Numerical solution of quadratic general Korteweg-de Vries equation by Galerkin quadratic finite element method.

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Abstract: In this work, we consider the quadratic generalized Kortewegde Vries (QGKdV) equation that is a mathematical model of waves on shallow water surfaces. Numerical solution of a Cauchy boundary-value problem with known exact solution is developed in details. Discretization is first accomplished by means of a quadratic finite element method. Then, the obtained system of first-order ordinary differential equations is discretized through a backward finite difference formula. Finally, the derived non linear algebraic system is solved by Newton's method with the Gauss elimination method as the inner iteration solver. Numerical results are presented in order to illustrate the efficiency of the present numerical treatment. In addition, a general form of multiple-soliton solution of QGKdV equation is obtained using the simplest equation method with Burgers equation as simplest equation.

Key–Words: KdV equation, Finite element method, Finite difference method.

1 Introduction

Korteweg-de Vries equation (KdV), the names of the mathematicians Diederik Johannes Korteweg and Gustav de Vries described the behavior of some types of waves in shallow waters by a nonlinear differential equation [11]. These mathematicians used their theory to explain wave propagation phenomena such as waves. They were able to determine many types of wave profiles such as nocturnal or solitons. This theory had a significant impact on modern mathematical problems of a nonlinear type in physics, electronics and biology especially in optical fibers. Moreover, Tsunamis are sea waves that cause gigantic walls of water devastating. The modeling of this phenomenon is possible thanks to the KdV equation. Although the solitons propagate in a depth of 4000 m (oceans), the wavelength of 100 km allows to approach the study in a shallow water.

Many researches concerning this problem have been treated by many authors in recent years. Such authors as [2],[13] which has established the analytical solutions to particular problems. Moreover, as well as concerning the numerical methods to solve the KdV equation, see [9, 16, 21] for finite difference method, as well [6, 7, 8],[10] for the method of finite elements. In addition, they have other methods to get the KdV numerical solutions, such as pseudo-spectral method [2], and heat balance integral method [12].

The paper is organized in five sections, in Section 2 we propose the QGKdV equation with the initial and the boundary conditions, especially, we study the so-called soliton solutions. In Section 3 a quadratic finite element method is applied for the discretization of the spatial variable, while the system of ordinary differential equations for the time variable is solved by means of an explicit finite difference method. In Section 4 the numerical results obtained for a test problem are presented with the known exact solution. In Section 5 we present our conclusions about the efficiency of Galerkin quadratic finite element method.

2 Statement of the model problems

We consider the KdV equation which is a nonlinear partial differential equation of third order with the boundary conditions and the initial condition given by

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 u}{\partial x^2} + u^4\right) = 0 & 0 \leq x \leq L, t > 0, \\ u(0,t) = u(L,t) = 0 \quad t > 0, \\ u(x,0) = (Asech^2(Bx+D))^{\frac{1}{3}} & 0 \leq x \leq L. \end{cases} \tag{1}$$

This problem has the exact solution [1], called "soliton" define as follows

$$u(x,t) = \left(Asech^2(Bx - Ct + D)\right)^{\frac{1}{3}},$$

where A, B, C and D are determined scalars so that the differential equation in (1) is satisfied. The calculations make it possible to find:

$$A = \frac{10}{9}B^2, \qquad C = \frac{4}{9}B^3, \qquad B, D \in \mathbb{R}. \tag{2}$$

2.1 Study the solition solution

It is well known that the QGKdV equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + u^4\right) = 0 \qquad x, t \in \mathbb{R}, \tag{3}$$

have explicit traveling wave solutions. Let

$$Q(x) = \left(\frac{5}{2\cosh^2(\frac{3}{2}x)}\right)^{\frac{1}{3}},$$

be the unique H^1 positive solution (up to translations) of

$$Q_{xx} + Q^4 = Q \ \ on \quad \mathbb{R}, \tag{4}$$

Then, for any $c > 0$, $x_0 \in \mathbb{R}$, the functions

$$R_{c,x_0}(t,x) = c^{\frac{1}{3}} Q(\sqrt{c}(x - x_0 - ct)),$$

is soliton solutions of the quadratic gKdV equations (3). The quadratic (gKdV) is a Hamiltonian system. In particular, three quantities are kept, at least formally:

$$\begin{aligned} \int_{\mathbb{R}} u(t,x)dx &= \int_{\mathbb{R}} u_0(x)dx, \\ \int_{\mathbb{R}} u(t,x)^2dx &= \int_{\mathbb{R}} u_0(x)^2dx, \qquad (mass\ L^2), \\ E(u(t)) &= \frac{1}{2}\int_{\mathbb{R}} u(t,x)_x^2dx - \frac{1}{5}\int_{\mathbb{R}} u(t,x)(t,x)^5dx \\ &= E(u_0) \qquad\qquad\qquad (energy). \end{aligned}$$

The natural energy space for the study of this equation is therefore H^1. Let us note, however, that the first conservation law is little used, because it is not a signed quantity, and that moreover it is not situated in the space of energy.

Let us recall some general results concerning the solutions of (3).

Theorem 1. *((Local existence in time, Kenig, Ponce et Vega ([5]))Let $u_0 \in H^1$. There exists $T = T(\|u_0\|_{H^1})$ and $u \in C^0([0;T[;H^1)$ solution of quadratic (gKdV)(3), unique in a suitable class. Such a solution keep the mass L^2 and the energy.*

Theorem 2. *(Stability of solitons [18]) Let $u_0 \in H^1(\mathbb{R})$ and let $u(t)$ be the globlal H^1 solution of quadratic (gKdV)(3) satisfying $u(0) = u_0$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|u_0 - Q\|_{H^1} \leq \delta$, then for all $t \in \mathbb{R}$, there exists $x(t) \in \mathbb{R}$ such that*

$$\|u(t, . + x(t)) - Q\|_{H^1} \leq \varepsilon.$$

To more details we guide the reader's to Benjamin[3],Bona [4], Weinstein[18].

Theorem 3. *(Asymptotic stability of solitons [19])Let $u_0 \in H^1(\mathbb{R})$ and let $u(t)$ be the globlal H^1 solution of quadratic (gKdV)(3) satisfying $u(0) = u_0$. There exists $\delta > 0$ such that if*

$$\|u_0 - Q\|_{H^1} \leq \delta,$$

then there exists c^+ close to1, and for all $t \in \mathbb{R}^+$, there exist $x(t) \in \mathbb{R}$ such that

$$\|u(t) - Q_{c^+}(.-x(t))\|_{H^1(x>t/10)} \to 0, as \quad t \to +\infty,$$

and

$$x'(t) \to c^+, as \quad t \to +\infty.$$

As well as concerning the multisoliton solutions, revise [20].

2.2 Multiple-soliton solution

In this section, we apply the simplest equation method to the KdV equation (3), more details were given in Refs. [], as follows:

Step 1. Consider the PDE (5) (QGKdV).

Step 2. The traveling wave variable

$$u(x,t) = U(\xi), \quad \xi = k_i\, x - \omega\, t, \qquad (5)$$

permits us to convert the PDE (5) into the following ODE form

$$-\omega\, U + k_i\, U^4 + k_i^3\, U_{\xi,\xi} = 0, \qquad (6)$$

where $U = U(\xi)$ is an unknown function to be determined later.

Step 3. The simplest equation method depends on expanding the traveling wave solutions $U(\xi)$ of Eq. (5) as a finite series

$$U(\xi) = \sum_{i=0}^{n} a_i\, f(\xi)^i \qquad a_m \neq 0, \qquad (7)$$

where a_i are constants to be determined later and f are the functions that satisfy some simplest ordinary differential equations. These types of equations are called the simplest equations that are of a lesser order than Eqs. (6) and, the general solution of them is known (or we know the way of finding its general solution, or at least we know some particular solutions of this equation).

In this study, we use the Burgers' equation as the simplest equation, which is a completely integrable equation and a fundamental second ODE from fluid mechanics, to construct the N-soliton solutions [17, 15, 14]. Consider the Burgers' equation

$$u_t - \alpha\, u\, u_x - u_{xx} = 0, \;\; u = u(x,t) \qquad (8)$$

and its ODE form

$$\frac{df}{d\xi} = \psi - p\psi^2 + q, \qquad (9)$$

where the wave variables $f(\xi) = u(x,t)$ and $\xi = k\, x - c\, t$, the dispersion relation $c = -k^2$, $p = \frac{\alpha}{2k}$, and $q = \frac{\beta}{k^2}$. Eq. (9) has a general form of N-soliton solution as follows:

$$f(\xi) = \left(\frac{2}{\alpha}\right) \frac{\sum_{i=0}^{N} k_i\, e^{\xi_i}}{1 + \sum_{i=0}^{N} e^{\xi_i}}. \qquad (10)$$

Step 4. The parameter n of Eq. (7) is a positive integer and it can be found by balancing the highest order derivative $U_{\xi,\xi}$ with the nonlinear one U^4 in Eq. (6). So, we obtain $n = \frac{2}{3}$. To get analytical solution, n should be integer [14]. Therefore, it requires the following transformation

$$U = (V(\xi))^{\frac{1}{3}}, \qquad (11)$$

that transforms (6) to

$$9\,\omega\, V^3 + 9\, k_i\, V^3 + 3\, k_i^3\, VV_{\xi,\xi} - 2\, k_i^3\, V_\xi^2 = 0, \qquad (12)$$

where $V = V(\xi)$. Now, by balancing the terms V^3 and $VV_{\xi,\xi}$ in Eqs. (19) gives $n = 2$. Therefore, exact solution of (19) reads

$$V(\xi) = a_0 + a_1\, h(\xi) + a_2\, h(\xi)^2. \qquad (13)$$

Step 5. Substituting (13) with (9) results in a polynomial in $f(\xi)$, and equating all coefficient of the polynomial to zero yields a set of algebraic equations for $a_0,\ a_1,\ a_2,\ \alpha,\ \beta$. Solving these algebraic equations with the help of Mathematica software, the following results are obtained

$$\begin{aligned} a_0 &= -\frac{1}{9}\left(25k_i^2\right), \;\; a_1 = -\frac{25k_i^4}{9\beta}, \;\; a_2 = -\frac{125k_i^6}{18\beta^2}, \\ \alpha &= -\frac{5k_i^3}{\beta}, \;\; \omega = -k_i^3, \end{aligned}$$

where $k_i,\ i = 1, 2, \ldots$ are arbitrary constants and ω is the dispersion relation of QKdV equation.

Step 6. Now, substituting the results with the N-soliton solution (10) into (5) using (18), we get a general form of N-soliton solution of Eq. (5) as follows, see Figs. 1 and 2 for $n = 1$,

$$\begin{aligned} u = & \frac{-5}{9\left(\sum_{i=1}^{n} e^{xk_i - t\omega} + 1\right)^2} \\ & (2(\sum_{i=1}^{n} k_i e^{xk_i - t\omega})^2 \\ & +((\sum_{i=1}^{n} e^{xk_i - t\omega} + 1) \\ & (3\sum_{i=1}^{n} e^{xk_i - t\omega} + 5)\sum_{i=1}^{n} k_i^2 e^{xk_i - t\omega})/\sum_{i=1}^{n} e^{xk_i - t\omega})). \end{aligned} \qquad (14)$$

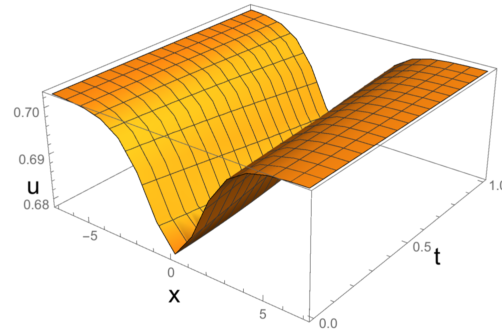

Figure 1: Soliton solution (14) of the QGKdV equation using Burgers' equation, at $k_i = 1$.

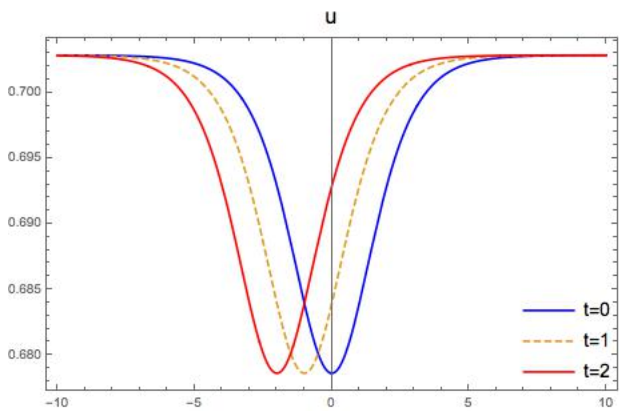

Figure 2: 2-D solitary solution (14) of the QGKdV equation using Burgers' equation, at $k_i = 1$.

3 Method of solution

3.1 Weak formulation of the continuous problem

Let $v(x) \in V = H_0^1(]0, L[)$. By multiplying the equation (1) by the function v and integrating for x over the interval $[0, L]$, we obtain

$$\int_0^L v(x) \left(\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} + u^4 \right) \right) dx = 0, \tag{15}$$

thanks to integration by parts, we obtain the following weak formulation: $\forall v \in V$

$$\int_0^L v(x) \frac{\partial u}{\partial t} dx - \int_0^L v(x) \frac{\partial^2 u}{\partial x^2} dx - \int_0^L v'(x) u^4 dx = 0. \tag{16}$$

3.2 Discreet formulation of Galerkin

Let $V_h \subset V$ of finite dimension N with the basis $\{\phi_1, ..., \phi_N\}$. If $u_h(x, t)$ denotes an approximate solution of $u(x, t)$ in the space $V_h \subset V$, then the Galerkin method gives (for t fixed)

$$\begin{cases} \text{Find } u_h(., t) \in V_h \text{ such as :} \\ \displaystyle\int_0^L v(x) \frac{\partial u_h}{\partial t} dx - \int_0^L v(x) \frac{\partial^2 u_h}{\partial x^2} dx - \int_0^L v'(x) u_h^4 dx = 0 \\ \forall v \in V_h. \end{cases} \tag{17}$$

Now, we assume that the approximate solution $u_h(x, t)$ can be expressed in the basis $\{\phi_i\}$ as follows:

$$u_h(x, t) = \sum_{j=1}^{N} U_j(t) \phi_j(x), \tag{18}$$

where $U_j(t)$ are undetermined coefficients.

Substituting (18) in the formulation (17) and taking $v = \phi_i$ $(i = 1, \cdots, N)$, after simplifications, (17) comes (for t fixed)

$$\begin{cases} \text{Find } U_1(t), \cdots, U_N(t) \text{ such as:} \\ \displaystyle\sum_{j=1}^{N} \left(\int_0^L \phi_j(x) \phi_i(x) dx \right) u_j'(t) \\ \displaystyle -\sum_{j=1}^{N} \left(\int_0^L \phi_j''(x) \phi_i'(x) dx \right) u_j(t) \\ \displaystyle -\int_0^L \left(\sum_{j=1}^{N} u_j(t) \phi_j(x) \right)^4 \phi_i'(x) dx = 0 \quad \forall i \in \{1, \cdots, N\}. \end{cases} \tag{19}$$

Thus, we obtain a system of nonlinear ordinary differential equations for the variable t. At this level, any discretization method can be applied to calculate the coefficients in the system equations (19). In the following, only the quadratic finite element method is adopted.

3.3 Finite Element Discretization

For a spatial step $h = \frac{L}{N+1}$, the interval $[0, L]$ is subdivided into $N+1$ subintervals $[x_i, x_{i+1}], i = 0, \cdots, N$, of length equals h. Thus, the nodes are such that $0 = x_0 < x_1 < \cdots < x_{N+1} = L$ with $x_i = ih$. The basic quadratic functions $\phi_1, \ldots, \phi_N$ are defined by:

$$\phi_i(x) = -\frac{1}{h^2} \begin{cases} (x_{i-1} - x)(x_{i+1} - x) & \text{if} \quad x \in [x_{i-1}, x_{i+1}], \\ 0 & \text{elsewhere,} \end{cases} \tag{20}$$

with $i = 1, \ldots, N$.

We note that for the nodes x_0 and x_{N+1} the basic functions are not considered because according to the boundary conditions, the values in $x = 0$ and $x = L$ are given.

This choice of the basic functions $\phi_i(x)$ makes it possible to rewrite (19) in the following form

$$\begin{aligned}
&\left(\int_{x_{i-1}}^{x_i} \phi_{i-1}(x)\phi_i(x)dx\right) U'_{i-1}(t)\\
&+\left(\int_{x_{i-1}}^{x_i} \phi_i^2(x)dx\right) U'_i(t)\\
&+\left(\int_{x_i}^{x_{i+1}} \phi_{i+1}(x)\phi_i(x)dx\right) U'_{i+1}(t)\\
&-\left(\int_{x_{i-1}}^{x_i} \phi''_{i-1}(x)\phi'_i(x)dx\right) U_{i-1}(t)\\
&-\left(\int_{x_{i-1}}^{x_{i+1}} \phi''_i(x)\phi'_i(x)dx\right) U_i(t)\\
&-\left(\int_{x_i}^{x_{i+1}} \phi''_{i+1}(x)\phi'_i(x)dx\right) U_{i+1}(t)\\
&-\int_{x_{i-1}}^{x_{i+1}} (U_{i-1}(t)\phi_{i-1}(x) + U_i(t)\phi_i(x)\\
&+ U_{i+1}(t)\phi_{i+1}(x))^4 \phi'_i(x)dx = 0,
\end{aligned} \tag{21}$$

for $i = 1, \cdots, N$.

The development of the four degree allows to write the last term of (21), which is the only nonlinear term, as follows:

$$\begin{aligned}
&- \int_{x_{i-1}}^{x_{i+1}} \Big(U_{i-1}(t)\phi_{i-1}(x) \qquad\qquad (22)\\
&+ U_i(t)\phi_i(x) + U_{i+1}(t)\phi_{i+1}(x)\Big)^4 \phi'_i(x)dx\\
&= -\left(\int_{(i-1)h}^{ih} \phi_{i-1}^4(x)\phi'_i(x)dx\right) U_{i-1}^4(t)\\
&- \left(\int_{(i-1)h}^{(i+1)h} \phi_i^4(x)\phi'_i(x)dx\right) U_i^4(t) \qquad\qquad (23)\\
&- \left(\int_{ih}^{(i+1)h} \phi_{i+1}^4(x)\phi'_i(x)dx\right) U_{i+1}^4(t)\\
&-6\left(\int_{(i-1)h}^{ih} \phi_{i-1}^2(x)\phi_i^2(x)\phi'_i(x)dx\right) U_{i-1}^2(t)U_i^2(t)\\
&-6\left(\int_{ih}^{(i+1)h} \phi_i^2(x)\phi_{i+1}^2(x)\phi'_i(x)dx\right) U_i^2(t)U_{i+1}^2(t)\\
&-4\left(\int_{(i-1)h}^{ih} \phi_{i-1}^3(x)\phi_i(x)\phi'_i(x)dx\right) U_{i-1}^3(t)U_i(t)\\
&-4\left(\int_{ih}^{(i+1)h} \phi_i^3(x)\phi_{i+1}(x)\phi'_i(x)dx\right) U_i^3(t)U_{i+1}(t)\\
&-4\left(\int_{(i-1)h}^{ih} \phi_{i-1}(x)\phi_i^3(x)\phi'_i(x)dx\right) U_{i-1}(t)U_i^3(t)\\
&-4\left(\int_{ih}^{(i+1)h} \phi_i(x)\phi_{i+1}^3(x)\phi'_i(x)dx\right) U_i(t)U_{i+1}^3(t).
\end{aligned}$$

After calculations, the integral terms are given by:

$$\int_{x_{i-1}}^{x_i} \phi_{i-1}\phi_i dx = \frac{11h}{30},$$

$$\int_{x_{i-1}}^{x_{i+1}} \phi_i^2 dx = \frac{16h}{15},$$

$$\int_{(i-1)h}^{ih} \phi_{i+1}\phi_i dx = \frac{11h}{30},$$

$$\int_{(i-1)h}^{ih} \phi''_{i-1}(x)\phi'_i(x)dx = \frac{2}{h^2},$$

$$\int_{(i-1)h}^{(i+1)h} \phi''_i(x)\phi'_i(x)dx = 0,$$

$$\int_{(i+1)h}^{ih} \phi''_{i+1}(x)\phi'_i(x)dx = -\frac{2}{h^2},$$

$$\int_{(i-1)h}^{ih} \phi_{i-1}^4(x)\phi_i' dx) = \frac{193}{315},$$

$$\int_{(i-1)h}^{(i+1)h} \phi_i^4(x)\phi_i' dx = 0,$$

$$\int_{ih}^{(i+1)h} \phi_{i+1}^4(x)\phi_i' dx = -\frac{193}{315},$$

$$\int_{(i-1)h}^{ih} \phi_{i-1}^2(x)\phi_i^2(x)\phi_i'(x) dx = \frac{103}{630},$$

$$\int_{ih}^{(i+1)h} \phi_{i+1}^2(x)\phi_i^2(x)\phi_i'(x) dx = -\frac{103}{630},$$

$$\int_{(i-1)h}^{ih} \phi_{i-1}^3(x)\phi_i(x)\phi_i'(x) dx = \frac{103}{420},$$

$$\int_{ih}^{(i+1)h} \phi_i^3(x)\phi_{i+1}(x)\phi_i'(x) dx = -\frac{193}{1260},$$

$$\int_{(i-1)h}^{ih} \phi_{i-1}(x)\phi_i^3(x)\phi_i'(x) dx = \frac{193}{1260},$$

$$\int_{ih}^{(i+1)h} \phi_i(x)\phi_{i+1}^3(x)\phi_i'(x) dx = -\frac{103}{420}.$$

Then, the equation (21) becomes

$$\begin{aligned}
&\left(\frac{11h}{30}\right)U_{i-1}'(t) + \left(\frac{16h}{15}\right)U_i'(t) + \left(\frac{11h}{30}\right)U_{i+1}'(t) \\
&- \left(\frac{2}{h^2}\right)U_{i-1}(t) + \left(\frac{2}{h^2}\right)U_{i+1}(t) - \left(\frac{193}{315}\right)U_{i-1}^4(t) \\
&+ \left(\frac{193}{315}\right)U_{i+1}^4(t) - \left(\frac{103}{105}\right)U_{i-1}^2(t)U_i^2(t) \\
&+ \left(\frac{103}{105}\right)U_i^2(t)U_{i+1}^2(t) - \left(\frac{103}{105}\right)U_{i-1}^3(t)U_i(t) \\
&+ \left(\frac{193}{315}\right)U_i^3(t)U_{i+1}(t) - \left(\frac{193}{315}\right)U_{i-1}(t)U_i^3(t) \\
&+ \left(\frac{103}{105}\right)U_i(t)U_{i+1}^3(t) = 0,
\end{aligned} \tag{24}$$

this system can be rewritten in the following matrix form:

$$MU'(t) + KU(t) + (G(U(t)))U(t) = 0, \tag{25}$$

where

$$M = (M_{ij}) = \int_0^L \phi_i(x)\phi_j(x) dx,$$

$$K = (K_{ij}) = \int_0^L \phi_i''(x)\phi_j'(x) dx,$$

$$U(t) = (U_1(t), \cdots, U_N(t))^t,$$
$$U'(t) = (U_1'(t), \cdots, U_N'(t))^t,$$

and the matrix $G(U(t))$ contains terms of order three with respect to the components of $U(t)$. This form will be explained further.

The system (25) is a system of ordinary differential equations of order one, which can be solved numerically by means of many methods. Here, we apply the finite difference method in the implicit form for the discretization of $U'(t)$. So, for a time step δt and the nodes are $t_j = j\delta t$ with $j = 0, 1, \cdots$, we obtain the following discrete system:

$$\frac{1}{\delta t}MU^{(j)} + KU^{(j)} + (G(U^{(j)}))U^{(j)} = \frac{1}{\delta t}MU^{(j-1)}, \tag{26}$$

where $U^{(j)}$ is the approximation of $U(t_j)$ and $U^{(0)}$ is given by the initial condition in (1). To solve (26) we apply the Newton method as the technique for the external iteration, and the Gauss elimination method for the internal iteration.

4 The numerical results

To illustrate the performance of the numerical method developed in this paper, a test problem of the type (1) is considered. All computations are done by taking the parameters $L = 2$, $B = \sqrt{0.9}$ and $D = -1$, while the steps are $h = 0.0125$ and $\delta t = 0.001$. The numerical solutions obtained at the nodes are compared with the exact solutions. The comparison is presented in the tables 1-2 for the values $t = 0.005$ and $t = 0.01$. To measure the difference between numerical solutions and exact solutions, the percentage error is used table 3. On the other hand, the figures 3-6 show that numerical solutions reproduce satisfactorily the behavior of the exact solution of the test problem.

Table 1: Comparison between the numerical and exact solutions at $t = 0.005$

x	Numerical solutions	Exact solutions
0.0	0.000000	0.000000
0.1	0.775213	0.784074
0.2	0.805622	0.819348
0.3	0.847203	0.853386
0.4	0.879001	0.885529
0.5	0.909311	0.915062
0.6	0.935602	0.941246
0.7	0.957018	0.963350
0.8	0.975035	0.980704
0.9	0.986871	0.992744
1.0	0.991033	0.999057
1.1	0.995112	0.999422
1.2	0.988002	0.993826
1.3	0.976800	0.982465
1.4	0.959113	0.965731
1.5	0.938503	0.944172
1.6	0.907744	0.918447
1.7	0.876541	0.881282
1.8	0.857001	0.857418
1.9	0.819851	0.823574
2.0	0.000000	0.000000

Table 2: Comparison between the numerical and exact solutions at $t = 0.01$

x	Numerical solutions	Exact solutions
0.0	0.000000	0.000000
0.1	0.784788	0.783360
0.2	0.820043	0.818652
0.3	0.854049	0.852721
0.4	0.886147	0.884909
0.5	0.915621	0.914502
0.6	0.941730	0.940760
0.7	0.963746	0.962953
0.8	0.980999	0.980407
0.9	0.992927	0.992558
1.0	0.999123	0.998989
1.1	0.999368	0.999473
1.2	0.993654	0.993995
1.3	0.982182	0.982747
1.4	0.965345	0.966115
1.5	0.943696	0.944646
1.6	0.917895	0.918998
1.7	0.888668	0.889894
1.8	0.856758	0.858077
1.9	0.822882	0.824266
2.0	0.000000	0.000000

5 Conclusion

In this paper, we deduce that finite element solutions satisfy the physical behavior of the problem for a KdV equation for a nonlinear term of four degree (Tsunamis). Finite element solutions for small moment problems have greater accuracy than other numerical solutions. So, solving the KdV equation for a nonlinear term of degree four by a finite element method allows us to achieve a relatively accurate resolution. Besides, multiple-soliton solutions of the QGKdV equation are obtained using the simplest equation method as simplest equations. In this study, the obtained solutions are significant and important in the study of nonlinear and dispersive waves problems. The results of this paper show that the applied methods are effective and powerful techniques to study many nonlinear evolution equations which have several applications in mathematical physics and engineering.

Table 3: Percentage of the Problem 1 for some selected values of x

x	t=0.005	t=0.01
0.2	1.6752	0.1699
0.4	0.7372	0.1399
0.6	0.5996	0.1031
0.8	0.5781	0.0604
1.0	0.8032	0.0134
1.2	0.5860	0.0343
1.4	0.6853	0.0797
1.6	1.1653	0.1014
1.8	0.1653	0.1537
2.0	0.0	0.0

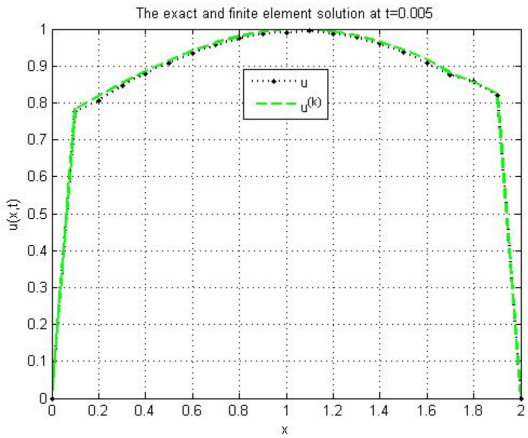

Figure 3: The exact and finite element solution at $t = 0.005$.

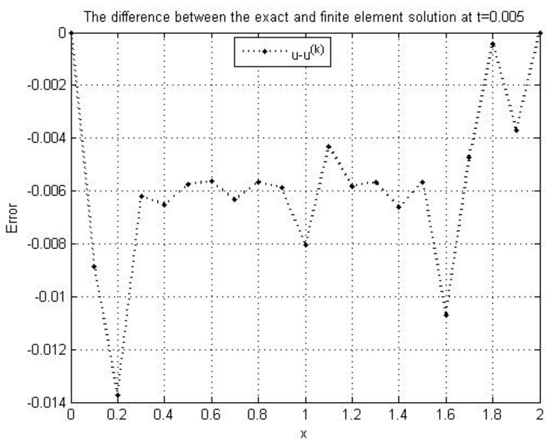

Figure 4: The difference between the exact and finite element solution at $t = 0.005$.

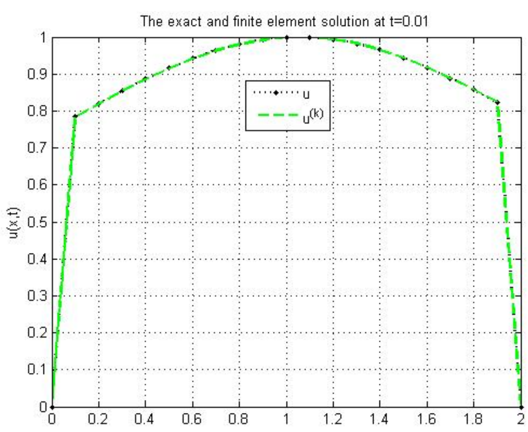

Figure 5: The exact and finite element solution at $t = 0.01$.

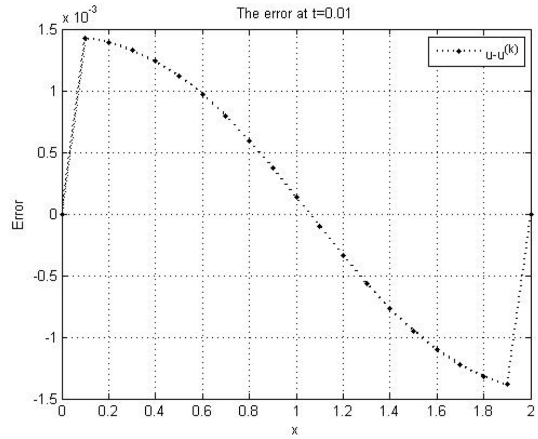

Figure 6: The difference between the exact and finite element solution at $t = 0.01$.

6 Appendix

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