g_1 + l_2 - b_1}, \qquad v_2 = 1.$$

If the system is in the state of the set A_1, then, returning to the set A, the system will only in states belonging $A_1 \cup (0; 0)$. In this case, over a finite time, the system will not be in states belonging A_2, and the system will be only in the states belonging to the spectral cycle such that this spectral cycle contains the state $(c_1 - l_1, 0)$. In this case, on the spectral cycle, the system is in the states, not belonging to the set A (both clusters move in these states), during g_2 time units and the system is in states, belonging to the set A_2 (only the cluster 1 moves in these states), during $l_1 - b_2$ time units, and therefore,

$$v_1 = 1,\ v_2 = \frac{g_2}{g_2 + l_1 - b_2},\ v_2 = 1.$$

Example 3. Suppose $c_1 = 4,\ l_1 = 2,\ c_2 = 6,\ l_2 = 2$.
In this case, we have

$$b_1 = 0, \qquad g_1 = 4, \qquad b_2 = 0, \qquad g_2 = 6.$$

There exist two spectral cycles. On one of these cycles, the clusters movewith velocities

$$v_1 = \frac{2}{3}, \qquad v_2 = 1.$$

The period of these cycles equals $T = 6$.
On the other spectral cycle, the clusters move with velocities $v_1 = 1,\ v_2 = \frac{3}{4}$.
The period of these cycles equals $T = 8$.

8 Conclusion

We has been proved that, if the condition of self-organization does not hold, then, depending on c_1, c_2, l_1, l_2, there are one or two spectral cycles.

Formulas for average velocities of clusters have been obtained. A necessary and sufficient condition for self-organization has been found.

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