On (B, m)-preinvex functions and optimality conditions for (B, m)-preinvex programming

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Abstract: A new class of generalized invex sets and generalized preinvex functions, called m-invex sets and (B, m)-preinvex functions are defined by modulating the m-convexity and B-preinvexity. Furthermore, some characterizations of (B, m)-preinvex functions are presented and the sufficient conditions of (B, m)-preinvex programming are established. In addition the nonlinear multi-objective programming with (B, m)-preinvex functions are considered and some relationships between vector critical point and weakly efficient solution for multi-objective programming with (B, m)-preinvexity are researched.

Key–Words: (B, m)-preinvex functions, m-invex sets, (B, m)-invex sets, (B, m)-preinvex programming, multi-objective programming

1 Introduction

Owing to the importance of the invexity and generalized preinvexity in the study of optimality to solve mathematical programming, researchers worked a lot on the generalized convex functions. For example, in earlier papers, Toder(1984)[14] introduced a class of functions called m-convex functions. Singh and Bector (1991)[12] studied a class of b-vex functions by relaxing the definition of convex function. Suneja $et\ al.$(1993)[13] gave a class of B-preinvex functions by relaxing the definitions of preinvex and B-vex functions. They also made some researches on characterizations of m-convex and B-preinvex functions.

Recently, these classes of generalized convex functions caused a lot of research interests. Especially for the research of preinvex functions. Such as, Luo and Wu(2008)[10] showed that the same result of preinvex functions and semistrictly preinvex functions or even more general ones of Yang $et\ al.$(2001)[15] can be obtained under weaker assumptions. Emam(2010)[2] discussed some their properties and obtained sufficient optimality criteria for nonlinear programming involving roughly B-invex functions and generalized roughly B-invex functions. For more results on generalized B-vex and preinvex functions, place refer to [8, 9, 15, 17] and closely related references therein.

Very Recently, more scholars began to do the generalized preinvex functions, like E-preinvex, D-preinvex and G-preinvex functions(see Refs.[3, 6, 16] and the references therein) and invex(Ref.[11]), (H_p, r)-invex functions(Ref.[7]). These scholars's researches promoted the development of the preinvex functions. Therefore, it is important to consider wider classes of generalized preinvex functions and also to seek practical characterizations for preinvexity and generalized preinvexity. We find that we can get a new class of generalized invex sets and preinvex functions by combining m-convexity and B-preinvexity together. We also find out some examples to prove the existence of these classes of sets and functions. So, we turn our attention to this new research.

Inspired by the research works[1, 2, 6, 9, 10], the purpose of this paper is to present a new class of generalized preinvex functions which is called (B, m)-preinvex functions and discuss some properties of this class of functions. We also give the sufficient conditions of optimality for both unconstrained and inequality constrained programming involving (B, m)-preinvexity. Moreover, we consider the nonlinear multi-objective programming with (B, m)-preinvex functions and study some relationships between vector critical point and weakly efficient solution for multi-objective programming with (B, m)-preinvexity.

The remainder of this paper is organized as follows. In Sect. 2, we first recall the definition of

m-convex functions, invex sets and B-preinvex functions, then the present paper defines the new m-invex sets and (B, m)-preinvex functions. We also give some examples to show that there exists this class sets and functions. In Sect. 3, some properties of m-invex sets and (B, m)-preinvex functions are presented. In Sect. 4, a new (B, m)-preinvex programming is studied and the sufficient conditions of optimality are established under the (B, m)-preinvexity. Sect. 5 studies some relationships between vector critical point and weakly efficient solution for multi-objective programming with (B, m)-preinvexity.

2 Notations and Preliminaries

In this section, for convenience, several definitions about m-convex functions, invex sets and B-preinvex functions, which will be needed in sequel, from Toder(1984)[14] and Suneja *et al.*(1993)[13] are summarized below. We use M, X to denote nonempty subsets of R^n, R_+ to denote the set of nonnegative real numbers.

Definition 1. *The function f: $[0, b] \to R$ is said to be m-convex if*

$$f\big(\lambda x + m(1-\lambda)y\big) \leq \lambda f(x) + m(1-\lambda)f(y) \quad (1)$$

holds for all $x, y \in [0, b]$, $\lambda \in [0, 1]$ and fixed $m \in (0, 1]$.

Definition 2. *Let $y \in M \subseteq R^n$. The set M is said to be invex with respect to $\eta : M \times M \to R^n$ at $y \in M$ if $y + \lambda\eta(x, y) \in M$ holds for each $x \in M$ and any $\lambda \in [0, 1]$.*

Definition 3. *Let $M \subseteq R^n$ be an invex set with respect to $\eta : R^n \times R^n \to R^n$. The function $f : M \to R$ is said to be B-preinvex at $y \in M$ with respect to η, $b_1 : M \times M \times [0, 1] \to R_+$ and $b_2 : M \times M \times [0, 1] \to R_+$, if for any $x \in M$ and $\lambda \in [0, 1]$,*

$$f\big(y + \lambda\eta(x, y)\big) \leq b_1(x, y, \lambda)f(y) + b_2(x, y, \lambda)f(x), \quad (2)$$

where

$$b_1(x, y, \lambda) + b_2(x, y, \lambda) = 1, b_1(x, y, 0) = 1 = b_2(x, y, 1).$$

The function $f(x)$ is said to be B-preinvex on M with respect to η, b_1 and b_2 if $f(x)$ is B-preinvex at each $y \in M$ with respect to η, b_1 and b_2; $f(x)$ is said to be strictly B-preinvex on M with respect to η, b_1 and b_2 if strict inequality holds in (2) for all $x, y \in M$.

Combining Definition 1, Definition 2 and Definition 3, we now introduce a new class generalized invex sets and a new class generalized preinvex functions, to be referred to as m-invex sets and (B, m)-preinvex functions, respectively. Some examples are provided to show that these generalized invex sets and preinvex functions are existed.

Definition 4. *Let $y \in X \subseteq R^n$. The set X is said to be m-invex with respect to $\eta : X \times X \to R^n$ at $y \in X$ if there exists fixed $m \in (0, 1]$, such that $my + \lambda\eta(x, y) \in X$, for each $x \in X$ and any $\lambda \in [0, 1]$.*

Remark 5. *When $m = 1$ and $my + \lambda\eta(x, y) \in X$ holds for each $x, y \in X$ and any $\lambda \in [0, 1]$, then the m-invex sets will become an invex set. When $\lambda = 0$, it is easy to see that $my \in X$.*

Example 6. *Let $m = \frac{1}{3}$ and $X = [-\pi/2, 0) \bigcup (0, \pi/2]$*

$$\eta(x, y) = \begin{cases} \cos(x-y), & \text{if } x \in (0, \pi/2], y \in (0, \pi/2]; \\ -\cos(x-y), & \text{if } x \in [-\pi/2, 0), y \in [-\pi/2, 0); \\ \cos(y), & \text{if } x \in [-\pi/2, 0), y \in (0, \pi/2]; \\ -\cos(y), & \text{if } x \in (0, \pi/2], y \in [-\pi/2, 0). \end{cases}$$

then X is an m-invex set with respect to η for $\lambda \in [0, 1]$ and $m = \frac{1}{3}$. It is obvious that X is not a convex set.

Definition 7. *Let $X \subseteq R^n$ be an m-invex set with respect to $\eta : R^n \times R^n \to R^n$. The function $f : X \to R$ is said to be a (B, m)-preinvex function with respect to η, $b_1 : X \times X \times [0, 1] \to R_+$ and $b_2 : X \times X \times [0, 1] \to R_+$ at $y \in X$, if for any $x \in X$, $\lambda \in [0, 1]$ and some fixed $m \in (0, 1]$*

$$f\big(my + \lambda\eta(x, y)\big) \leq mb_1(x, y, \lambda)f(y) + b_2(x, y, \lambda)f(x), \quad (3)$$

where

$$b_1(x, y, \lambda) + b_2(x, y, \lambda) = 1, \quad b_1(x, y, 0) = 1 = b_2(x, y, 1). \quad (4)$$

The function $f(x)$ is said to be (B, m)-preinvex on X with respect to η, b_1 and b_2 if $f(x)$ is B-preinvex at each $y \in X$ with respect to η, b_1 and b_2; $f(x)$ is said to be strictly (B, m)-preinvex on X with respect to η, b_1 and b_2 if strict inequality holds in (3) for all $x, y \in X$.

Remark 8. *When $m = 1$ and the formula (3) holds for any $x, y \in X$, then the (B, m)-preinvex function $f(x)$ reduces to B-preinvex on X. But the (B, m)-preinvex function is not necessarily a convex function, see the example below.*

Example 9. *Let $f(x) = -|x|$ and*

$$\eta(x,y) = \begin{cases} x - \frac{1}{2}y, & \text{if } x \geq 0, y \geq 0; \\ x - \frac{1}{2}y, & \text{if } x \leq 0, y \leq 0; \\ \frac{1}{2}y - x, & \text{if } x \leq 0, y \geq 0; \\ \frac{1}{2}y - x, & \text{if } x \geq 0, y \leq 0. \end{cases}$$

Then $f(x)$ is a (B,m)-preinvex function with respect to η on R, where $m = \frac{1}{2}$, $b_1(x,y,\lambda) = 1 - \lambda$ and $b_2(x,y,\lambda) = \lambda$. However, it is obvious that $f(x) = -|x|$ is not a convex function on R.

Definition 10. *Given $S \subseteq R^n \times R$, S is said to be a (B,m)-invex set if there exists $\eta : R^n \times R^n \to R^n$, $b_1 : X \times X \times [0,1] \to R_+$, $b_2 : X \times X \times [0,1] \to R_+$ and some fixed $m \in (0,1]$ such that for any pair of $(x,\alpha), (y,\beta) \in S$*

$$\big(my+\lambda\eta(x,y), mb_1(x,y,\lambda)\beta+b_2(x,y,\lambda)\alpha\big) \in S \tag{5}$$

holds for any $\lambda \in [0,1]$, where $x, y \in R^n$ and $\alpha, \beta \in R$.

3 Properties of (B,m)-preinvex functions

In this section, we drive some properties of (B,m)-preinvex functions and (B,m)-invex sets. In the following, Some basic results are presented without proof.

Theorem 11. *If $f_i : X \to R$ $(i = 1,2,\cdots,n)$ are (B,m)-preinvex functions with respect to the same η, b_1 and b_2 for some fixed $m \in (0,1]$, then the function*

$$f = \sum_{i=1}^{n} a_i f_i, a_i \geq 0, (i = 1,2,\cdots,n)$$

is also a (B,m)-preinvex function with respect to the same η, b_1 and b_2 for fixed $m \in (0,1]$.

Theorem 12. *If $f_i : X \to R$ $(i = 1,2,\cdots,n)$ are (B,m)-preinvex functions with respect to η, b_{1i} and b_{2i} for some fixed $m \in (0,1]$, then the function*

$$f = \max\{f_i, i = 1,2,\cdots,n\}$$

is also a (B,m)-preinvex function with respect to the η, $b_1 = \max\{b_{1i}, i = 1,2,\cdots,n\}$ and $b_2 = \max\{b_{2i}, i = 1,2,\cdots,n\}$ for fixed $m \in (0,1]$.

Theorem 13. *Let $f : X \to R$ be a (B,m)-preinvex function with respect to η, b_1 and b_2 for some fixed $m \in (0,1]$, and let $g : W \to R$ ($W \subseteq R$) be a positively homogenous and nondecreasing function, where $rang(f) \subseteq W$. Then the composite function $g(f)$ is a (B,m)-preinvex function with respect to η, b_1 and b_2 on X for fixed $m \in (0,1]$.*

Proof: Since f is a (B,m)-preinvex function, then for all $x, y \in X$

$$f\big(my+\lambda\eta(x,y)\big) \leq mb_1(x,y,\lambda)f(y)+b_2(x,y,\lambda)f(x)$$

holds for any $\lambda \in [0,1]$. Since g is a positively homogenous and nondecreasing function, then

$$\begin{aligned} g\big(f(my+\lambda\eta(x,y))\big) &\leq g\big(mb_1(x,y,\lambda)f(y) \\ &\quad + b_2(x,y,\lambda)f(x)\big) \\ &= mb_1(x,y,\lambda)g\big(f(y)\big) \\ &\quad + b_2(x,y,\lambda)g\big(f(x)\big), \end{aligned}$$

which follows that $g(f)$ is a (B,m)-preinvex function with respect to η, b_1 and b_2 on X for some fixed $m \in (0,1]$.

Theorem 14. *If $g_i : R^n \to R$ $(i = 1,2,\cdots,n)$ are (B,m)-preinvex functions with respect to η, b_{1i} and b_{2i} for some fixed $m \in (0,1]$, then the set $M = \{x \in R^n : g_i(x) \leq 0, i = 1,2,\cdots,n\}$ is an m-invex set.*

Proof: Since $g_i(x)$, $(i = 1,2,\cdots,n)$ are (B,m)-preinvex functions, then for all $x, y \in R^n$

$$g_i\big(my+\lambda\eta(x,y)\big) \leq mb_{1i}(x,y,\lambda)g_i(y)+b_{2i}(x,y,\lambda)g_i(x),$$

holds for any $\lambda \in [0,1]$. When $x, y \in M$, we know $g_i(x) \leq 0$ and $g_i(y) \leq 0$, from the above inequality, it yields that

$$g_i\big(my+\lambda\eta(x,y)\big) \leq 0, \quad i = 1,2,\cdots,n.$$

That is, $my+\lambda\eta(x,y) \in M$. Hence, M is an m-invex set.

In what following, we give some characterizations of (B,m)-preinvex function $f : X \to R$ in terms of their epigraph $E(f)$, which is given by

$$E(f) = \big\{(x,\alpha) : x \in X, \alpha \in R, f(x) \leq \alpha\big\}. \tag{6}$$

Theorem 15. *A function $f : X \to R$ is a (B,m)-preinvex function with respect to η, b_1 and b_2 for some fixed $m \in (0,1]$, if and only if $E(f)$ is a (B,m)-invex set with respect to η, b_1 and b_2.*

Proof: Suppose that f is a (B,m)-preinvex function. Let (x_1,α_1), $(x_2,\alpha_2) \in E(f)$. Then, $f(x_1) \le \alpha_1$, $f(x_2) \le \alpha_2$. It follows that

$$\begin{aligned} f\big(mx_2+\lambda\eta(x_1,x_2)\big) &\le mb_1(x_1,x_2,\lambda)f(x_2) \\ &\quad + b_2(x_1,x_2,\lambda)f(x_1) \\ &\le mb_1(x_1,x_2,\lambda)\alpha_2 \\ &\quad + b_2(x_1,x_2,\lambda)\alpha_1. \end{aligned}$$

That is $\big(mx_2 + \lambda\eta(x_1,x_2), mb_1(x,y,\lambda)\alpha_2 + b_2(x,y,\lambda)\alpha_1\big) \in E(f)$. Hence, by Definition 6, $E(f)$ is a (B,m)-invex set with respect to η, b_1 and b_2 for fixed $m \in (0,1]$.

Conversely, let's assume that $E(f)$ is a (B,m)-invex set and $x_1,x_2 \in X$, then $\big(x_1,f(x_1)\big),\big(x_2,f(x_2)\big) \in E(f)$. Thus, for $\lambda \in [0,1]$ and fixed $m \in (0,1]$, it yields that $\Big(mx_2 + \lambda\eta(x_1,x_2), mb_1(x_1,x_2,\lambda)f(x_2) + b_2(x_1,x_2,\lambda)f(x_1)\Big) \in E(f)$.

This implies that

$$\begin{aligned} f\big(mx_2+\lambda\eta(x_1,x_2)\big) &\le mb_1(x_1,x_2,\lambda)f(x_2) \\ &\quad + b_2(x_1,x_2,\lambda)f(x_1). \end{aligned}$$

That is, f is a (B,m)-preinvex function with respect to b and the proof of Theorem 15 is completed.

Theorem 16. *If X_i, $i \in I = \{1,2,\cdots,n\}$ is a family of (B,m)-invex sets in $R^n \times R$ with respect to the same η, b_1 and b_2 for some fixed $m \in (0,1]$, then the intersection $\bigcap_{i\in I} X_i$ is a (B,m)-invex set.*

Theorem 17. *If $\{f_i(x)|i \in I\}$ is a family of numerical functions on X, and each f_i is a (B,m)-preinvex function with respect to the same map η, b_1, b_2 for some fixed $m \in (0,1]$, then the numerical function $f = \sup_{i\in I} f_i(x)$ is a (B,m)-preinvex function with respect to η, b_1 and b_2.*

The proofs of Theorem 16 and Theorem 17 are not particularly difficult, so no proofs will be given here.

4 (B,m)-preinvex programming

Before studying the the sufficient conditions of optimality for both unconstrained and inequality constrained programming involving (B,m)-preinvexity, we first present the necessary condition of (B,m)-preinvex functions under the differentiability and some assumptions.

Assumption 1 The function $f(x)$ satisfies that $f\big(my+\lambda\eta(x,y)\big) \ge mf\big(y+\frac{\lambda}{m}\eta(x,y)\big)$ holds for any $x,y \in X$, $\lambda \in (0,1]$ and some fixed $m \in (0,1]$. For fixed $x,y \in X$, when $\lambda \to 0^+$, $f\big(my+\lambda\eta(x,y)\big) \to mf(y)$.

Assumption 2 The limit $\lim_{\lambda\to 0^+} \frac{b_2(x,y,\lambda)}{\lambda}$ exists for fixed $x,y \in X$. Let $\bar{b}(x,y) = \lim_{\lambda\to 0^+} \frac{b_2(x,y,\lambda)}{\lambda}$.

The following theorems are studied under the above assumptions.

Theorem 18. *Let X be a nonempty m-invex set in R^n with respect to η, and let $f : X \to R$ be a differentiable (B,m)-preinvex function on X with respect to η, b_1 and b_2 for fixed $m \in (0,1]$. Then for any $x,y \in X$*

$$\nabla f(y)^T\eta(x,y) \le \bar{b}(x,y)\big(f(x) - f(my)\big). \quad (7)$$

Proof: Since f is a (B,m)-preinvex function on X with respect to η, b_1 and b_2, combining the equality (4), then for all $x,y \in X$, $\lambda \in [0,1]$ and fixed $m \in (0,1]$

$$\begin{aligned} f(my+\lambda\eta(x,y)) &\le mb_1(x,y,\lambda)f(y) \\ &\quad + b_2(x,y,\lambda)f(x) \\ &= m\big(1-b_2(x,y,\lambda)\big)f(y) \\ &\quad + b_2(x,y,\lambda)f(x). \end{aligned} \quad (8)$$

By the differentiability of f and according to Assumption 1, we have that

$$\begin{aligned} f\big(my+\lambda\eta(x,y)\big) &\ge mf\big(y+\frac{\lambda}{m}\eta(x,y)\big) \\ &= mf(y) + \lambda\nabla f(\varepsilon)^T\eta(x,y) \end{aligned} \quad (9)$$

where $\varepsilon = y + \theta\lambda\eta(x,y)$ and $0 < \theta < 1$. Combining the above inequality(8) and the inequality(9), it follows that

$$\lambda\nabla f(\varepsilon)^T\eta(x,y) \le b_2(x,y,\lambda)\big(f(x) - mf(y)\big). \quad (10)$$

From Assumption 1, dividing the inequality (10) by λ and taking $\lambda \to 0^+$, then $\theta \to 0^+$, it is easy verify that

$$\begin{aligned} \nabla f(y)^T\eta(x,y) &\le \lim_{\lambda\to 0^+} \frac{b_2(x,y,\lambda)}{\lambda}\big(f(x) - f(my)\big) \\ &= \bar{b}(x,y)\big(f(x) - f(my)\big). \end{aligned}$$

holds for all $x,y \in X$ and fixed $m \in (0,1]$. The statement in Theorem 18 is completed.

Corollary 19. *Let X be a nonempty m-invex set in R^n with respect to η, and let $f : X \to R$ be a differentiable strictly (B, m)-preinvex function on X with respect to η, b_1 and b_2 for fixed $m \in (0, 1]$. Then for any $x, y \in X$*

$$\nabla f(y)^T \eta(x, y) < \bar{b}(x, y)\big(f(x) - f(my)\big). \quad (11)$$

Here, the proof of Corollary 19 will be omitted.

By using the associated results above, we consider the nonlinear unconstraint problem (P).

$$(P) : \min\{f(x), x \in X\}, \quad (12)$$

where X is a nonempty m-invex set in R^n with respect to η and $f(x)$ is differentiable (B, m)-preinvex function on X with respect to η, b_1 and b_2 for fixed $m \in (0, 1]$.

Theorem 20. *Let X be a nonempty m-invex set in R^n with respect to η and $f(x)$ be a differentiable (B, m)-preinvex function on X with respect to η, b_1 and b_2 for some fixed $m \in (0, 1]$. If $\bar{x} \in X$ and the inequality*

$$\nabla f(\bar{x})^T \eta(x, \bar{x}) \geq 0 \quad (13)$$

holds for each $x \in X$ and any $\lambda \in [0, 1]$, then $m\bar{x}$ is the optimal solution to the optimal problem (P) with respect to f on X.

Proof: Since f is a differentiable (B, m)-preinvex function on X, according to Theorem 18, it yields that

$$\nabla f(y)^T \eta(x, \bar{x}) \leq \bar{b}(x, \bar{x})(f(x) - f(m\bar{x})). \quad (14)$$

According to the equality (4), we have $b_1, b_2 \geq 0$ and $\bar{b}(x, \bar{x}) \geq 0$. So, when the inequality

$$\nabla f(\bar{x})^T \eta(x, \bar{x}) \geq 0$$

holds for each $x \in X$ and any $\lambda \in [0, 1]$, it follows that $f(x) - f(m\bar{x}) \geq 0$ for every $x \in X$. Therefore, $m\bar{x} \in X$ is the optimal solution. This completes the proof.

Example 21. *From Example 9, we know $f(x) = -|x|$ is a (B, m)-preinvex function on R with respect to η, $b_1 = 1 - \lambda$ and $b_2 = \lambda$ for $m = \frac{1}{2}$. So we consider the following unconstraint problem:*

$$\min\{f(x) = -|x|, x \in R\}.$$

Combining Assumption 1 and Assumption 2,

$$\begin{aligned} f\big(my + \lambda\eta(x, y)\big) &= -|my + \lambda\eta(x, y)| \\ &= -m|y + \frac{\lambda}{m}\eta(x, y)| \\ &= mf(y + \frac{\lambda}{m}\eta(x, y)); \end{aligned}$$

$$\lim_{\lambda \to 0_+} \frac{b_2(x, y, \lambda)}{\lambda} = \lim_{\lambda \to 0_+} \frac{\lambda}{\lambda} = 1.$$

Hence, function $f(x)$ satisfies Assumption 1 and Assumption 2. According to Theorem 20,

$$\nabla f(\bar{x})^T \eta(x, \bar{x}) = \begin{cases} -(x - \frac{1}{2}\bar{x}), & \text{if } x \geq 0, \bar{x} \geq 0; \\ x - \frac{1}{2}\bar{x}, & \text{if } x \leq 0, \bar{x} \leq 0; \\ \frac{1}{2}\bar{x} - x, & \text{if } x \leq 0, \bar{x} \geq 0; \\ -(\frac{1}{2}\bar{x} - x), & \text{if } x \geq 0, \bar{x} \leq 0. \end{cases} \quad (15)$$

If $\nabla f(\bar{x})^T \eta(x, \bar{x}) \geq 0$ holds for each $x \in R$ and any $\lambda \in [0, 1]$, then $\bar{x} = \infty$. what is more, when $x \to \infty$, $f(x) \to -\infty$. In order to facilitate observation, now we consider the unconstraint problem on $X = [0, 2]$ and give the optimal solution of $f(x)$ on $[-2, 2]$.

On the basis of formula (15), it is easy to show that, when $m\bar{x} = 2$ and $m\bar{x} = -2$, $f(x)$ has the optimal solution.

Corollary 22. *Let X be a nonempty m-invex set in R^n with respect to η and $f(x)$ be a differentiable strictly (B, m)-preinvex function on X with respect to η, b_1 and b_2 for fixed $m \in (0, 1]$. If $\bar{x} \in X$ and the inequality*

$$\nabla f(\bar{x})^T \eta(x, \bar{x}) \geq 0, \quad (16)$$

holds for each $x \in X$, any $\lambda \in [0, 1]$, then $m\bar{x}$ is the unique optimal solution to the optimal problem (P).

Proof: Since $f(x)$ is a strictly (B, m)-preinvex function, from Corollary 1, we have that

$$\nabla f(\bar{x})^T \eta(x, \bar{x}) < \bar{b}(x, \bar{x})\big(f(x) - f(m\bar{x})\big)$$

So, when the inequality

$$\nabla f(\bar{x})^T \eta(x, \bar{x}) \geq 0$$

holds for each $x \in X$ and any $\lambda \in [0, 1]$, it follows that $f(x) - f(m\bar{x}) > 0$ for every $x \in X$. Therefore, $m\bar{x} \in X$ is the unique optimal solution, which ends the proof.

In the following, we apply the associated results to the nonlinear programming with inequality constraints as follows:

$$(P_g) : \min\{f(x) : g_i(x) \leq 0, x \in X, i \in I\}, \quad (17)$$

where X is a nonempty m-invex set in R^n with respect to η and $f(x), g_i(x)$ $(i \in I)$ are differentiable (B, m)-preinvex functions on X with respect to η, b_1 and b_2 and η, b_{1i}, $b_{2i}(i \in I)$ for fixed $m \in (0, 1]$, respectively. Denote the feasible set of (P_g) by $M = \{x \in X : g_i(x) \leq 0, i \in I\}$.

Theorem 23. *(Karush-Kuhn-Tucker Sufficient Conditions) Let X be a nonempty m-invex set and let $f(x)$, $g_i(x)$ $(i \in I)$ be differentiable (B, m)-preinvex functions on X. Assume that $x^* \in M$ is a KKT point of (P_g), i.e., there exists multipliers $u_i \geq 0$ $(i \in I)$ such that*

$$\nabla f(x^*) + \sum_{i \in I} u_i \nabla g_i(x^*) = 0, u_i g_i(x^*) = 0. \quad (18)$$

Then mx^* is an optimal solution of the problem (P_g).

Proof: For any $x \in M$, we have that

$$g_i(x) \leq 0 = g_i(x^*), i \in I(x^*) = \{i \in I : g_i(x^*) = 0\}.$$

Therefore, according to Theorem 18 and by the (B, m)-preinvexity of g_i $(i \in I(x^*))$, it is easy to show that

$$\nabla g_i(x^*)^T \eta(x, x^*) \leq \bar{b}(x, x^*)(g_i(x) - m g_i(x^*)) \leq 0. \quad (19)$$

On the basis of the equality (18), it follows that

$$\begin{aligned} \nabla f(x^*)^T \eta(x, x^*) &= -\sum_{i \in I} u_i \nabla g_i(x^*)^T \eta(x, x^*) \\ &= -\sum_{i \in I(x^*)} u_i \nabla g_i(x^*)^T \eta(x, x^*). \end{aligned}$$

On account of $u_i \geq 0$ $(i \in I)$ and combining the inequality (19), it yields that

$$\nabla f(x^*)^T \eta(x, x^*) \geq 0.$$

According to Theorem 20, we have that $f(x) - f(mx^*) \geq 0$ for every $x \in M$. Therefore, mx^* is an optimal solution of the problem (P_g) which ends the proof.

5 (B,m)-preinvex multi-objective programming

The unconstrained multi-objective optimization problem with (B, m)-preinvexity can be represented as follows:

$$\begin{aligned} (MP): \ & \min f(x) = (f_1(x), f_2(x), \cdots, f_k(x)) \\ & s.t.\ x \in X \subseteq R^n, \end{aligned} \quad (20)$$

where X is a nonempty m-invex set in R^n with respect to η, $f_j(x)$ $(j \in K = \{1, 2, \cdots, k\})$ are differentiable (B, m)-preinvex functions on X with respect to the same η and b_{1j}, b_{2j} $(j \in K)$ for fixed $m \in (0, 1]$, respectively.

As we know, in the multi-objective programming there does not necessarily exist a point which maybe optimal for all objectives. In the following, we give the concepts of efficient solution and weakly efficient solution of problem (MP) as follows:

Definition 24. *A feasible point x^* of problem (MP), is said to be an efficient solution if and only if there does not exist another $x \in X$ such that $f_j(x) \leq f_j(x^*)$ for every $j \in K$ with strict inequality holding for at least one j.*

Definition 25. *A feasible point x^* of problem (MP), is said to be a weakly efficient solution if and only if there does not exist another $x \in X$ such that $f_j(x) < f_j(x^*)$ for every $j \in K$.*

Remark 26. *It is easy to verify that every efficient point is a weakly efficient solution.*

Osuna-Gomez $et\ al.$(1999)[5] provided a concept analogous to the stationary point or vector critical point (VCP) for the scalar function. Now we get a similar relationship between VCP and weakly efficient solution for problem (MP) with (B, m)-preinvexity.

Theorem 27. *Let x^* be a vector critical point (VCP) to problem (MP) and $f(x)$ be (B, m)-preinvex at x^* with respect to η, b_1 and b_2, then mx^* is a weakly efficient solution for (MP).*

Proof: If x^* is a vector critical point, then there exists $\bar{\lambda} \geq 0$ such that $\bar{\lambda}^T \nabla f(x^*) = 0$. By Gordan′s theorem, the system

$$\nabla f(x^*)^T u < 0 \quad (21)$$

does not have a solution at $u \in R^n$. According to the (B, m)-preinvexity of $f(x)$ at x^* and Theorem 18, it follows that

$$\nabla f(x^*)^T \eta(x, x^*) \leq \bar{b}(x, x^*)(f(x) - f(mx^*)).$$

Suppose that there exists an $x \in X$ such that $f(x) - f(mx^*) < 0$, we can get that $\nabla f(x^*)^T \eta(x, x^*) < 0$, which inconsistent with previous conclusion which is that the system (21) does not have a solution. Then, there will not exist any $x \in X$ such that $f(x) - f(mx^*) < 0$. Therefore mx^* is a weakly efficient solution for (MP).

In what following, the constrained multi-objective optimization problem (CMP) is given as follows:

$$\begin{aligned} (CMP): \ & \min f(x) = (f_1(x), f_2(x), \cdots, f_k(x)) \\ & s.t.\ \ g_i(x) \leq 0, i \in I = \{1, 2, \cdots, n\} \\ & \quad\ \ x \in X \subseteq R^n, \end{aligned}$$

(22)

where X is a nonempty m-invex set with respect to η, $f_j(x)$ $(j \in K)$ are differentiable (B, m)-preinvex functions on X with respect to the same η, b_1 and b_2, $g_i(x)$ $(i \in I)$ are differentiable (B, m)-preinvex functions on X with respect to η and b_{1i}, b_{2i} $(i \in I)$ for fixed $m \in (0, 1]$, respectively.

Theorem 28. *Let X be a nonempty m-invex set with respect to η, and let $f_j(x)$ $(j \in K)$ and $g_i(x)$ $(i \in I)$ be differentiable (B, m)-preinvex functions on X. Suppose that there exists a feasible $x^* \in M$ of (CMP) and multipliers $\bar{\lambda}_j > 0$ $(j \in K)$, $u_i \geq 0$ $(i \in I)$ such that*

$$\sum_{j\in K} \bar{\lambda}_j \nabla f_j(x^*) + \sum_{i\in I} u_i \nabla g_i(x^*) = 0, u_i g_i(x^*) = 0. \tag{23}$$

Then mx^* is a properly efficient solution of the problem (CMP).

Proof: For any $x \in M$, we have that

$$g_i(x) \leq 0 = g_i(x^*), i \in I(x^*) = \{i \in I : g_i(x^*) = 0\}.$$

Through the equality (23), we have that $u_i = 0$ for $i \notin I(x^*)$. Then, it follows that

$$\sum_{i\in I} u_i \nabla g_i(x^*) = \sum_{i\in I(x^*)} u_i \nabla g_i(x^*). \tag{24}$$

According to Theorem 18 and by the (B, m)-preinvexity of g_i $(i \in I(x^*))$, it is easy to show that

$$\nabla g_i(x^*)^T \eta(x, x^*) \leq \bar{b}_i(x, x^*)(g_i(x) - m g_i(x^*)) \leq 0. \tag{25}$$

On account of $\bar{\lambda}_j > 0$ $(j \in K)$, $u_i \geq 0$ $(i \in I)$ and combining the above equality (24), inequality (25) and the (B, m)-preinvexity of $f_j(x)$ $(j \in K)$, it yields that

$$\begin{aligned}\sum_{j\in K} \bar{\lambda}_j \bar{b}(x, x^*)(f_j(x) - f_j(mx^*)) &\geq \sum_{j\in K} \bar{\lambda}_j \nabla f_j(x^*)^T \eta(x, x^*) \\ &= -\sum_{i\in I(x^*)} u_i \nabla g_i(x^*)^T \eta(x, x^*) \\ &\geq -\sum_{i\in I(x^*)} u_i \bar{b}_i(x, x^*)(g_i(x) - m g_i(x^*)) \\ &\geq 0.\end{aligned}$$

Combining the fact $\bar{b}(x, x^*) \geq 0$, that is

$$\sum_{j\in K} \bar{\lambda}_j f_j(x) - \sum_{j\in K} \bar{\lambda}_j f_j(mx^*) \geq 0.$$

holds for all $x \in M$. It follows that mx^* minimizes $\sum_{j\in K} \bar{\lambda}_j f_j(x)$ subject to $g_i(x) \leq 0,\ i \in I$. Therefore, from Theorem 1 of [4], mx^* is a proper efficient solution of the problem (CMP) which ends the proof.

6 Conclusion

In this paper, we have introduced a new class of generalized convex functions which referred to as (B, m)-preinvex functions by combining m-convex functions and B-preinvex functions. Moreover, we have studied the basic properties of (B, m)-preinvex functions. In addition, the optimality conditions of single objective programming and multi-objective programming involving (B, m)-preinvexity have been presented. Emphasis here is that this work is the promotion of m-convex functions, B-preinvex functions and B-preinvex programming.

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