The Bounded Solutions of a Fourth Order Model Equation

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Abstract: The objective of this paper is to construct bounded solutions of a model equation, which governs two-dimensional steady capillary-gravity waves of an ideal fluid flow with Bond number near 1/3 and Froude number close to one.

Key-Words: Lyapunov's Center Theorem, Schauder fixed point theorem , Bounded Solution.

1 Introduction

Progressive capillary-gravity waves on an irrotional incompressible inviscid fluid of constant density with surface tension in a two-dimensional channel of finite depth have been studied since nineteen century. Assume that a coordinate system moving with the wave at a speed is chosen so that in reference to it the wave motion is steady. Let H be the depth of water, g the acceleration of gravity, T the coefficient of surface tension, and ρ the constant density of the fluid. Then there are two nondimensional numbers which are important and defined as $F = c^2/(gH)$, the Froude number, and $\tau = T/(\rho g H^2)$, the Bond number.

When F is not close to 1, the linear theory of water waves is applicable. But when F approaches to 1, the solutions of linearized equations of water waves will grow to infinity (Peters and Stoker [12]). Therefore for F close to 1 nonlinear effect must be taken into account and thus $F = 1$ is a critical value. The first study of a solitary wave on water with surface tension is due to Korteweg and DeVries [10] after whom the K-dV equation with surface tension effect is named. A stationary K-dV equation with Bond number not near $1/3$ can also be formally derived by different approaches. However, if τ is close to 1, the formal derivation of the stationary K-dV equation fails. Thus $\tau = 1/3$ is also a critical value.

It becomes apparent that the problems for F near 1 and for τ near $1/3$ depend on each other and are difficult because they are not only strongly nonlinear, but also very delicate. Since the full nonlinear equations for the water waves are too complicated to study, it is of interest to study model equations. In Hunter and Vanden -Broeck's work [8], a fifth order ordinary differential equation considered as a perturbed stationary K-dV equation was obtained with the assumption that $F = 1 + F_2\epsilon^2$, $\tau = 1/3 + \tau_1\epsilon$ and ϵ is a small positive parameter. By integrating the fifth order ordinary differential equation once and set the con-stant of integration to be zero, then the model equation becomes

$$2F_2\eta - \frac{3}{2}\eta^2 + \tau_1\eta_{xx} - \frac{1}{45}\eta_{xxxx} = 0 \qquad (1)$$

Equation (1) has been studied extensively by many authors [1-8] and several types of solutions have been found, such as periodic solutions [1, 5, 6, 7], solitary wave solutions [2-8], generalized solitary wave solutions (solitary waves with osciallatory tails at infinity) in the parameter region $\tau_1 < 0$ and $F_2 > 0$ [1,8], etc.

2 Problem Formulation

We add a bump $y = b(x)$ at the bottom of the two-dimensional ideal fluid flow and then derive a forced model equation

$$2F_2\eta - \frac{3}{2}\eta^2 + \tau_1\eta_{xx} - \frac{1}{45}\eta_{xxxx} = b \qquad (2)$$

We follow Zufiria [18] to construct a Hamiltonian associated to (2).

When $\mathbf{b} = 0$, we rewrite (2) as

$$\eta_{xxxx} - 45\tau_1\eta_{xx} - 90F_2\eta + \frac{135}{2}\eta^2 = 0 . \qquad (3)$$

We multiply $-\eta_x$ to (3) and integrate the resulting equation, then equation (3) has first integral as

$$H = 45F_2\eta^2 + \frac{1}{2}\eta_{xx}^2 - \eta_{xxx}\eta_x + \frac{45}{2}\tau_1\eta_x^2 - \frac{45}{2}\eta^3 , \qquad (4)$$

where H is a constant. Introducing the change of variables

$$\left.\begin{aligned} q_1 &= \eta, \quad p_1 = \eta_{xxx} - 45\tau_1\eta_{x,} \\ q_2 &= \eta_{xx}, \ p_2 = \eta_{x,} \end{aligned}\right\}$$

then (4) becomes

$$H\left(q_1,q_2,p_1,p_2\right)=45F_2q_1^2+\frac{1}{2}q_2^2 - p_1p_2-\frac{45}{2}\tau_1p_2^2-\frac{45}{2}q_1^3, \qquad (5)$$

and we have

$$\frac{dz}{dx} = J\nabla_z H(z) = Az + g(z) \equiv f(z,\mu), \ (6)$$

where $\mu = (\tau_1, F_2) \in \mathbf{R}^2$,

$$z = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \in \mathbf{R}^4, \ J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ (7)$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -45\tau_1 \\ -90F_2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, g(z) = \begin{pmatrix} 0 \\ 0 \\ \frac{135}{2}q_1^2 \\ 0 \end{pmatrix}. \ (8)$$

Therefore (5) is a two degree of freedom Hamiltonian with two parameters τ_1 and F_2. Because different parameters (τ_1, F_2) in (5) give rise to different eigenvalues λ for the linearized system of (6) at the origin, we divide the parameter plane (τ_1, F_2) into following nine cases

Case 0 $(\tau_1 = 0, F_2 = 0)$: $\lambda=0,0,0,0$.

Case 1 $(\tau_1 \in \mathbf{R}, F_2 > 0)$: $\lambda=\pm r, \pm wi;\ r,w>0$.

Case 2 $(\tau_1 < 0, F_2 = 0)$: $\lambda=0,0,\pm wi;\ w>0$.

Case 3 $(\tau_1 < 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 > 0)$:
$\lambda = \pm w_1 i, \pm w_2 i;\ w_1 > w_2 > 0$.

Case 4 $(\tau_1 < 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 = 0)$:
$\lambda = \pm wi, \pm wi;\ w > 0$

Case 5 $(\tau_1 \in \mathbf{R}, F_2 < 0,\ (45\tau_1)^2 + 360F_2 < 0)$:
$\lambda = \pm a \pm bi;\ a,b > 0$

Case 6 $(\tau_1 > 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 = 0)$:
$\lambda = \pm r, \pm r;\ r > 0$

Case 7 $(\tau_1 > 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 > 0)$:
$\lambda = \pm r_1, \pm r_2;\ r_1 > r_2 > 0$

Case 8 $(\tau_1 > 0, F_2 = 0)$: $\lambda=0,0,\pm r;\ r > 0$.

We rewrite (2) as follows,

$$\eta_{xxxx} - 45\tau_1\eta_{xx} - 90F_2\eta = -45(\mathbf{b}(x)) + \frac{3}{2}\eta^2) \equiv f, \ (9)$$

3 Problem Solution

In this section, we would like to discuss the bounded solutions of model equation (9).

3.1 Case 1

In this subsection, we shall construct a bounded solution for equation (9) in Case 1. We construct this half-periodic and half-solitary-wave solution as follows : On interval $(-\infty, x_1)$, we let b(x) = 0 and use Lyapunov's Center Theorem to show that a periodic solution $\eta_P(x)$ exists initiating at $x = x_1$ to the left. On $[x_1, x_2]$, we shall use Schauder fixed point theorem to prove there exist a bounded solution $\eta_C(x)$ for equation (9) subject to initial values $(\eta_P(x_1),\ \eta'_P(x_1),\ \eta''_P(x_1),\ \eta'''_P(x_1))$ at $x = x_1$. On (x_2, ∞) , we also let b(x) = 0 and show that equation (9) with initial values at $x = x_2$ has a solution $\eta_R(x)$, which decay to zero exponentially at positive infinity by using a theorem from [6]. Then we combine $\eta_P(x), \eta_C(x)$ and $\eta_R(x)$ to have a solution of equation (9). Since the proof of existence of bounded solution $\eta_C(x)$ and $\eta_R(x)$ on $[x_1, x_2]$ and (x_2, ∞) are the same as in [17], in the following, we shall focus on the existence of $\eta_P(x)$ on interval $(-\infty, x_1)$. First, we state Lyapunov's Center Theorem :

Theorem 1 *Assume that a system with a non - degenerate integral has an equilibrium point with exponents* $\pm wi,\ \lambda_3, \cdots, \lambda_m$ *where* $iw \neq 0$ *is pure imaginary. If* λ_j / iw *is never an integer for* $j = 3, \cdots, m$, *then there exists a one-parameter family of periodic orbits emanating from the equilibrum point. Moreover, when approaching the equilibrum point along the family, the periods tend to* $2\pi / w$.

When b(x) = 0, equation (2) possesses a Hamiltonian (5) H and an equilibrium at the origin. In **Case 1**, the engenvalues of the linearized systems of (9) are $\pm wi$ and $\pm r$ where $w=(-(45\tau_1-((45\tau_1)^2+360F_2)^{\frac{1}{2}})/2)^{\frac{1}{2}}>0$ and $r=((45\tau_1+((45\tau_1)^2+360F_2)^{\frac{1}{2}}/2)^{\frac{1}{2}}>0$. Thus, by Theorem, there exists a periodic motion of period close to $2\pi/w$ in the nonlinear system of differential equations with the Hamiltonian H. Since the amplitude of the periodic motions are small and depends on initial conditions, we can write the periodic solutions in the form [13]

$$z_P(x;\epsilon)=\epsilon e^{Ax}a+O(\epsilon^2) \tag{10}$$

where ϵ is a small parameter, A is the same as in (8), and a is a fixed nonzero vector such that $z_P(0;\epsilon)/\epsilon \to a$ when $\epsilon\to 0$. We rewrite (10) in eigenvector coordinates as

$$\hat{z}_P(x;\epsilon)=\epsilon e^{\Lambda x}\hat{a}+(\epsilon^2) \tag{11}$$

where $\hat{z}_P(x;\epsilon)=P^{-1}z_P(x;\epsilon), \hat{a}=P^{-1}a, \Lambda=diag(-wi,wi,-r,r)$, and P is a 4×4 matrix with the column vectors ξ_1,ξ_2,ξ_3, and ξ_4 corresponding to the unit eigenvectors of eigenvalues $-wi, wi, -r$ and r. We see that $\hat{a}$ must be in the form $(\hat{a}_1,\hat{a}_2,0,0)$, otherwise (11) will not be periodic. Therefore, vector $a\in R^4$ lies in the two dimensional eigenspace $S_a=\{\xi_1,\xi_2\}$ where ξ_2 is the conjugate of ξ_1.

On interval $(-\infty,x_1)$, by Theorem and the discussion above, there is a one-parameter family of periodic solutions in the form (10) with initial values $z_P(x_1;\epsilon)$ having the properties that $z_P(x_1;\epsilon)/\epsilon \to e^{Ax_1}a$ as $\epsilon\to 0$ and $a\in S_a\cap R^4$. The solution η_R on (x_2,∞) can be found by Theorem as in [17]. As in [17], the bounded solution $\eta_C(x)$ on (x_1,x_2) is obtained by Schauder fixed point theorem and it is required that the initial values at $x=x_1$ and the bump b both must be sufficiently small. Now, we write the first component of $z_P(x;\epsilon^*;x_1)$ as $\eta_P(x;\epsilon^*;x_1)$ to obtain the solution of (9) on $(-\infty,x_1)$. As in [17], we combine $\eta_P(x;\epsilon^*;x_1)$, $\eta_C(x;x_1)$, $\eta_R(x;T_2)$ and to be a solution of equation (9) in **Case 1**, which is periodic on interval $(-\infty,x_1)$ and decays to zero exponentially at positive infinity on interval (x_2,∞).

3.2 Case 3

The idea to investgate the solutions of equation (9) for the parameters τ_1,F_2 corresponding to Case 3 is to combine solutions on three different intervals $(-\infty,x_1),[x_1,x_2]$, and (x_2,∞). On $[x_1,x_2]$, we shall prove there exists bounded solutions of equation (9) with initial values at $x=x_1$ by Schauder fixed point theorem. On intervals $(-\infty,x_1)$ and (x_2,∞), we let b(x) = 0 and show that equation (9) has periodic solutions. Then these solutions can be combined to become a C4 solution of equation (9).

From section 2, we know that the eigenvalues of the linearized systems of equation (9) in Case 3 are two pairs of pure imaginaries, $\pm w_1 i$ and $\pm w_2 i$, with $w_1>w_2>0$. When w_1/w_2 is irrational, Lyapunov's Center Theorem can be used to construct periodic solutions on intervals $(-\infty,x_1)$ and (x_2,∞). There exist two one-parameter families of periodic orbits emanating from the fixed point $z=0$. If we let $w=w_1$ in Lyapunov's Center Theorem, then the periods of this one-parameter periodic family tend to $2\pi/w_1$ when the fixed point is approached along the family. We call this family as short-period family since $w_1>w_2$. If we let $w=w_2$ in Lyapunov's Center Theorem, then the periods of this one-parameter periodic family tend to $2\pi/w_2$ when the fixed point is approached along the family. We call this family as long-period family.

We write the short-period family of periodic solutions in the form

$$z_{w_1}(x;\epsilon)=\epsilon e^{Ax}a_{w_1}+O(\epsilon^2) \tag{12}$$

where ϵ is a small parameter,

$$A=\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 90F_2 & 0 & 45\tau_1 & 0\end{pmatrix}$$

, and a_{w_1} is a fixed nonzero vector such that $z_{w_1}(0;\epsilon)/\epsilon \to a_{w_1}$ when $\epsilon\to 0$.

We rewrite (9) in eigenvector coordinates as

$$\hat{z}_{w_1}(x;\epsilon)=\epsilon e^{\Lambda x}\hat{a}_{w_1}+O(\epsilon^2) \tag{13}$$

where $\hat{z}_{w_1}(x;\epsilon)=P^{-1}z_{w_1}(x;\epsilon), \hat{a}_{w_1}=P^{-1}a, \Lambda=diag(-w_1 i,w_1 i,-w_2 i,w_2 i)$, and P is a 4×4 matrix with the column vectors ξ_1,ξ_2,ξ_3, and ξ_4 corresponding to the unit eigenvectors of eigenvalues $-w_1 i, w_1 i, -w_2 i,$ and $w_2 i$ respectively. We see that $\hat{a}_{w_1}$ must be in the form $(\hat{a}_{w_{11}}, \hat{a}_{w_{12}}, 0, 0)$ since the periods tend to $2\pi/w_1$ when we approach the fixed point zero along the family. Therefore, vector $a_{w_1} \in R^4$ lies in the two dimensional eigenspace $S_{w_1} = \{\xi_1, \xi_2\}$ where ξ_2 is the conjugate of ξ_1. Thus, on intevral $(-\infty, x_1)$, we have periodic solutions (13) in short-period family with initial values $z_{w_1}(x_1;\epsilon)$ having the properties that $z_{w_1}(x_1;\epsilon)/\epsilon \to e^{Ax_1} a_{w_2}$ as $\epsilon \to 0$ and $a_{w_2} \in S_{w_2} \cap R^4$. On intevral (x_2, ∞), we also obtain periodic solutions (13) in short - period family with initial values $z_{w_2}(x_2;\epsilon)$ having the properties that $z_{w_1}(x_2;\epsilon)/\epsilon \to e^{Ax_2} a_{w_1}$ as $\epsilon \to 0$ and $a_{w_1} \in S_{w_1} \cap R^4$.

By the same arguments as above on periodic solutions of short-period family, we have periodic solutions $z_{w_2}(x;\epsilon) = \epsilon e^{Ax} a_{w_2} + O(\epsilon^2)$ in long-period family on intevral $(-\infty, x_1)$ with initial values $z_{w_2}(x_1;\epsilon)$ having the properties that $z_{w_2}(x_1;\epsilon)/\epsilon \to e^{Ax_1} a_{w_2}$ as $\epsilon \to 0$ and $a_{w_2} \in S_{w_2} \cap R^4$ where $S_{w_2} = \{\xi_3, \xi_4\}$ and ξ_4 is the conjugate of ξ_3. On intevral (x_2, ∞), we also have periodic solutions $z_{w_2}(x;\epsilon)$ in long-period family with initial values $z_{w_2}(x_2;\epsilon)$ having the properties that $z_{w_2}(x_2;\epsilon)/\epsilon \to e^{Ax_2} a_{w_2}$ as $\epsilon \to 0$ and $a_{w_2} \in S_{w_2} \cap R^4$.

As in [17], the bounded solution $\eta_C(x)$ on $[x_1, x_2]$ is obtained by Schauder fixed point theorem and it is required that the initial values at x = x_1 and the bump b both must be sufficiently small such that MY and M_b satisfy (98) and (100) in [17]. These requirements could be met by choosing a small bump b and sufficiently small ϵ, say ϵ^*. Now, we write the first component of $z_{w_1}(x;\epsilon_1^*;x_1)$ or $z_{w_2}(x;\epsilon_2^*;x_1)$ as $\eta_L(x;\epsilon_L;x_1)$ to be the solution of (9) on $(-\infty, x_1)$. In [16], we showed that the zero solution is stable for Case 3, thus bounded $\eta_R(x;\epsilon_R;x_2)$ on interval (x_2, ∞) can be obtained if $(\eta_c(x_2), \eta_c'(x_2), \eta_c''(x_2), \ \eta_c'''(x_2))$ is small and this could be done as disscussed in [17]. As in [17], we combine $\eta_L(x;\epsilon_L;x_1)$, $\eta_C(x;x_1)$, and $\eta_R(x;\epsilon_R;x_2)$ to obtain a solution of equation (9) in Case 3 with w_1/w_2 irrational, which is periodic on interval $(-\infty, x_1)$ and bounded on $[x_1, \infty]$.

3.3 Case 4

In this subsection, we would like to discuss the solutions of equation (9) for the parameters τ_1, F_2 corresponding to **Case 4**. As in previous subsections, we shall show the existence of solutions of equation (9) on three different intervals $(-\infty, x_1), [x_1, x_2]$, and (x_2, ∞). Then combine these solutions to become a C^4 solution of equation (9).

First we show there exist peroidic solutions of equation (9) with $\mathbf{b}(x) = 0$ by a theorem from Meyer [13]. In [13], Meyer discussed the bifurc- ation occurring in restricted 3-body problem. The Hamiltonian he concerned depends on a parameter $\bar{\mu}$ and has the properties that the eigenvalues of the associated linearized operator are (I) $\pm i w_2, \pm i w_2$ if $\bar{\mu} > 0$ where $w_1, w_2 \in \mathbf{R}, w_1 \neq w_2$, and $w_1 w_2 \neq 0$. (e.g., In **Case 3**) (II) $\pm iw, \pm iw$ if $\bar{\mu} = 0$ where $w \in \mathbf{R}$ and $w \neq 0$, with two two-dimensional Jordan blocks. (e.g., In **Case 4**) (III) $\pm a \pm ib$ if $\bar{\mu} < 0$ where $a, b \in \mathbf{R}$ and $ab \neq 0$.(e.g., In **Case 5**). Meyer transforms the perturbed Hamiltonian to Sokol'skii's normal form

$$H(\bar{\mu}) = w\Gamma_1 + \delta\Gamma_2 + \bar{\mu}(a\Gamma_1 + b\Gamma_3) + \frac{1}{2}(c\Gamma_1^2 + 2d\Gamma_1\Gamma_3 + e\Gamma_3^2) + \cdots, \tag{14}$$

where

$$\Gamma_1 = i(z_2 z_4 - z_1 z_3), \quad \Gamma_2 = z_1 z_2, \quad \Gamma_3 = z_3 z_4. \tag{15}$$

With higher-order terms in H are functions of Γ_1, Γ_2 and Γ_3 only. Then he proved the following results.

Theorem 2 *Consider a Hamiltonian of the form (14) with* $w \neq 0,\ \delta = \pm 1,\ b \neq 0,\ e \neq 0.$ *Assume* $\delta e > 0.$ *There exist two Lyapunov families of periodic orbits emanating from the origin when* $\delta b \bar{\mu}$ *is small and positive. These families persist when* $\bar{\mu} = 0$ *as two distinct families of periodic orbits emanating from the origin. As* $\delta b \bar{\mu}$ *becomes negative, the two families detach from the origin as a single family and recede from the origin.*

Theorem 2 can be used to show a Hamiltonian Holf biurcation near **Case 4** since the Hamiltonian system (5) has the same properties as Meyer [13] discussed if we assume

$$F_2 = -\frac{45}{8}\tau_1^2 + \bar{\mu},$$

with $\tau_1 < 0$ and the parameter $\bar{\mu} \approx 0$. After transforming (5) to Sokol'skii's normal form (14), we have

$$w=(-\frac{45}{2}\tau_1)^{\frac{1}{2}},\quad \delta=-1,\quad b=\frac{90}{w^2\delta},\quad e=-\frac{38475}{32w^8}.$$

Since $\delta e>0$, we have the same results as **Theorem 2** described. When $\bar{\mu}>0$ is small and we are in the region **Case 3** but near **Case 4**, there are two Lyapunov periodic families emanating from the origin. These families persist when $\bar{\mu}=0$ as two distinct families of periodic orbits emanating from the origin. As $\bar{\mu}<0$ is small and we are in the region **Case 5** but near **Case 4**, the two families detach from the origin as a single family and recede from the origin.

The periodic solutions derived by **Theorem 2** with $\bar{\mu}=0$ which corresponds to **Case 4** can be used as $\eta_L(x)$ in interval $(-\infty,x_1]$. The existence of $\eta_C(x)$ on $[x_1,x_2]$ can also be proved by the same arguments in [17]. On interval $(x_2,\infty]$, since the zero solution of equation (9) with $\mathbf{b}(x)=0$ is almost stable, bounded $\eta_R(x)$ for large x is obtained provided that $(\eta_R(x_2)=\eta_C(x_2),\eta_R'(x_2)=\eta_C'(x_2),\eta_R''(x_2)=\eta_C''(x_2),\eta_R'''(x_2)=\eta_C'''(x_2))$ is small and this could be done as we disscussed in section [17].

As in [17], we match $\eta_L(x),\eta_C(x)$, and $\eta_R(x)$ at $x=x_1$ and $x=x_2$ to obtain a solution of equation (9) in **Case 4** which is periodic on interval $(-\infty,x_1)$ and bounded on $[x_1,\infty)$ for large x.

3.4 Case 5

In this subsection, we shall construct bounded solutions of the model equation (9) for **Case 5**.

Our idea is to investgate the solutions of equation (9) on three different intervals $(-\infty,-T_1),[-T_1,T_2]$, and (T_2,∞), where T_1 and T_2 are positive constants and will be specified later. On intervals $(-\infty,-T_1)$ and (T_2,∞), we try to show that equation (9) with initial values at $x=-T_1$ on $(-\infty,-T_1)$ and initial values at $x=T_2$ on (T_2,∞) has bounded solutions $\eta_L(x)$ and $\eta_R(x)$, respectively, which decay to zero exponentially at negative and positive infinity by using a theorem from [6]. On $[-T_1,T_2]$, we shall use Schauder fixed point theorem to prove there exist bounded solutions $\eta_C(x)$ of equation (9) subject to initial values $(\eta_L(-T_1)$, $-\eta_L'(-T_1)$, $\eta_L''(-T_1)$, $-\eta_L'''(-T_1))$ at $x=-T_1$. Then we combine $\eta_C(x)$, $\eta_L(x)$, and $\eta_R(x)$ to obtain a solution of equation (9).

3.4.1 Solutions on $(-\infty,-T_1)$ and (T_2,∞)

On interval (T_2,∞), we rewrite (9) as a system of first order differential equations,

$$\frac{dz}{dx}=Az+g(z), \tag{16}$$

where $z(x)=(\eta(x),\eta'(x),\eta''(x),\eta'''(x))^t$,

$$A=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\90F_2&0&45\tau_1&0\end{pmatrix}$$

$$\text{and}\quad g(x,z)=\begin{pmatrix}0\\0\\0\\-\frac{135}{2}\eta^2-45\mathrm{b}(x)\end{pmatrix}. \tag{17}$$

In the following, we shall use a theorem from [6] to prove that (16) with some restriction on the initial values at $x=T_2$ has bounded solutions. The theorem is stated as follows :

We consider the asymptotic behavior of the solutions of equation

$$\frac{dz}{dx}=Az+f(x,z), \tag{18}$$

where A is a constant matrix and f is a continuous vector function defined for $x\ge x_0$, $|z|<c$. Then the underlying vector space X can be uniquely represented as the direct sum of three suspaces X_{-1}, X_0, X_1 invariant under A on which all characteristic roots of A have real parts respectively less than, equal to, greater than μ. We shall denote by P_i the corresponding projection of X onto $X_i(i=-1,0,1)$.

Theorem 3 *Suppose that at least one characteristic root of A has real part $\mu<0$ and*

$$f(x,z)=o(|z|)\quad \mathit{for}\quad x\to\infty,\ |z|\to 0 \tag{19}$$

holds.

Then there exist positive constants k, K depending only on A and positive constants T, ρ depending also on f such that if $x^* \geq T$ *and if* $\xi_{-1} \in X_{-1}, \xi_0 \in X_0$ *satisfy*

$$|\xi_{-1}| \leq k\,|\xi_0|, \quad 0 < |\xi_0| < \frac{\rho}{2K}, \qquad (20)$$

then the equation (18) has at least one solution z(x) for $x \geq x^*$ *satisfying*

$$P_{-1}z(x^*) = \xi_{-1}, \quad P_0 z(x^*) = \xi_0, \qquad (21)$$

$|z(x)| \leq \rho$ *for* $x \geq x^*$ and

$$\mu = \lim_{x\to\infty} x^{-1} log\,|z(x)|. \qquad (22)$$

For **Case 5,** there exists at least one eigenvalue with negative real part and $g(x, z)$ in (17) satisfies (19) since $\mathbf{b}(x)$ is compact on $[x_1, x_2]$. Hence, by Theorem 1, there are bounded solutions $z_R(x)$ of equation (16) subject to the initial values $z_R(T_2)$ that satisfy (20) and (21) with $T_2 \geq T$. Then we have $\eta_R(x)$, the first component of $z_R(x)$, as the solution of (9) subject to the initial values $z_R(T_2) = (\eta_R(T_2), \eta_R'(T_2), \eta_R''(T_2), \eta_R'''(T_2))^t$ on interval (T_2, ∞).

For interval $(-\infty, -T_1)$, we let $x = -\hat{x}$ and put it in (9), then equation (9) does not change except that the independent variable is replaced by $\hat{x}$. Thus, by **Theorem 3** again, there exist bounded solutions $z_L(\hat{x})$ of equation (16) subject to the initial value $z_L(T_1) = (\eta_L(T_1), \eta_L'(T_1), \eta_L''(T_1), \eta_L'''(T_1))$ that satisfy (20) and (21) with $T_1 \geq T$. Hence, by substituting $\hat{x} = -x$, we obtain $\eta_L(x)$, the first component of $z_L(x)$, to be the solution of (9) subject to the initial values $z_L(-T_1) = (\eta_L(-T_1), -\eta_L'(-T_1), \eta_L''(-T_1), -\eta_L'''(-T_1))^t$ on interval $(-\infty, -T_1)$.

Next, we shall prove there is a bounded solution $\eta_C(x)$ of (9) subject to initial value $(\eta_L(-T_1), -\eta_L'(-T_1), \eta_L''(-T_1), -\eta_L'''(-T_1))$ at $x = -T_1$ on interval $[-T_1, T_2]$ and the end point value, $(\eta_C(T_2), \eta_C'(T_2), \eta_C''(T_2), \eta_C'''(T_2))$, which also satisfies (20) and (21).

3.4.2 Solutions on $[-T_1, T_2]$

From (9) and posing initial values at $x = -T_1$, we have:

$$\eta_{xxxx} - 45\tau_1\eta_{xx} - 90F_2\eta = -45(\mathbf{b}(x) + \tfrac{3}{2}\eta^2) \equiv f(\eta), \quad x \geq -T_1,$$
$$\eta(-T_1) = P, \quad \eta_x(-T_1) = Q,$$
$$\eta_{xx}(-T_1) = R, \quad \eta_{xxx}(-T_1) = S, \qquad (23)$$

where

$$P = \eta_L(-T_1), Q = -\eta_L'(-T_1), R = \eta_L''(-T_1), S = -\eta_L'''(-T_1).$$

To analyze the solutions of (23), we transform the ordinary differential equation (23) to an integral equation. First we solve the homogeneous equation of (23) :

$$Y_{xxxx} - 45\tau_1 Y_{xx} - 90F_2 Y = 0, \qquad x \geq -T_1$$
$$Y(-T_1) = P,\ Y_x(-T_1) = Q,\ Y_{xx}(-T_1) = R,$$
$$Y_{xxx}(-T_1) = S \qquad (24)$$

Next, we use $Y(x)$ in (24) and let $\eta = S + Y$ to convert equation (23) as follows :

$$S_{xxxx} - 45\tau_1 S_{xx} - 90F_2 S = f, \qquad x \geq -T_1$$
$$S(-T_1) = 0,\ S_x(-T_1) = 0,\ S_{xx}(-T_1) = 0,$$
$$S_{xxx}(-T_1) = 0. \qquad (25)$$

Let the causal Green's function of equation (25) be $G(x, t)$, then we have

$$S(x) = \int_{-T_1}^{\infty} G(x,t)\, f(\eta(t))dt \qquad (26)$$

Thus we transform the differential equation (23) to the integral equation :

$$\eta(x) = Y(x) + \int_{-T_1}^{x} G(x-t)$$
$$\left\{-45(\mathbf{b}(t) + \frac{3}{2}\eta^2(t))\right\} dt = Q(\eta)(x) \qquad (27)$$

To prove the existence of a bounded solution of equation (9) initiating at $x = -T_1$ on the interval $[-T_1, T_2]$, we need to show that the operator defined by the right-hand side of (27) has a fixed point. In other words, we try to find a function $\hat{\eta}$ such that $Q(\hat{\eta})(x) = \hat{\eta}(x)$ for all $x \in [-T_1, T_2]$. We take the domain of Q to be

$$K \underset{def}{=} \{\eta \in C([-T_1,T_2];\mathbf{R}) \mid |\eta(x)| \le M \text{ for } x \in [-T_1,T_2]\}, \quad (28)$$

where M is some positive real number and should be chosen in such a way that Q maps K into itself.

It is clear that the function $x \mapsto Q(\eta)(x)$ is conti- nuous. In order to prove that Q maps K into itself it remains only to analyze the size of $|Q(\eta)(x)|$. If $\eta \in K$, then we have for all $x \in [-T_1, T_2]$

$$|Q(\eta)(x)| \le M_Y + M_G M_x (M_{\mathbf{b}} + \frac{3}{2} M^2) \quad (29)$$

where

$$M_Y = \max_{x \in [-T_1,T_2]} |Y(x)|, \quad M_G = \max_{x,t \in [-T_1,T_2]} |G(x,t)|$$

and

$$M_x = x_2 - x_1, \quad M_{\mathbf{b}} = \sup_{x \in [-T_1,T_2]} |\mathbf{b}(x)|.$$

If we assume that the right-hand side of (29) $\le M$, then we have

$$\frac{3}{2} M_G (M - M^+)(M - M^-) \le 0$$

where

$$M^{\pm} = \frac{1 \pm (1 - 6 M_G M_x (M_G M_{\mathbf{b}} M_x + M_Y))^{\frac{1}{2}}}{3 M_G M_x} \quad (30)$$

and

$$(1 - 6 M_G M_x (M_G M_{\mathbf{b}} M_x + M_Y)) \ge 0. \quad (31)$$

The inequality (31) can be satisfied if we choose bump **b** and the initial values in (23) such that both $M_{\mathbf{b}}$ and M_Y are sufficiently small. Hence, if we take $M \in [M^-, M^+]$ and inequality (31) is also satisfyied, then $|Q(\eta)(x)| \le M$ for all $-T_1 \le x \le T_2$, and Q maps K into itself.

The set K is a bounded, closed, and convex subset of the Banach space $C([-T_1, T_2])$. To apply Schauder's theorem it suffices, therefore, to show that Q is a compact map of K into itself. By the Arzelà-Ascoli Theorem and by what we have already proved, this amounts to showing that the set $\{Q(\eta) \mid \eta \in K\}$ is equicontinuous. The following simple estimate acco- mplishes the task. Let $-T_1 \le \xi \le x$, then

$$\begin{aligned}
|Q(\eta)(x) - Q(\eta)(\xi)| \;&\le\; |Y(x) - Y(\xi)| \\
&+ \left|\int_{\xi}^{x} G(x-t) f(\eta(t)) dt\right| \\
&+ \left|\int_{-T_1}^{\xi} (G(x-t) - G(\xi - t)) f(\eta(t)) dt\right| \\
&\le |Y(x) - Y(\xi)| + \\
&\sup_{|\eta| \le M} |f(\eta)| \left\{\int_0^{x-\xi} |G(x - \xi - t)| dt\right\} + \\
&\sup_{|\eta| \le M} |f(\eta)| \left\{\int_0^{\xi + T_1} |G(x - \xi + t) - G(t)| dt\right\}.
\end{aligned}$$

Since the function Y and G are continuous, we conclude that the set $\{Q(\eta) \mid \eta \in K\}$ is equicontinuous on $[-T_1, T_2]$. An application of the Schauder Theorem tells us that there exists a fixed point η_C of Q.

To combine $\eta_L(x), \eta_C(x)$ and $\eta_R(x)$ to be a solution of equation (9), it requires that the end point values, $(\eta_C(T_2), \eta_C'(T_2), \eta_C''(T_2), \eta_C'''(T_2))$ which will be used as the initial values of $z_R(x)$ on (x_2, ∞), satisfy (20) and (21) in Theorem 3. This needs the right hand side of (29) to be small and this could be done by having $M_Y, M_{\mathbf{b}}$ and M sufficiently small. Observing (30), the positive number M^- could be as small as we want by choosing sufficiently small $M_{\mathbf{b}}$ and M_Y, and thus M could be as small as required. Therefore, we obtain an bounded solution of (9) in **Case 5.**

4 Conclusion

We construct bounded solutions of a model equation, which governs two-dimensional steady capillary-gravity waves of an ideal fluid flow with Bond number near 1/3 and Froude number close to one.

References:

[1] Amick, C.J. & Mcleod, J.B., ”A singular perturbation problem in waterwaves,” *Stability Appl. Anal. Continuous Media* **1**, 127-148 (1991)

[2] Amick, C.J. & Toland, J.F., ”Solitary waves with surface tension I: Trajectories homoclinic to periodic orbits in four dimensions,” *Arch. Rat. Mech. Anal.* **118**, 37-69 (1992)

[3] Amick, C.J. & Toland, J.F., ”Homoclinic orbits in the dynamic phasespace analogy of an elastic strut,” *Euro. Jnl of Applied Mathematics* **3**, 97-114 (1992)

[4] B. Buffoni, A.R. Champneys, A.R., Toland, J.F., ”Bifurcation and colescence of a plethora of homoclinic orbits for a Hamiltonian system,” *J.Dyn. Differential Equations* **8,** 221-281 (1996)

[5] Champneys, A.R. ”Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics,” *Physica D* **112**, 158-186 (1998)

[6] Coppel, W.A. ”Stability and Asymptotic Behavior of Differential equations,” *D.C. HEATH AND COMPANY*, (1965)

[7] Coppola, V.T. & Rand, R.H. ”Computer Algebra Implemention of Lie Transforms for Hamiltonian Systems: Application to the Nonlinear Stability of *L4*,” *ZAMM.Z.angew. Math. Mech.* **69**, 275-284 (1989)

[8] Hunter J.K., and Vanden-Broeck, J.M ., ”Solitary and periodic gravity-capillary waves of finite amplitude,” *J. Fluid Mech.* **134**, 205-219 (1983)

[9] Iooss, G. & Kirchgässner, K. ”Water waves for small surface tension: an approach via normal form,” *Proceedings of the Royal of Edinburgh* **122A**, 267-299 (1992)

[10] Korteweg, P.J. and de Vries, G. ”On the change of the form of long waves advancing in a rectangular canal, and on a new type of long stationary waves,” *Phil. Mag.* **39**, 422-443 (1895)

[11] Markeev, A.P. ”Stability of a canonical system with two degrees of freedom in the presence of resonance,” *Journals of Applied Mathematics and Mechanics* **32**, 766-722 (1968)

[12] Merkin, D.R. ”Intrduction to the Theory of Stability,” *Springer-Verlag*, **TAM 24**, (1997)

[13] Meyer, K.R. & Hall, G.R. ”Intrduction to Hamiltonian Dynamical Systems and the N-Body Problem,” *Springer-Verlag*, **90**, (1990)

[14] Peters, A.D.; Stoker, J.J. ”Solitary waves in liquids having non-constant density.,” *Comm. Pure Appl. Math.* **13**, 115-164 (1960)

[15] Sokol’skii, A.G. ”On the stability of an autonomous Hamiltonian system with two degrees of freedom in the case of equal frequencies,” *J. Appl. Math.Mech.*, **38**, 741-749 (1975)

[16] Tsai, Chung-Hsien & Guo, Shy-Jen ” The stability of zero solution of a model equation for steady capillary-gravity waves with the bond number close to 1/3.,” *Proceedings of the 6th WSEAS International Conference on FLUID MECHANICS (FLUIDS'09),* pp.90-93, Ningbo, China, January 10-12 (2009).

[17] Tsai, Chung-Hsien & Guo, Shy-Jen ” Unsymmetric Solitary Wave Solutions of a Fifth Order Model Equation for Steady Capillary-Gravity Waves over a Bump with the Bond Number Near 1/3,” *Proceedings of the 6th WSEAS International Conference on FLUID MECHANICS (FLUIDS'09)*, pp.94-99, Ningbo, China January 10-12 (2009).

[18] Tsai, Chung-Hsien & Guo, Shy-Jen ”Stability of Fixed Points of a Fifth Order Equation,” *18th International Conference on APPLIED MATHEMATICS (AMATH '13)*, pp.94-99, Budapest, Hungary December 189-193 (2013).

[19] Zufiria, J.A. ”Weakly nonlinear non-symmetric gravity waves on water of finite depth,” *J. fluid Mech.* **180**, 371-385 (1987)