Dynamic Feedback Stabilization of Timoshenko beam with internal input delays

GENQI Xu
Tianjin University
Department of Mathematics
92 Weijin Road, Tianjin 300072
CHINA
gqxu@tju.edu.cn

A. JALILI RAHMATI
Sharif University of Technology
Department of Mathematical Sciences
P.O.Box 11155-9415,Tehran
IRAN
ali.jalilir@yahoo.com

F. BADPAR
Sharif University of Technology
Department of Energy Engineering
Azadi Ave. Tehran
IRAN
fbadpar@yahoo.com

Abstract: In this paper, we consider the exponential stabilization problem of a Timoshenko beam with interior local controls with input delays. In the past, most of the stabilization results for the Timoshenko beam were on the boundary control with input delays. In the present paper we shall extend the method treating the boundary control with delays to the case of interior local control with delays. Essentially we design a new dynamic feedback control laws that stabilizes exponentially the system. Detail of the design procedure of the dynamic feedback controller and analysis of the exponential stability are given.

Key–Words: Timoshenko beam, input delay, dynamic feedback controllers, exponential stabilization.

1 Introduction

In the past few years, many of researchers of various fields of science such as mathematics and mechanical engineering have been investigating Timoshenko beam due to its high application in industry. Designing a control law which could stabilize Timoshenko beam system is one of the most fascinating problems that has engaged many of mathematicians and engineers in branch of vibrations. For instance, Kim and Rendary [1] studied the following Timoshenko beam system:

$$\begin{cases} \rho w_{tt}(x,t) = K(w_{xx}(x,t) - \varphi_x(x,t)),\ x \in (0,L), \\ I_\rho \varphi_{tt}(x,t) = EI\varphi_{xx}(x,t) \\ \qquad\qquad + K(w_x(x,t) - \varphi(x,t)),\ x \in (0,L), \\ w(0,t) = \varphi(0,t) = 0, \\ K(w_x(L,t) - \varphi(L,t)) = u_1(t), \\ EI\varphi_x(L,t) = u_2(t), \end{cases}$$

They applied two boundary controls $u_1(t) = -\alpha\frac{\partial w}{\partial t}(L,t)$ and $u_2(t) = -\beta\frac{\partial \varphi}{\partial t}(L,t)$ on the free endpoint and obtained the uniformly stability of the closed loop system; Xu and Feng [2] studied the same system and concluded the Riesz basis property of the closed loop system. Other works on this subject we refer to Xu and Feng [2], Xu and Yung [3], Xu [4] and references therein.

Observe that above design of the feedback control law strongly depends on the precise control time t. If there is any small time delay, the feedback control law may be invalid, this fact was found at first by Datko et al. in 1985. Datko et al. [5] studied the effect of time delay in boundary control for the following wave equation

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - 2aw_t(x,t) - a^2 w(x,t), \\ w(0,t) = 0, \\ w_x(1,t) = -kw_t(1,t-\epsilon), \qquad t > \epsilon, \end{cases}$$

and deduced that these feedback control laws were not robust with respect of time delay. Other counterexample we refer to Datko [6] (1988) and [7] (1993) for further instances.

After these works, researchers are starting to turn their attention to systems with time delay. Xu et al. [8] (2006) studied wave equation with boundary control along with time delay as follows:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t),\ x \in (0,1), \\ w(0,t) = 0, \\ w_x(1,t) = -k\mu w_t(1,t) - k(1-\mu)w_t(1,t-\tau), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \\ w_t(1,t-\tau) = f(t-\tau) \end{cases}$$

with $k > 0$. They showed that the closed loop system will be exponentially stable if $\mu > \frac{1}{2}$ for all $\tau > 0$, unstable if $\mu < \frac{1}{2}$, and at most asymptotically stable if $\mu = \frac{1}{2}$. One could refer to Nicaise and Pignotti [9](2006), Nicaise and Valien [10] (2007) and more other works that have been done on wave equation and other models [11][12]. Summarizing these works we can conclude the $\frac{1}{2}$-stability criterion. Essentially speaking these works do not include the design of controller, this is because the controller can be regarded as the input-delay $\alpha u(t) + \beta u(t-\tau)$ with feedback control law $u(t) = -w_t(1,t)$. Obviously, $\mu < \frac{1}{2}$, these controller fail to work. So, for $\mu < \frac{1}{2}$, the key issue is to design a new feedback control law that could stabilize the system exponentially.

Shang et al. [13](2012) started the boundary controller design for a system with input delay. They studied the following Euler-Bernoulli beam equation:

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0,\ x \in (0,1) \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \\ w_{xxx}(1,t) = \alpha u(t) + \beta u(t-\tau), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x) \end{cases}$$

and designed a dynamic feedback controller that consists of two parts, the partial state predictor that transform the delayed system into a undelayed system, design of the collocated feedback controller that gives a control signal. In such a way they proved that the closed loop system is exponentially stable provided $|\alpha| \neq |\beta|$ and all $\tau > 0$, and at most asymptotically stable for $|\alpha| = |\beta|$. Xu and Wang [15], Wang and Xu[16] extended this design to the Timoshenko beam and wave equation respectively. Han and Xu [17] extended the controller design to the case of output-based model. Liu and Xu [17], Shang and Xu [18] improved the control design to fit the distributed delay, [19] and [20] for case of output-based models.

In all the aforementioned works, the obtained results mainly are on the boundary control with delay. In the present paper, our aim is to extend the design approach of controller from the boundary control to interior distributed control. Here we mainly consider a Timoshenko beam with time delay in the internal control. More precisely, we study the following Timoshenko beam:

$$\begin{cases} \rho w_{tt}(x,t) = K(w_{xx}(x,t) - \varphi_x(x,t)) \\ \quad + a(x)[\alpha_1 u_1(x,t) + \beta_1 u_1(x,t-\tau)],\ x \in (0,1) \\ I_\rho \varphi_{tt}(x,t) = EI\varphi_{xx}(x,t) + K(w_x(x,t) - \varphi(x,t)) \\ \quad + b(x)[\alpha_2 u_2(x,t) + \beta_2 u_2(x,t-\tau)],\ x \in (0,1) \\ w(0,t) = \varphi_(0,t) = 0, \\ K(w_x(1,t) - \varphi(1,t)) = 0, \\ EI\varphi_x(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x) \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) \\ u_1(x,\theta) = f_1(x,\theta), \quad u_2(x,\theta) = f_2(x,\theta) \end{cases} \tag{1}$$

where $a(x)$ and $b(x)$ are nonnegative and piecewisely continuous functions and satisfy the condition that there exists an interval $[c_1, c_2] \subset [0,1]$ such that

$$a(x) > a_0 > 0, \quad b(x) > b_0 > 0, \quad x \in [c_1, c_2] \tag{2}$$

This is an extensive model of Timoshenko beam with the distributed controls and input delays. For examples, if $\beta_1 = \beta_2 = 0$, Shi and Fend [21], Soufyane and Whebe [22] studied the exponential stability of the system under the collocated feedback control law. If $\alpha_j > \beta_j > 0$, Raposo et al. [23] studied the exponential stability of the system under the collocated feedback control law. In this paper, we shall remove the restriction on α_j and β_j, $j = 1,2$, and design a dynamic controller for the above system, that could stabilize the system exponentially for all $|\alpha_j| \neq |\beta_j| (j = 1,2); \forall \tau > 0$.

The rest of this paper is organized as follows. In section 2, we describe the design procedure of controller, that includes the three steps: the first step is to transform the system (1) into a system without delays; the second step is to design the collocated feedback controllers for the resulted system, and generates the control signals; the third step is to feed the control signal into the system (1), and hence forms a closed loop system. Since there are great calculations in this section, to simplify contents, we postponed some detail calculations to appendix. In section 3, we prove the main results of this paper. Essentially we prove that system (1) under the controls is exponentially stable for any $|\alpha_j| \neq |\beta_j|$,$j = 1,2$. In section 4, we conclude this paper.

2 Design of Dynamic Feedback Controller

In this section we describe the design procedure of dynamic feedback controllers for the system (1). The design idea is similar to that used in boundary control with delays.

Step 1. We find a transform that transforms the system (1) into the objective system:

$$\begin{cases} p_{1,t}(x,t) = q_1(x,t) \\ \quad + \alpha_1 \int_0^1 H_1(x,\tau,y)a(y)u_1(y,t)dy \\ \quad + \alpha_2 \int_0^1 H_2(x,\tau,y)b(y)u_2(y,t)dy, \\ p_{2,t}(x,t) = q_2(x,t) \\ \quad + \alpha_1 \int_0^1 H_3(x,\tau,y)a(y)u_1(y,t)dy \\ \quad + \alpha_2 \int_0^1 H_4(x,\tau,y)b(y)u_2(y,t)dy, \\ q_{1,t}(x,t) = \frac{K}{\rho}(p_{1,xx}(x,t) - p_{2,x}(x,t)) \\ \quad + \alpha_1 \int_0^1 H_5(x,\tau,y)a(y)u_1(y,t)dy \\ \quad + \alpha_2 \int_0^1 H_6(x,\tau,y)b(y)u_2(y,t)dy \\ \quad + \frac{\beta_1}{\rho}a(x)u_1(x,t), \\ q_{2,t}(x,t) = \frac{EI}{I_\rho}p_{2,xx}(x,t) \\ \quad + \frac{K}{I_\rho}(p_{1,x}(x,t) - p_2(x,t)) \\ \quad + \alpha_1 \int_0^1 H_7(x,\tau,y)a(y)u_1(y,t)dy \\ \quad + \alpha_2 \int_0^1 H_8(x,\tau,y)b(y)u_2(y,t)dy \\ \quad + \frac{\beta_2}{I_\rho}b(x)u_2(x,t), \\ p_1(0,t) = p_2(0,t) = q_1(0,t) = q_2(0,t) = 0, \\ K(p_{1,x}(1,t) - p_2(1,t)) = 0, \\ EIp_{2,x}(1,t) = 0, \\ p_1(x,0) = G_1(w_0,\varphi_0,w_1,\varphi_1)(x) \\ \quad - \beta_1 \int_{-\tau}^0 \int_0^1 H_1(x,s,y)a(y)f_1(y,s)dyds \\ \quad - \beta_2 \int_{-\tau}^0 \int_0^1 H_2(x,s,y)b(y)f_2(y,s)dyds, \\ p_2(x,0) = G_2(w_0,\varphi_0,w_1,\varphi_1)(x) \\ \quad - \beta_1 \int_{-\tau}^0 \int_0^1 H_3(x,s,y)a(y)f_1(y,s)dyds \\ \quad - \beta_2 \int_{-\tau}^0 \int_0^1 H_4(x,s,y)b(y)f_2(y,s)dyds, \\ q_1(x,0) = G_3(w_0,\varphi_0,w_1,\varphi_1)(x) \\ \quad + \beta_1 \int_{-\tau}^0 \int_0^1 H_5(x,s,y)a(y)f_1(y,s)dyds \\ \quad + \beta_2 \int_{-\tau}^0 \int_0^1 H_6(x,s,y)b(y)f_2(y,s)dyds \\ q_2(x,0) = G_4((w_0,\varphi_0,w_1,\varphi_1)(x), \\ \quad + \beta_1 \int_{-\tau}^0 \int_0^1 H_7(x,s,y)a(y)f_1(y,s)dyds \\ \quad + \beta_2 \int_{-\tau}^0 \int_0^1 H_8(x,s,y)b(y)f_2(y,s)dyds. \end{cases} \tag{3}$$

To realize the transform, we suppose that the state of the system (1) is valid. Similarly we introduce the partial state predictive system as

$$\begin{cases} \rho\widehat{w}_{ss}(x,s,t) - K(\widehat{w}_{xx} - \widehat{\varphi}_x)(x,s,t) \\ = \beta_1 a(x)u_1(x,t-\tau+s),\ x\in(0,1), s\in(0,\tau); \\ I_\rho\widehat{\varphi}_{ss}(x,s,t) - EI\widehat{\varphi}_{xx}(x,s,t) - K(\widehat{w}_x - \widehat{\varphi})(x,s,t) \\ = \beta_2 b(x)u_2(x,t-\tau+s),\ x\in(0,1), s\in(0,\tau); \\ \widehat{w}(0,s,t) = \widehat{\varphi}(0,s,t) = 0, \\ K(\widehat{w}_x - \widehat{\varphi})(1,s,t) = 0, \\ EI\widehat{\varphi}_x(1,s,t) = 0, \\ \widehat{w}(x,0,t) = w(x,t),\ \widehat{w}_s(x,0,t) = w_t(x,t), \\ \widehat{\varphi}(x,0,t) = \varphi(x,t),\ \widehat{\varphi}_s(x,0,t) = \varphi_t(x,t). \end{cases} \tag{4}$$

and take

$$\begin{cases} p_1(x,t) = \widehat{w}(x,\tau,t), \\ q_1(x,t) = \widehat{w}_s(x,\tau,t) \\ p_2(x,t) = \widehat{\varphi}(x,\tau,t), \\ q_2(x,t) = \widehat{\varphi}_s(x,\tau,t). \end{cases} \tag{5}$$

Then (p_1,p_2,q_1,q_2) satisfy the equation (3), and the functions $G_k(w_0,\varphi_0,w_1,\varphi_1)(x), k = 1,2,3,4$, $H_j(x,s,y), j = 1,2,\cdots,8$, are given in Appendix.

Step 2. We design feedback control law for system (3) that may stabilize the system. Hence we get a control signal.

In order to get the right control signal, we consider the energy function of (3)

$$\mathcal{E}(t) = \frac{1}{2}\int_0^1 [K(p_{1,x}(x,t) - p_2(x,t))^2 dx + \frac{1}{2}\int_0^1 EIp_{2,x}^2(x,t)]dx + \frac{1}{2}\int_0^1 [\rho q_1^2(x,t) + I_\rho q_2^2(x,t)]dx.$$

A direct calculation gives

$$\begin{aligned} \frac{d\mathcal{E}(t)}{dt} = \\ &\alpha_1 \int_0^1 a(y)u_1(y,t)\Big[\int_0^1 K(p_{1,x}(x,t) - p_2(x,t)) \\ &(\partial_x H_1(x,\tau,y) - H_3(x,\tau,y))dx\Big]dy \\ &+\alpha_1 \int_0^1 a(y)u_1(y,t)dy \int_0^1 EIp_{2,x}(x,t) \\ &\partial_x H_3(x,\tau,y)dx \\ &+\alpha_1 \int_0^1 a(y)u_1(y,t)dy \int_0^1 \rho q_1(x,t)H_5(x,\tau,y)dx \\ &+\alpha_1 \int_0^1 a(y)u_1(y,t)dy \int_0^1 I_\rho q_2(x,t)H_7(x,\tau,y)dx \\ &+\beta_1 \int_0^1 a(y)u_1(y,t)q_1(y,t)dx \\ &+\alpha_2 \int_0^1 b(y)u_1(y,t)\Big[\int_0^1 K(p_{1,x}(x,t) - p_2(x,t)) \\ &(\partial_x H_2(x,\tau,y) - H_4(x,\tau,y))dx\Big]dy \\ &+\alpha_2 \int_0^1 b(y)u_2(y,t)dy \int_0^1 EIp_{2,x}(x,t) \\ &\partial_x H_4(x,\tau,y)dx \\ &+\alpha_2 \int_0^1 b(y)u_2(y,t)dy \int_0^1 \rho q_1(x,t)H_6(x,\tau,y)dx \\ &+\alpha_2 \int_0^1 b(y)u_2(y,t)dy \int_0^1 I_\rho q_2(x,t)H_8(x,\tau,y)dx \end{aligned}$$

$$+\beta_2\int_0^1 b(y)u_2(y,t)q_2(y,t)dy.$$

Subsequently, the feedback control laws can be considered as follows:

$$\begin{aligned}
&u_1(y,t) = -U_1(p_1,p_2,q_1,q_2)(y,t)\\
= \ & -\Big[\beta_1 q_1(y,t)+\alpha_1\int_0^1 K(p_{1,x}(x,t)-p_2(x,t))\\
&(\partial_x H_1(x,\tau,y)-H_3(x,\tau,y))dx\\
&+\alpha_1\int_0^1 EIp_{2,x}(x,t)\partial_x H_3(x,\tau,y)dx\\
&+\alpha_1\int_0^1 \rho q_1(x,t)H_5(x,\tau,y)dx\\
&+\alpha_1\int_0^1 I_\rho q_2(x,t)H_7(x,\tau,y)dx\Big]; \qquad (6)
\end{aligned}$$

$$\begin{aligned}
&u_2(y,t) = -U_2(p_1,p_2,q_1,q_2)(y,t)\\
= \ & -\Big[\beta_2 q_2(y,t)+\alpha_2\int_0^1 K(p_{1,x}(x,t)-p_2(x,t))\\
&(\partial_x H_2(x,\tau,y)-H_4(x,\tau,y))dx\\
&+\alpha_2\int_0^1 EIp_{2,x}(x,t)\partial_x H_4(x,\tau,y)dx\\
&+\alpha_2\int_0^1 \rho q_1(x,t)H_6(x,\tau,y)dx\\
&+\alpha_2\int_0^1 I_\rho q_2(x,t)H_8(x,\tau,y)dx\Big]. \qquad (7)
\end{aligned}$$

Based on the feedback control laws, we have

$$\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} &= -\int_0^1 a(y)U_1^2(p_1,p_2,q_1,q_2)(y,t)\\
&-\int_0^1 b(y)U_2^2(p_1,p_2,q_1,q_2)(y,t)dy \le 0.
\end{aligned}$$

Under this feedback control law, the closed loop system associated with (3) is

$$\left\{\begin{array}{l}
p_{1,t}(x,t)=q_1(x,t)\\
-\alpha_1\int_0^1 H_1(x,\tau,y)a(y)U_1(y,t)dy\\
-\alpha_2\int_0^1 H_2(x,\tau,y)b(y)U_2(y,t)dy,\\
p_{2,t}(x,t)=q_2(x,t)\\
-\alpha_1\int_0^1 H_3(x,\tau,y)a(y)U_1(y,t)dy\\
-\alpha_2\int_0^1 H_4(x,\tau,y)b(y)U_2(y,t)dy,\\
q_{1,t}(x,t)=\frac{K}{\rho}(p_{1,xx}(x,t)-p_{2,x}(x,t))\\
-\alpha_1\int_0^1 H_5(x,\tau,y)a(y)U_1(y,t)dy\\
-\alpha_2\int_0^1 H_6(x,\tau,y)b(y)U_2(y,t)dy\\
-\frac{\beta_1}{\rho}a(x)U_1(x,t),\\
q_{2,t}(x,t)=\frac{EI}{I_\rho}p_{2,xx}(x,t)\\
+\frac{K}{I_\rho}(p_{1,x}(x,t)-p_2(x,t))\\
-\alpha_1\int_0^1 H_7(x,\tau,y)a(y)U_1(y,t)dy\\
-\alpha_2\int_0^1 H_8(x,\tau,y)b(y)U_2(y,t)dy\\
-\frac{\beta_2}{I_\rho}b(x)U_2(x,t),\\
p_1(0,t)=p_2(0,t)=q_1(0,t)=q_2(0,t)=0,\\
K(p_{1,x}(1,t)-p_2(1,t))=0,\\
EIp_{2,x}(1,t)=0,\\
p_1(x,0)=G_1(w_0,\varphi_0,w_1,\varphi_1)(x)\\
-\beta_1\int_{-\tau}^0\int_0^1 H_1(x,s,y)a(y)f_1(y,s)dyds\\
-\beta_2\int_{-\tau}^0\int_0^1 H_2(x,s,y)b(y)f_2(y,s)dyds,\\
p_2(x,0)=G_2(w_0,\varphi_0,w_1,\varphi_1)(x)\\
-\beta_1\int_{-\tau}^0\int_0^1 H_3(x,s,y)a(y)f_1(y,s)dyds\\
-\beta_2\int_{-\tau}^0\int_0^1 H_4(x,s,y)b(y)f_2(y,s)dyds,\\
q_1(x,0)=G_3(w_0,\varphi_0,w_1,\varphi_1)(x)\\
+\beta_1\int_{-\tau}^0\int_0^1 H_5(x,s,y)a(y)f_1(y,s)dyds\\
+\beta_2\int_{-\tau}^0\int_0^1 H_6(x,s,y)b(y)f_2(y,s)dyds\\
q_2(x,0)=G_4((w_0,\varphi_0,w_1,\varphi_1)(x)\\
+\beta_1\int_{-\tau}^0\int_0^1 H_7(x,s,y)a(y)f_1(y,s)dyds\\
+\beta_2\int_{-\tau}^0\int_0^1 H_8(x,s,y)b(y)f_2(y,s)dyds.
\end{array}\right. \qquad (8)$$

Step 3. We feed the control signals both (6) and (7) back to the system (1) and get that the following system:

$$\left\{\begin{array}{l}
\rho w_{tt}(x,t)=K(w_{xx}(x,t)-\varphi_x(x,t))\\
\quad -a(x)[\alpha_1U_1(P,Q)(x,t)+\beta_1U_1(P,Q)(x,t-\tau)],\\
I_\rho\varphi_{tt}(x,t)=EI\varphi_{xx}(x,t)+K(w_x(x,t)-\varphi(x,t))\\
\quad -b(x)[\alpha_2U_2(P,Q)(x,t)+\beta_2U_2(P,Q)(x,t-\tau)],\\
w(0,t)=\varphi_(0,t)=0,\\
K(w_x(1,t)-\varphi(1,t))=0,\\
EI\varphi_x(1,t)=0,\\
w(x,0)=w_0(x),\quad w_t(x,0)=w_1(x),\\
\varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\\
u_1(x,\theta)=f_1(x,\theta),\ u_2(x,\theta)=f_2(x,\theta).
\end{array}\right. \qquad (9)$$

Our main results are as the following.

Theorem 1 *If the system (8) is exponentially stable, then the system (9) also is exponentially stable; If the system 8 is asymptotically stable, so is (9).*

Theorem 2 *Assume that $a(x), b(x)$ satisfy the condition (2),and assume that $\frac{K}{\rho}\neq\frac{EI}{I_\rho}$. Then for any $\tau>0$, the energy function of the system (8) decays exponentially provided that $|\alpha_j|\neq|\beta_j|(j=1,2)$.*

3 Proofs of main results

In this section we prove Theorems 1 and 2. For simplicity of notation, we introduce some spaces.

Let $H^k[0,1]$ be the usual sobolev space of order k. Set $V_e^k[0,1]=\{f\in H^k[0,1]\mid f(0)=0\}$.

Let space be

$$\mathcal{H}=V_e^1[0,1]\times V_e^1[0,1]\times L_\rho^2[0,1]\times L_{I_\rho}^2[0,1]$$

equipped with inner product

$$\langle F,G\rangle_{\mathcal{H}} = \int_0^1 K(f_1(x)-f_2(x))(g_1'(x)-g_2(x))dx + \int_0^1 EIf_2'(x)g_2'(x)dx + \int_0^1 \rho f_3(x)g_3(x)dx + \int_0^1 I_\rho f_4(x)g_(x)dx$$

for $F=(f_1,f_2,f_3,f_4), G=(g_1,g_2,g_3,g_4)\in\mathcal{H}$.

It is easy to check that $\mathcal{H}$ is a real Hilbert space.

Lemma 3 *(See[11, Lemma 2.1]) Let the differential operator in $L^2_\rho[0,1]\times L^2_{I_\rho}[0,1]$ be defined by*

$$\mathcal{J}\begin{bmatrix} w \\ \varphi \end{bmatrix} = -\begin{bmatrix} \frac{K}{\rho}(w''(x)-\varphi'(x)) \\ \frac{EI}{I_\rho}\varphi''(x)+\frac{K}{I_\rho}(w'(x)-\varphi(x)) \end{bmatrix}$$
$$\mathcal{D}(\mathcal{J}) = \Big\{(w(x),\varphi(x))\in H^2(0,1)\times H^2(0,1)\ \Big|\ w(0)=\varphi(0)=0, K(w'(1)-\varphi(1))=0, EI\varphi'(1)=0\Big\}$$

Then $\mathcal{J}$ is a self-adjoin and positive definite operator with compact resolvent in $L^2_\rho[0,1]\times L^2_{I_\rho}[0,1]$, its eigenvalues are

$$0<\lambda_1<\lambda_2\le\ldots\le\lambda_n\le\ldots$$

and the eigenfunctions $\Phi_n(x)=(w_n(x),\varphi_n(x))^T$ corresponding to λ_n are real functions and form a normalised orthogonal basis for $L^2_\rho[0,1]\times L^2_{I_\rho}[0,1]$.

Note that

$$(\mathcal{J}(w,\varphi),(w,\varphi))_{L^2_\rho\times L^2_{I_\rho}}$$
$$=\int_0^1 K(w'(x)-\varphi(x))^2dx+\int_0^1 EI(\varphi'(x))^2dx$$

so, we can rewrite $\mathcal{H}$ as

$$\mathcal{H}=\mathcal{D}(\mathcal{J}^{\frac12})\times H,\quad H=L^2_\rho[0,1]\times L^2_{I_\rho}[0,1].$$

Set

$$\begin{aligned}
&X(x,t)=(w(x,t),\varphi(x,t))^T,\\
&X_t(x,t)=(w_t(x,t),\varphi_t(x,t))^T,\\
&P(x,t)=(p_1(x,t),p_2(x,t))^T,\\
&Q(x,t)=(q_1(x,t),q_2(x,t)^T,\\
&U(x,t)=(u_1(x,t),u_2(x,t))^T,\\
&U(P,Q)=(U_1(p_1,p_2,q_1,q_2),U_2(p_1,p_2,q_1,q_2))^T.
\end{aligned}$$

Then the system (9) can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} X(x,t)\\ X_t(x,t)\end{pmatrix}=\begin{pmatrix}0 & I\\ -\mathcal{J} & 0\end{pmatrix}\begin{pmatrix} X(x,t)\\ X_t(x,t)\end{pmatrix} -\begin{pmatrix} 0 \\ A(x)[\Delta_1U(x,t)+\Delta_2U(x,t-\tau)]\end{pmatrix} \tag{10}$$

and the system (8) can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} P(x,t)\\ Q(x,t)\end{pmatrix}=\begin{pmatrix}0 & I\\ -\mathcal{J} & 0\end{pmatrix}\begin{pmatrix} P(x,t)\\ Q(x,t)\end{pmatrix}- \begin{pmatrix} Sin(\tau\mathcal{J})A(.)\Delta_1U(P,Q) \\ Cos(\tau\mathcal{J})A(.)\Delta_1U(P,Q)-A(x)\Delta_2U(P,Q)\end{pmatrix} \tag{11}$$

where $Sin(\tau\mathcal{J})$ and $Cos(\tau\mathcal{J})$ are defined as in Appendix.

3.1 The Proof of Theorem 1

In this subsection we prove that the system (9) (i.e., the system (1) with controls (6) and (7) has the same stability as that of the system (8). In the sequel, the notations $\mathcal{J}$, $A(x)$, Δ_j are the same as in Appendix.

We begin with considering the error of both systems (1) and (4):

$$e(x,s,t)=X(x,s+t)-\widehat{X}(x,s,t).$$

It is easy to see that $e(x,s,t)$ satisfies the following vector-valued equation:

$$\begin{cases} e_{ss}(x,s,t)+\mathcal{J}e(x,s,t) \\ = A(x)\Delta_1U(x,t+s),\ x\in(0,1),s\in(0,\tau),\\ e(0,s,t)=0, \qquad s\in(0,\tau),t>0\\ \Gamma_Ne(.,s,t)=0, \qquad s\in(0,\tau),t>0\\ e(x,0,t)=0, \quad e_s(x,0,t)=0, \qquad x\in(0,1).\end{cases} \tag{12}$$

Consequently, the energy function of the error system is

$$\begin{aligned}\varepsilon(s,t) &= \frac12\|(e(.,s,t),e_s(.,s,t))\|^2_{\mathcal{H}}\\ &= \frac12[(\mathcal{J}^{\frac12}e(.,s,t),\mathcal{J}^{\frac12}e(.,s,t))_{L^2_\rho\times L^2_{I_\rho}} + (e_s(.,s,t),e_s(.,s,t))_{L^2_\rho\times L^2_{I_\rho}}]\end{aligned}$$

Based on the equation (12), we can calculate

$$\frac{\partial\varepsilon(s,t)}{\partial s} = (A(.)\Delta_1U(.,t+s),e_s(.,s,t))_{L^2_\rho\times L^2_{I_\rho}}$$

and hence

$$\varepsilon(\tau,t)=\int_0^\tau(A(.)\Delta_1U(\cdot,t+s),e_s(\cdot,s,t))_{L^2_\rho\times L^2_{I_\rho}}ds.$$

Note that

$$Cos(t\mathcal{J})F = \sum_{n=1}^{\infty} \cos\sqrt{\lambda_n} t (F, \Phi_n)_{L^2_\rho \times L^2_{I_\rho}} \Phi_n(x)$$

and

$$e_s(x,s,t) = \int_0^s Cos((s-r)\mathcal{J})A(\cdot)\Delta_1 U(\cdot, t+r)dr.$$

Thus we have

$$\begin{aligned} &(A(.)\Delta_1 U(.,t+s), e_s(.,s,t))_{L^2_\rho \times L^2_{I_\rho}} \\ = &\int_0^s \sum_{n=1}^{\infty} \cos\sqrt{\lambda_n}(s-r) \\ &(A(.)\Delta_1 U(.,t+r), \Phi_n)_{L^2_\rho \times L^2_{I_\rho}} \\ &(A(\cdot)\Delta_1 U(\cdot, t+s), \Phi_n)_{L^2_\rho \times L^2_{I_\rho}} dr. \end{aligned}$$

Using Cauchy-Schwartz inequality, by a detailed estimate, we can get that there are positive constants N_j,$j = 1,2,3$, depended only on α_1, α_2 such that

$$\begin{aligned} &\varepsilon(\tau,t) \le \\ &\tau N_1 \int_0^\tau \left(\int_0^1 a(x)u_1(x,t+s)dx\right)^2 ds \\ &+2\tau N_2 \int_0^\tau \left(\int_0^1 a(x)u_1(x,t+s)dx\right) \times \\ &\left(\int_0^1 b(x)u_2(x,t+s)dx\right) ds \\ &+\tau N_3 \int_0^\tau \left(\int_0^1 b(x)u_2(x,t+s)dx\right)^2 ds \\ \le\ &\tau N_1 \int_0^1 a(x)dx \int_0^\tau \int_0^1 a(x)u_1^2(x,t+s)dxds \\ &+\tau N_3 \int_0^1 b(x)dx \int_0^\tau \int_0^1 b(x)u_2^2(x,t+s)dxds \\ &+\tau N_2 \int_0^1 a(x)dx \int_0^\tau \int_0^1 a(x)u_1^2(x,t+s)dxds \\ &+\tau N_2 \int_0^1 b(x)dx \int_0^\tau \int_0^1 b(x)u_2^2(x,t+s)dxds \\ &\le M \int_t^{t+\tau} \int_0^1 \left[a(x)u_1^2(x,s) + b(x)u_2^2(x,s)\right] dxds \end{aligned}$$

where $M = \max\{(\tau N_1 + \tau N_2)||a||_1, (\tau N_2 + \tau N_3)||b||_1\}$.

Let $\mathcal{E}(t)$ be the energy function of system (8). From the section 2 we see that

$$\begin{aligned} \frac{d\mathcal{E}(t)}{dt} &= -\int_0^1 a(y)U_1^2(p_1,p_2,q_1,q_2)(y,t)dy \\ &-\int_0^1 b(y)U_2^2(p_1,p_2,q_1,q_2)(y,t)dy \le 0. \end{aligned}$$

So

$$\begin{aligned} &\mathcal{E}(t+\tau) + \int_0^\tau \int_0^1 a(y)U_1^2(p_1,p_2,q_1,q_2)(y,s)dy \\ &+\int_0^\tau \int_0^1 b(y)U_2^2(p_1,p_2,q_1,q_2)(y,s)dyds = \mathcal{E}(t). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\varepsilon(\tau,t) = \frac{1}{2} \\ &||(X(\cdot,t+\tau), X_t(\cdot,t+\tau)) - (\widehat{X}(\cdot,\tau,t), \widehat{X}_s(\cdot,\tau,t))||^2_{\mathcal{H}} \\ &= \frac{1}{2}||(X(\cdot,t+\tau), X_t(\cdot,t+\tau)) - (P(\cdot,t), Q(\cdot,t))||^2_{\mathcal{H}} \\ &\le M(\mathcal{E}(t) - \mathcal{E}(t+\tau)). \end{aligned}$$

The desired result follows from above inequality. □

3.2 The Proof of Theorem 2

Note that the vector from of system (3) is

$$\begin{cases} P_t(x,t) = Q(x,t) + Sin(\tau\mathcal{J})A(.)\Delta_1 U(.,t) \\ Q_t(x,t) = -\mathcal{J}P(x,t) + Cos(\tau\mathcal{J})A(.)\Delta_1 U(.,t) \\ \quad + A(x)\Delta_2 U(x,t) \\ P(x,0) = P_0(x), \\ Q(x,0) = Q_0(x). \end{cases} \tag{13}$$

We determine its dual system as follows:

$$\begin{aligned} &\int_0^T (\mathcal{J}^{\frac12}P_t(t), \mathcal{J}^{\frac12}W(t))_H + (Q_t(t), V(t))_H dt \\ &= \Big(\mathcal{J}^{\frac12}P(t), \mathcal{J}^{\frac12}W(t))_H + (Q(t), V(t)\Big)_H \\ &-\int_0^T \Big(\mathcal{J}^{\frac12}P(t), \mathcal{J}^{\frac12}W_t(t)\Big)_H + \Big(Q(t), V_t(t)\Big)_H dt \\ &= \int_0^T \Big(\mathcal{J}^{\frac12}[Q(x,t) + Sin(\tau\mathcal{J})A(.)\Delta_1 \\ &\qquad U(.,t)], \mathcal{J}^{\frac12}W(t)\Big)_H dt \\ &+\int_0^T \Big(-\mathcal{J}P(x,t) + Cos(\tau\mathcal{J})A(.)\Delta_1 U(.,t) \\ &\qquad + A(.)\Delta_2 U(.,t), V(t)\Big)_H dt \\ &= \int_0^T \Big(\mathcal{J}^{\frac12}Q(.,t), \mathcal{J}^{\frac12}W(t)\Big)_H dt \\ &+\int_0^T \Big(\mathcal{J}^{\frac12}Sin(\tau\mathcal{J})A(.)\Delta_1 U(.,t), \mathcal{J}^{\frac12}W(t)\Big)_H dt \\ &+\int_0^T (-\mathcal{J}P(x,t), V(t))_H dt \\ &+\int_0^T \Big(Cos(\tau\mathcal{J})A(.)\Delta_1 U(.,t) \end{aligned}$$

$$
\begin{aligned}
&+ A(.)\Delta_2 U(.,t), V(t)\Big)_H dt \\
&= -\int_0^T (\mathcal{J}^{\frac{1}{2}} P(x,t), \mathcal{J}^{\frac{1}{2}} V(t))_H dt \\
&+ \int_0^T (Q(x,t), \mathcal{J} W(t))_H dt \\
&+ \int_0^T (U(.,t), \Delta_1 A(.) Sin(\tau\mathcal{J}) W(t))_H dt \\
&+ \int_0^T (U(.,t), \Delta_1 A(.) Cos(\tau\mathcal{J}) V(t))_H dt \\
&+ \int_0^T (U(.,t), \Delta_2 A(.) V(t))_H dt
\end{aligned}
$$

where we have used the property of diagonal matrix of $A(x)$, Δ_1 and Δ_2. So its dual observation system is

$$
\begin{cases}
W_t(x,t) = V(x,t), \qquad x \in (0,1), t > 0 \\
V_t(x,t) = -\mathcal{J} W(x,t), \qquad x \in (0,1), t > 0 \\
W(x,0) = W_0, \qquad V(x,0) = V_0 \\
Y(x,t) = C(W,V) \\
= A(x)\Delta_2 V(x,t) + A(x)\Delta_1 Sin(\tau\mathcal{J})\mathcal{J} W(.,t) \\
\quad + A(x)\Delta_1 Cos(\tau\mathcal{J}) V(.,t).
\end{cases} \tag{14}
$$

So the closed loop system (8) can be resulted by the collocated feedback of system (14)

$$
\begin{pmatrix} P_t(x,t) \\ Q_t(x,t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\mathcal{J} & 0 \end{pmatrix} \begin{pmatrix} P(x,t) \\ Q(x,t) \end{pmatrix} - C^* C \begin{pmatrix} P(x,t) \\ Q(x,t) \end{pmatrix}. \tag{15}
$$

Therefore, the exponential stabilization of the system (8) is equivalent to the exact observability of the system (14) in finite time (see [24]).

To study the exact observability of system (14), we need the following Lemma.

Lemma 4 *(See [16, Proposition 3.4], or [11, Lemma 4.1]) Let H be a separable Hilbert space, and $\mathcal{J}$ be an unbounded positive definite operator. Assume that $\mathcal{J}$ satisfies the following conditions:*

- *$\mathcal{J}$ has compact resolvent and its spectrum is $\sigma(\mathcal{J}) = \{\lambda_n; n \in \mathbb{N}\}$.*
- *the spectra of $\mathcal{J}$ satisfy the separable condition*

$$
\inf_{m \neq n} \{\sqrt{\lambda_n} - \sqrt{\lambda_m}\} = \delta > 0
$$

- *the corresponding eigenvectors $\{\Phi_n; n \in \mathbb{N}\}$ with $\|\Phi_n\|_H = 1$ form a normalised orthogonal basis for H.*

Let Y be a Hilbert space. Assume that $C : \mathcal{D}(\mathcal{J}^{\frac{1}{2}}) \times H \to Y$ is an admissible observation operator for $\begin{pmatrix} 0 & I \\ -\mathcal{J} & 0 \end{pmatrix}$. Then the second order linear system

$$
\begin{cases}
K_{tt}(t) + \mathcal{J} K(t) = 0 \\
K(0) = K_0 \quad K_t(0) = K_1 \\
Y(t) = C(K, K_t)
\end{cases}
$$

is exactly observable in finite time on the energy space $\mathcal{H} = \mathcal{D}(\mathcal{J}^{\frac{1}{2}}) \times H$ if and only if

$$
\inf_{n \in \mathbb{N}} \left\| C\left(\frac{1}{i\sqrt{\lambda_n}} \Phi_n, \Phi_n\right) \right\|_Y^2 > 0 \tag{16}
$$

We are now in a position to check the exact observability of the system (14) in finite time. In our model, the space $Y = H = L^2_\rho[0,1] \times L^2_{I_\rho}[0,1]$. $\mathcal{J}$ is defined as in Lemma 3. If $\frac{K}{\rho} \neq \frac{EI}{I_\rho}$, then the condition 2 of Lemma 4 is fulfilled. The Lemma 3 asserts that $\Phi_n = (w_n(x), \varphi_n(x))^T$ satisfy the condition 3. Obviously, C is admissible operator. According to Lemma 4, we only need to check the condition (16). Since

$$
\begin{aligned}
& C\left(\frac{1}{i\sqrt{\lambda_n}} \Phi_n, \Phi_n\right) \\
& = A(x)\Delta_2 \Phi_n(x) \\
& + A(x)\Delta_1 (\mathcal{J}^{\frac{1}{2}} Sin(\tau\mathcal{J}))\mathcal{J}^{\frac{1}{2}} \frac{1}{i\sqrt{\lambda_n}} \Phi_n \\
& + A(x)\Delta_1 (Cos(\tau\mathcal{J}))\Phi_n \\
= & \begin{pmatrix} \frac{a(x)}{\rho} & \\ & \frac{b(x)}{I_\rho} \end{pmatrix} \begin{pmatrix} \beta_1 & \\ & \beta_2 \end{pmatrix} \begin{pmatrix} w_n(x) \\ \varphi_n(x) \end{pmatrix} \\
& + \begin{pmatrix} \frac{a(x)}{\rho} & \\ & \frac{b(x)}{I_\rho} \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \times \\
& \begin{pmatrix} -i \sin \tau\sqrt{\lambda_n} w_n(x) \\ -i \sin \tau\sqrt{\lambda_n} \varphi_n(x) \end{pmatrix} \\
& + \begin{pmatrix} \frac{a(x)}{\rho} & \\ & \frac{b(x)}{I_\rho} \end{pmatrix} \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} \times \\
& \begin{pmatrix} \cos \tau\sqrt{\lambda_n} w_n(x) \\ \cos \tau\sqrt{\lambda_n} \varphi_n(x) \end{pmatrix} \\
= & \begin{pmatrix} \frac{a(x)}{\rho} [\beta_1 + \alpha_1 e^{-i\tau\sqrt{\lambda_n}}] w_n(x) \\ \frac{b(x)}{I_\rho} [\beta_2 + \alpha_2 e^{-i\tau\sqrt{\lambda_n}}] \varphi_n(x) \end{pmatrix},
\end{aligned}
$$

we have

$$
\begin{aligned}
& \left\| C\left(\frac{1}{i\sqrt{\lambda_n}} \Phi_n, \Phi_n\right) \right\|_Y^2 \\
= & \frac{1}{\rho} \int_0^1 |a(x)| [\beta_1 + \alpha_1 e^{-i\tau\sqrt{\lambda_n}}] w_n(x)|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{I_\rho}\int_0^1 |b(x)[\beta_2+\alpha_2 e^{-i\tau\sqrt{\lambda_n}}]\varphi_n(x)|^2dx\\
=\ &\frac{|\beta_1+\alpha_1 e^{-i\tau\sqrt{\lambda_n}}|^2}{\rho^2}\int_0^1 \rho|a(x)w_n(x)|^2dx\\
&+\frac{|\beta_2+\alpha_2 e^{-i\tau\sqrt{\lambda_n}}|^2}{I_\rho^2}\int_0^1 I_\rho|b(x)\varphi_n(x)|^2dx.
\end{aligned}
$$

If $|\alpha_j|\neq|\beta_j|$, it holds that

$$
\begin{aligned}
&\inf_n|\beta_1+\alpha_1 e^{-i\tau\sqrt{\lambda_n}})|>0,\\
&\inf_n|\beta_2+\alpha_2 e^{-i\tau\sqrt{\lambda_n}})|>0.
\end{aligned}
$$

Set

$$
\begin{aligned}
M=\min\Bigg\{&\frac{1}{\rho^2}\inf_n|\beta_1+\alpha_1 e^{-i\tau\sqrt{\lambda_n}})|^2,\\
&\frac{1}{I_\rho^2}\inf_n|\beta_2+\alpha_2 e^{-i\tau\sqrt{\lambda_n}})|^2\Bigg\}.
\end{aligned}
$$

Then applying the condition (2) we have

$$
\begin{aligned}
&\left\|C\Big(\frac{1}{i\sqrt{\lambda_n}}\Phi_n,\Phi_n\Big)\right\|_Y^2\geq M\\
&\Big(\int_0^1\rho a^2(x)w_n^2(x)dx+\int_0^1 I_\rho b^2(x)\varphi_n^2(x)dx\Big)\\
&\geq M\left(\int_{c_1}^{c_2}\rho a_0^2w_n^2(x)dx+\int_{c_1}^{c_2}I_\rho b_0^2\varphi_n^2(x)dx\right)\\
&\geq M\min\{a_0^2,b_0^2\}\times\\
&\quad\left(\int_{c_1}^{c_2}\rho w_n^2(x)dx+\int_{c_1}^{c_2}I_\rho\varphi_n^2(x)dx\right)
\end{aligned}
$$

Note that

$$
1=||\Phi_n||_H^2=\int_0^1\rho w_n^2(x)dx+\int_0^1 I_\rho\varphi_n^2(x)dx.
$$

We can assert that

$$
\inf_n\left(\int_{c_1}^{c_2}\rho w_n^2(x)dx+\int_{c_1}^{c_2}I_\rho\varphi_n^2(x)dx\right)>0.
$$

Therefore, we have the following results.

Theorem 5 *Suppose that $\frac{K}{\rho}\neq\frac{EI}{I_\rho}$. If $|\alpha_j|\neq|\beta_j|$, $j=1,2$, then the system (14) is exactly observable in finite time.*

4 Conclusion

In this paper we extend the dynamic control design from the boundary control with delays to the interior local controls with delay for a Timoshenko beam. The new control strategy stabilizes exponentially the system for any time delay τ provided that $|\alpha_j|\neq|\beta_j|$, $j=1,2$.

Our contribution in this paper includes the following two aspects:

1) We found out the transform

$$
\begin{aligned}
&\begin{pmatrix}P(t)\\ Q(t)\end{pmatrix}=\\
&\begin{pmatrix}Cos(\tau\mathcal{J}) & Sin(\tau\mathcal{J})\\ -\mathcal{J}Sin(\tau\mathcal{J}) & Cos(\tau\mathcal{J})\end{pmatrix}\begin{pmatrix}K(t)\\ K_t(t)\end{pmatrix}\\
&+\int_{t-\tau}^t\begin{pmatrix}Sin((t-s)\mathcal{J})\\ Cos((t-s)\mathcal{J})\end{pmatrix}B\Lambda_2U(s)ds
\end{aligned}
$$

That transfer the delayed second order system:

$$
\begin{cases}K_{tt}(t)+\mathcal{J}K(t)=B(\Lambda_1U(t)+\Lambda_2U(t-\tau))\\ K(0)=K_0\quad K_t(0)=K_1\end{cases}
$$

into a undelayed system:

$$
\begin{cases}P_t(t)=Q(t)+Sin(\tau\mathcal{J})B\Lambda_1U(t),\\ Q_t(t)=-\mathcal{J}P(t)+Cos(\tau\mathcal{J})B\Lambda_1U(t)\\ \qquad+B\Lambda_2U(t),\\ P(0)=P_0,Q(0)=Q_0.\end{cases}
$$

The partial state predictive system is a realization of this transform.

2) We find out the dual observation system of the (P,Q)-system:

$$
\begin{cases}W_t(t)=V(t),\\ V_t(t)=-\mathcal{J}W(t),\\ W(0)=W_0,V(0)=V_0,\\ Y(t)=C(W,V)\\ =\Lambda_1^*B^*V(t)+\Lambda_1^*B^*\mathcal{J}^{\frac12}Sin(\tau\mathcal{J})\mathcal{J}^{\frac12}W(t)\\ +\Lambda_1^*B^*Cos(\tau\mathcal{J})V(t).\end{cases}
$$

Hence the negative feedback law $U(t)=-Y(t)$ give the closed loop system:

$$
\begin{cases}P_t(t)=Q(t)-Sin(\tau\mathcal{J})B\Lambda_1C(P,Q)(t)\\ Q_t(t)=-\mathcal{J}P(t)-Cos(\tau\mathcal{J})B\Lambda_1C(P,Q)(t)\\ \quad+B\Lambda_2C(P,Q)(t)\\ P(0)=P_0,Q(0)=Q_0.\end{cases}
$$

This discoveries make us take directly transform for second order system, and find out the closed loop system. Therefore, the final work is to prove the exponential stability of the closed loop system. In the present paper we mainly apply the dual principle to transfer the exponential stabilization into the exact observability in finite time. For more complicated model, we need to prove the observation inequality for the dual system. In the future, we shall focus our attention on the exponential stability of the closed loop system.

APPENDIX

In this appendix we use the operator manner to describe the procedure in section 2, and find out all functions in formula. To this end, we begin with introducing useful lemmas and theorems.

Lemma 6 *(See[11, Lemma 2.1]) Let the differential operator in $L^2_\rho[0,1] \times L^2_{I_\rho}[0,1]$ be defined by*

$$\mathcal{J}\left[\begin{array}{c} w \\ \varphi \end{array}\right] = -\left[\begin{array}{c} \frac{K}{\rho}(w''(x)-\varphi'(x)) \\ \frac{EI}{I_\rho}\varphi''(x) + \frac{K}{I_\rho}(w'(x)-\varphi(x)) \end{array}\right]$$
$$\mathcal{D}(\mathcal{J}) = \Big\{(w(x),\varphi(x)) \in H^2(0,1)\times H^2(0,1) \mid w(0)=\varphi(0)=0, K(w'(1)-\varphi(1))=0, EI\varphi'(1)=0\Big\}$$

Then $\mathcal{J}$ is a self-adjoin and positive definite operator with compact resolvent in $L^2_\rho[0,1] \times L^2_{I_\rho}[0,1]$, its eigenvalues are

$$0 < \lambda_1 < \lambda_2 \le \ldots \le \lambda_n \le \ldots$$

and the eigenfunctions $\Phi_n(x) = (w_n(x), \varphi_n(x))^T$ corresponding to λ_n are real functions and form a normalised orthogonal basis for $L^2_\rho[0,1] \times L^2_{I_\rho}[0,1]$.

Remark 7 *(See[11, Remark 2.1]) Let $d_1 = \sqrt{\frac{K}{\rho}}$ and $d_2 = \sqrt{\frac{EI}{I_\rho}}$. Then the spectrum of operator has two branches that have asymptotic expressions as follows:*

$$\begin{cases} \lambda_{n,1} = d_1^2\left[\left(n-\frac{1}{2}\right)\pi\right]^2 + o(n^{-1}); \\ \lambda_{n,2} = d_2^2\left[\left(n-\frac{1}{2}\right)\pi\right]^2 + o(n^{-1}), \quad n\in\mathbb{N}. \end{cases}$$

We write equations (1) in the vector form:

$$\begin{aligned} &\left(\begin{array}{c} w_{tt}(x,t) \\ \varphi_{tt}(x,t) \end{array}\right) - \left(\begin{array}{cc} \frac{K}{\rho}\partial_{xx} & -\frac{K}{\rho}\partial_x \\ \frac{K}{I_\rho}\partial_x & \frac{EI}{I_\rho}\partial_{xx} + \frac{K}{I_\rho} \end{array}\right)\left(\begin{array}{c} w(x,t) \\ \varphi(x,t) \end{array}\right) \\ &= \left(\begin{array}{cc} \frac{a(x)}{\rho} & 0 \\ 0 & \frac{b(x)}{I_\rho} \end{array}\right)\left(\begin{array}{cc} \alpha_1 & 0 \\ 0 & \alpha_2 \end{array}\right)\left(\begin{array}{c} u_1(x,t) \\ u_2(x,t) \end{array}\right) \\ &+ \left(\begin{array}{cc} \frac{a(x)}{\rho} & 0 \\ 0 & \frac{b(x)}{I_\rho} \end{array}\right)\left(\begin{array}{cc} \beta_1 & 0 \\ 0 & \beta_2 \end{array}\right)\left(\begin{array}{c} u_1(x,t-\tau) \\ u_2(x,t-\tau) \end{array}\right) \\ &\left(\begin{array}{c} w(0,t) \\ \varphi(0,t) \end{array}\right) = 0, \\ &\left(\begin{array}{cc} K\partial_x & -K \\ 0 & EI\partial_x \end{array}\right)\left(\begin{array}{c} w(x,t) \\ \varphi(x,t) \end{array}\right)_{x=1} = 0, \\ &\left(\begin{array}{c} w(x,0) \\ \varphi(x,0) \end{array}\right) = \left(\begin{array}{c} w_0(x) \\ \varphi_0(x) \end{array}\right), \\ &\left(\begin{array}{c} w_t(x,0) \\ \varphi_t(x,0) \end{array}\right) = \left(\begin{array}{c} w_1(x) \\ \varphi_1(x) \end{array}\right). \end{aligned}$$

Set $X(x,t) = (w(x,t),\varphi(x,t))^T$ and $U(x,t) = (u_1(x,t),u_2(x,t))^T$. Define 2×2 matrices by

$$\Delta_1 = \left(\begin{array}{cc} \alpha_1 & 0 \\ 0 & \alpha_2 \end{array}\right), \quad \Delta_2 = \left(\begin{array}{cc} \beta_1 & 0 \\ 0 & \beta_2 \end{array}\right),$$

and define an operator Γ_N from $H^2(0,1)\times H^2(0,1)$ to $\mathbb{R}^2$ and a matrix function, by

$$\Gamma_N = \left(\begin{array}{cc} K\partial_x & -K \\ 0 & EI\partial_x \end{array}\right), \; A(x) = \left(\begin{array}{cc} \frac{a(x)}{\rho} & 0 \\ 0 & \frac{b(x)}{I_\rho} \end{array}\right).$$

With help of these notations, we can rewrite the system (1) into

$$\begin{cases} X_{tt}(x,t) + \mathcal{J}X(x,t) \\ = A(x)(\Delta_1 U(x.t) + \Delta_2 U(x,t-\tau)), \; t>0, \\ X(0,t) = 0, \qquad \Gamma_N X(1,t) = 0, \\ X(x,0) = X_0(x) = (w_0(x),\varphi_0(x))^T, \\ X_t(x,0) = X_1(x) = (w_1(x),\varphi_1(x))^T \end{cases} \tag{17}$$

and the partial state predictive system (4) into

$$\begin{cases} \widehat{X}_{ss}(x,s,t) + \mathcal{J}\widehat{X}(x,s,t) \\ = A(x)\Delta_2 U(x,t+s-\tau), \;\; s\in(0,\tau), x\in(0,1) \\ \widehat{X}(0,s,t) = 0, \qquad \Gamma_N\widehat{X}(1,s,t) = 0, \\ \widehat{X}(x,0,t) = X(x,t), \\ \widehat{X_s}(x,0,t) = X_t(x,t) \end{cases} \tag{18}$$

By Lemma 6, we can define two family of the bounded linear operators on $L^2_\rho[0,1] \times L^2_{I_\rho}[0,1]$ by

$$\begin{aligned} Sin(t\mathcal{J})P &= \sum_{n=1}^{\infty}\frac{\sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}(P,\Phi_n)_{L^2_\rho\times L^2_{I_\rho}}\Phi_n \\ Cos(t\mathcal{J})P &= \sum_{n=1}^{\infty}\cos\sqrt{\lambda_n}t(P,\Phi_n)_{L^2_\rho\times L^2_{I_\rho}}\Phi_n. \end{aligned} \tag{19}$$

Clearly,

$$\frac{d}{dt}Sin(t\mathcal{J})P = Cos(t\mathcal{J})P,$$

$$\frac{d}{dt}Cos(t\mathcal{J})P = -\mathcal{J}Sin(t\mathcal{J})P.$$

We define the vector-valued function $X(x,t)$ by

$$\begin{aligned}X(x,t) &= Cos(t\mathcal{J})X_0 + Sin(t\mathcal{J})X_1\\ &+ \int_0^t Sin((t-s)\mathcal{J})A(\cdot)\Delta_1 U(\cdot,s)ds\\ &+ \int_0^t Sin((t-s)\mathcal{J})A(\cdot)\Delta_2 U(\cdot,s-\tau)ds.\end{aligned}$$

It is easy to see that

$$\begin{aligned}X_t(x,t) &= -\mathcal{J}Sin(t\mathcal{J})X_0 + Cos(t\mathcal{J})X_1\\ &+ \int_0^t Cos((t-s)\mathcal{J})A(x)\Delta_1 U(x,s)ds\\ &+ \int_0^t Cos((t-s)\mathcal{J})A(x)\Delta_2 U(x,s-\tau)ds,\\ X_{tt}(x,t) &= -\mathcal{J}X(x,t)\\ &+ A(x)[\Delta_1 U(x,t) + \Delta_2 U(x,t-\tau)].\end{aligned}$$

So $X(x,t)$ is a solution of system of equation (1). Similarly, the function

$$\begin{aligned}\widehat{X}(x,s,t) &= Cos(s\mathcal{J})X(.,t) + Sin(s\mathcal{J})X_t(.,t)\\ &+ \int_0^s Sin((s-r)\mathcal{J})A(\cdot)\Delta_2 U(\cdot,t+r-\tau)dr\end{aligned}$$

is a formal solution of (4).

Denote the state of (4) at the moment $s=\tau$ by

$$\begin{cases} P(x,t) = (p_1(x,t),p_2(x,t))^T = \widehat{X}(x,\tau,t)\\ Q(x,t) = (q_1(x,t),q_2(x,t))^T = \widehat{X}_s(x,\tau,t)\end{cases}$$

Therefore, we have

$$\begin{aligned}\begin{pmatrix} P(x,t)\\ Q(x,t)\end{pmatrix} &= \begin{pmatrix}\widehat{X}(x,\tau,t)\\ \widehat{X}_s(x,\tau,t)\end{pmatrix}\\ &= \begin{pmatrix} Cos(\tau\mathcal{J}) & Sin(\tau\mathcal{J})\\ -\mathcal{J}Sin(\tau\mathcal{J}) & Cos(\tau\mathcal{J})\end{pmatrix}\begin{pmatrix} X(x,t)\\ X_t(x,t)\end{pmatrix}\\ &+ \int_0^\tau \begin{pmatrix} Sin((\tau-r)\mathcal{J})\\ Cos((\tau-r)\mathcal{J})\end{pmatrix} A(.)\Delta_2 U(\cdot,t+r-\tau)dr\\ &= \begin{pmatrix} Cos(\tau\mathcal{J}) & Sin(\tau\mathcal{J})\\ -\mathcal{J}Sin(\tau\mathcal{J}) & Cos(\tau\mathcal{J})\end{pmatrix}\begin{pmatrix} X(x,t)\\ X_t(x,t)\end{pmatrix}\\ &+ \int_{t-\tau}^t \begin{pmatrix} Sin((t-s)\mathcal{J})\\ Cos((t-s)\mathcal{J})\end{pmatrix} A(.)\Delta_2 U(.,s)ds\end{aligned}$$

i.e.,

$$\begin{aligned}\begin{pmatrix} P(x,t)\\ Q(x,t)\end{pmatrix} &= \begin{pmatrix} Cos(\tau\mathcal{J}) & Sin(\tau\mathcal{J})\\ -\mathcal{J}Sin(\tau\mathcal{J}) & Cos(\tau\mathcal{J})\end{pmatrix}\begin{pmatrix} X(x,t)\\ X_t(x,t)\end{pmatrix}\\ &+ \int_{t-\tau}^t \begin{pmatrix} Sin((t-s)\mathcal{J})\\ Cos((t-s)\mathcal{J})\end{pmatrix} A(.)\Delta_2 U(.,s)ds\end{aligned} \tag{20}$$

is the transform from (1) to (3). The corresponding initial condition is given by

$$\begin{aligned}&\begin{pmatrix} P(x,0)\\ Q(x,0)\end{pmatrix}\\ &= \begin{pmatrix} Cos(\tau\mathcal{J}) & Sin(\tau\mathcal{J})\\ -\mathcal{J}Sin(\tau\mathcal{J}) & Cos(\tau\mathcal{J})\end{pmatrix}\begin{pmatrix} X_0(x)\\ X_1(x)\end{pmatrix}\\ &+ \int_{-\tau}^0 \begin{pmatrix} -Sin(s\mathcal{J})\\ Cos(s\mathcal{J})\end{pmatrix} A(.)\Delta_2 U(.,s)ds\\ &= \begin{pmatrix} Cos(\tau\mathcal{J})X_0 + Sin(\tau\mathcal{J})X_1\\ -\int_{-\tau}^0 Sin(s\mathcal{J})A(.)\Delta_2 U(.,s)\mathrm{d}s\\ \\ -\mathcal{J}Sin(\tau\mathcal{J})X_0 + Cos(\tau\mathcal{J})X_1\\ +\int_{-\tau}^0 Cos(s\mathcal{J})A(.)\Delta_2 U(.,s)\mathrm{d}s\end{pmatrix}.\end{aligned}$$

In addition, we observe that

$$\begin{aligned}\begin{pmatrix} P_t(x,t)\\ Q_t(x,t)\end{pmatrix} &= \begin{pmatrix} Cos(\tau\mathcal{J}) & Sin(\tau\mathcal{J})\\ -\mathcal{J}Sin(\tau\mathcal{J}) & Cos(\tau\mathcal{J})\end{pmatrix}\times\\ &\begin{pmatrix} 0 & I\\ -\mathcal{J} & 0\end{pmatrix}\begin{pmatrix} X(x,t)\\ X_t(x,t)\end{pmatrix}\\ &+ \begin{pmatrix} Sin(\tau\mathcal{J})A(.)[\Delta_1 U(.,t)]\\ Cos(\tau\mathcal{J})A(.)[\Delta_1 U(.,t)] + A(.)\Delta_2 U(.,t)\end{pmatrix}\\ &+ \int_{t-\tau}^t \begin{pmatrix} Cos((t-s)\mathcal{J})\\ -\mathcal{J}Sin((t-s)\mathcal{J})\end{pmatrix} A(.)\Delta_2 U(.,s)ds\\ &= \begin{pmatrix} 0 & I\\ -\mathcal{J} & 0\end{pmatrix}\begin{pmatrix} P(x,t)\\ Q(x,t)\end{pmatrix}\\ &+ \begin{pmatrix} \sin(\tau\mathcal{J})A(.)\Delta_1 U(.,t)\\ \cos(\tau\mathcal{J})A(.)\Delta_1 U(.,t) + A(x)\Delta_2 U(x,t)\end{pmatrix}.\end{aligned}$$

Therefore, the (P,Q) satisfy the following equation

$$\begin{cases} P_t(x,t) = Q(x,t) + Sin(\tau\mathcal{J})A(.)\Delta_1 U(.,t)\\ Q_t(x,t) = -\mathcal{J}P(x,t) + Cos(\tau\mathcal{J})A(.)\Delta_1 U(.,t)\\ \quad + A(x)\Delta_2 U(x,t)\\ \Gamma_N P(1,t) = 0, P(0,t) = Q(0,t) = 0,\\ P(x,0) = P_0(x), Q(x,0) = Q_0(x)\end{cases} \tag{21}$$

or equivalently,

$$\begin{cases} p_{1,t}(x,t) = q_1(x,t) \\ +\alpha_1 \int_0^1 H_1(x,\tau,y)a(y)u_1(y,t)dy \\ +\alpha_2 \int_0^1 H_2(x,\tau,y)b(y)u_2(y,t)dy, \\ p_{2,t}(x,t) = q_2(x,t) \\ +\alpha_1 \int_0^1 H_3(x,\tau,y)a(y)u_1(y,t)dy \\ +\alpha_2 \int_0^1 H_4(x,\tau,y)b(y)u_2(y,t)dy, \\ q_{1,t}(x,t) = \frac{K}{\rho}(p_{1,xx}(x,t) - p_{2,x}(x,t)) \\ +\alpha_1 \int_0^1 H_5(x,\tau,y)a(y)u_1(y,t)dy \\ +\alpha_2 \int_0^1 H_6(x,\tau,y)b(y)u_2(y,t)dy \\ +\frac{\beta_1}{\rho}a(x)u_1(x,t), \\ q_{2,t}(x,t) = \frac{EI}{I_\rho}p_{2,xx}(x,t) \\ +\frac{K}{I_\rho}(p_{1,x}(x,t) - p_2(x,t)) \\ +\alpha_1 \int_0^1 H_7(x,\tau,y)a(y)u_1(y,t)dy \\ +\alpha_2 \int_0^1 H_8(x,\tau,y)b(y)u_2(y,t)dy \\ +\frac{\beta_2}{I_\rho}b(x)u_2(x,t), \\ p_1(0,t) = p_2(0,t) = q_1(0,t) = q_2(0,t) = 0, \\ K(p_{1,x}(1,t) - p_2(1,t)) = 0, \\ EIp_{2,x}(1,t) = 0, \\ p_1(x,0) = G_1(w_0,\varphi_0,w_1,\varphi_1)(x) \\ -\beta_1 \int_{-\tau}^0 \int_0^1 H_1(x,s,y)a(y)u_1(y,s)dyds \\ -\beta_2 \int_{-\tau}^0 \int_0^1 H_2(x,s,y)b(y)u_2(y,s)dyds, \\ p_2(x,0) = G_2(w_0,\varphi_0,w_1,\varphi_1)(x) \\ -\beta_1 \int_{-\tau}^0 \int_0^1 H_3(x,s,y)a(y)u_1(y,s)dyds \\ -\beta_2 \int_{-\tau}^0 \int_0^1 H_4(x,s,y)b(y)u_2(y,s)dyds, \\ q_1(x,0) = G_3(w_0,\varphi_0,w_1,\varphi_1)(x) \\ +\beta_1 \int_{-\tau}^0 \int_0^1 H_5(x,s,y)a(y)u_1(y,s)dyds \\ +\beta_2 \int_{-\tau}^0 \int_0^1 H_6(x,s,y)b(y)u_2(y,s)dyds, \\ q_2(x,0) = G_4(w_0,\varphi_0,w_1,\varphi_1)(x) \\ +\beta_1 \int_{-\tau}^0 \int_0^1 H_7(x,s,y)a(y)u_1(y,s)dyds \\ +\beta_2 \int_{-\tau}^0 \int_0^1 H_8(x,s,y)b(y)u_2(y,s)dyds. \end{cases} \tag{22}$$

The functions appearing in (22) are

$$\begin{cases} H_1(x,s,y) = \sum_{n=1}^{\infty} \frac{\sin(\sqrt{\lambda_n}s)}{\sqrt{\lambda_n}} w_n(y)w_n(x), \\ H_2(x,s,y) = \sum_{n=1}^{\infty} \frac{\sin(\sqrt{\lambda_n}s)}{\sqrt{\lambda_n}} \varphi_n(y)w_n(x)), \\ H_3(x,s,y) = \sum_{n=1}^{\infty} \frac{\sin(\sqrt{\lambda_n}s)}{\sqrt{\lambda_n}} w_n(y)\varphi_n(x), \\ H_4(x,s,y) = \sum_{n=1}^{\infty} \frac{\sin(\sqrt{\lambda_n}s)}{\sqrt{\lambda_n}} \varphi_n(y)\varphi_n(x); \\ H_5(x,s,y) = \partial_s H_1(x,s,y) \\ \quad = \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n}s)w_n(y)w_n(x), \\ H_6(x,s,y) = \partial_s H_2(x,s,y) \\ \quad = \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n}s)\varphi_n(y)w_n(x)), \end{cases}$$

and

$$\begin{cases} H_7(x,s,y) = \partial_s H_3(x,s,y) \\ = \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n}s)w_n(y)\varphi_n(x), \\ H_8(x,s,y) = \partial_s H_4(x,s,y) \\ = \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n}s)\varphi_n(y)\varphi_n(x) \end{cases}$$

and functions $G_k(X_0, X_1)$:

$$G_1(X_0,X_1) = \sum_{n=1}^{\infty} \Bigg[\cos\tau\sqrt{\lambda_n}(X_0,\Phi_n)_{L^2_\rho \times L^2_{I_\rho}} + \frac{\sin\tau\sqrt{\lambda_n}}{\lambda_n}(X_1,\Phi_n)_{L^2_\rho \times L^2_{I_\rho}} \Bigg] w_n(x)$$

$$G_2(X_0,X_1) = \sum_{n=1}^{\infty} \Bigg[\cos\tau\sqrt{\lambda_n}(X_0,\Phi_n)_{L^2_\rho \times L^2_{I_\rho}} + \frac{\sin\tau\sqrt{\lambda_n}}{\lambda_n}(X_1,\Phi_n)_{L^2_\rho \times L^2_{I_\rho}} \Bigg] \varphi_n(x)$$

$$G_3(X_0,X_1) = \sum_{n=1}^{\infty} \Bigg[\cos\tau\sqrt{\lambda_n}(X_1,\Phi_n)_{L^2_\rho \times L^2_{I_\rho}} - \sqrt{\lambda_n}\sin\tau\sqrt{\lambda_n}(X_0,\Phi_n)_{L^2_\rho \times L^2_{I_\rho}} \Bigg] w_n(x)$$

$$G_4(X_0,X_1) = \sum_{n=1}^{\infty} \Bigg[\cos\tau\sqrt{\lambda_n}(X_1,\Phi_n)_{L^2_\rho \times L^2_{I_\rho}} - \sqrt{\lambda_n}\sin\tau\sqrt{\lambda_n}(X_0,\Phi_n)_{L^2_\rho \times L^2_{I_\rho}} \Bigg] \varphi_n(x).$$

Acknowledgements: This research is supported by the Natural Science Foundation of China grant NSFC-61174080.

References:

[1] J. U. Kim, Y. Renardy, Boundary Control of the Timoshenko Beam, *SIAM Journal on Control and Optimization*, (1987), 25(6): pp. 1417-13.

[2] G. Q. Xu, D. X. Feng, The Riesz basis property of a Timoshenko beam with boundary feedback and application, *IMA Journal of Applied Mathematics*, (2002), 67, pp.357-370.

[3] G. Q. Xu, S. P. Yung, Exponential decay rate for a Timoshenko beam with boundary damping, *Journal of Optimization Theory and Applications*, (2004), 123, pp.669-693.

[4] G. Q. Xu, Boundary feedback exponential stabilization of a Timoshenko beam with both ends free, *International Journal of Control*, (2005), 78, pp.286-297.

[5] R. Datko, J. Lagnese, M. P. Polis, An example of the effect of time delays in boundary feedback stabilization of wave equations, *SIAM Journal on Control and Optimization*, (1986), 24, pp.152-156.

[6] R. Datko, Not all feedback stabilized hyperbolic systems are robust with respect to small time delay in their feedbacks, *SIAM Journal on Control and Optimization*, (1988), 26, pp.697-713.

[7] R. Datko, Two examples of ill-posedness with respect to small delays in stabilized elastic systems, *IEEE Transactions on Automatic Control*, (1993), 38, pp.163-166.

[8] G. Q. Xu, S. P. Yung, L. K. Li, Stabilization of wave systems with input delay in the boundary control, *ESAIM: Control, Optimization and Calculus of Variations*, (2006), 12, pp.770-785.

[9] S. Nicaise, C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM Journal on Control and Optimization*, (2006), 45(5), pp.1561-1585.

[10] S. Nicaise, J. Valein, Stabilization of the wave equation on 1-d networks with a delay term in the nodal feedbacks, *Networks and Heterogeneous Media*, (2007), 2(3), pp.425-479.

[11] Y. F. Shang, G. Q. Xu,Y. L. Chen, Stability analysis of Euler-Bernoulli beam with input delay in the boundary control, *Asian Journal of Control*, (2012), 14, pp.186-196.

[12] Z. J. Han, G. Q. Xu, Exponential stability of Timoshenko beam system with delay terms in boundary feedbacks. *ESAIM:Control, Optimization and Calculus of Variations*, (2010),17, pp.552-574.

[13] Y. F. Shang, G. Q. Xu, Stabilization of an Euler-Bernoulli beam with input delay in the boundary control, *Systems and Control Letters*, (2012), 61, pp.1069-1078.

[14] G. Q. Xu, H. X. Wang, Stabilisation of Timoshenko beam system with delay in the boundary control, *International Journal of Control*, (2013), 86(6): pp. 1165-1178.

[15] H. Wang and G. Q. Xu, Exponential stabilization of 1-d wave equation with input delay, *WSEAS Trans. Math.*, 12, 2013, pp.1001-1013.

[16] Z. J. Han and G. Q. Xu, Output-based stabilization of EulerCBernoulli beam with time-delay in boundary input, *IMA Journal of Mathematical Control and Information*, (2014), 31 (4): pp.533-550. doi:10.1093/imamci/dnt030

[17] Y. F. Shang and G. Q. Xu, Dynamic feedback control and exponential stabilization of a compound system, Journal of Mathematical Analysis and Applications, 422, (2015), pp. 858-879.

[18] X. F. Liu and G. Q. Xu, Exponenntial stabilization for Timoshenko beam with distributed delay in the boundary control, *Abstract and Applied Analysis*, (2013), doi:10.1155/2013/726794.

[19] X. F. Liu and G. Q. Xu, Output-based stabilization of Timoshenko beam with the boundary control and input distributed delay, *Journal of Dynamic and Control System*, (2015), doi: 10.1007/s10883-015-9293-4

[20] Y. F. Shang, G. Q. Xu, X. Li, Output-based stabilization for a one-dimensional wave equation with distributed input delay in the boundary control, *IMA Journal of Mathematical Control and Information*, (2014), doi:10.1093/imamci/dnu030

[21] D. H. Shi, D. X. Feng, Exponential decay of Timoshenko beam with locally distributed feedback, *IMA Journal of Mathematical Control and Information*, (2001), 18(3): pp.395-403.

[22] A. Soufyane, A. Whebe, Uniform stabilization for the Timoshenko beam by a locally distributed damping, *Electronic Journal of Differential Equations*, (2003).

[23] C. A. Raposo, J. A. D. Chuquipoma, J. A. J. Avila, M. L. Santos, Exponential decay and numerical solution for a Timoshenko system with delay term in the internal feedback, *International Journal of Analysis and Applications*, (2013), 3(1), pp.1-13.

[24] B. Z. Guo, Y. H. Luo, Controllability and stability of a second-order hyperbolic system with collocated sensor/ actuator, *Systems and Control Letters*, (2002), 46, pp.45-65.