Successive iteration and positive solutions for nonlocal higher-order fractional differential equations with p-Laplacian

QIUYAN ZHONG
Jining Medical College
Department of Information Engineering
Hehua Road No. 16, 272067, Jining
CHINA
zhqy197308@163.com

XINGQIU ZHANG
Jining Medical College
Department of Information Engineering
Hehua Road No. 16, 272067, Jining
CHINA
zhxq197508@163.com

Abstract: In this paper, we investigate the existence of positive solutions for higher-order fractional differential equations with p-Laplacian operator and nonlocal boundary conditions. By means of the properties of the corresponding Green function together with monotone iterative technique, we obtain not only the existence of positive solutions for the problems, but also establish iterative schemes for approximating the solutions. The nonlinearity permits singularities at $t = 0$ and/or $t = 1$.

Key–Words: Fractional differential equations; Integral boundary value problem; p-Laplacian; Positive solution; Successive iteration

1 Introduction

The purpose of this paper is to consider the existence of positive solutions for the following singular fractional differential equations involving p-Laplacian operator (PFDE, for short) and integral conditions

$$\begin{cases} D^{\beta}_{0+}(\varphi_p(D^{\alpha}_{0+}u(t))) + a(t)f(t, u(t)) = 0, t \in J, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D^{\alpha}_{0+}u(0) = 0,\ u^{(i)}(1) = \lambda \displaystyle\int_0^{\eta} h(t)u(t)\mathrm{d}t, \end{cases} \tag{1}$$

where $D^{\alpha}_{0+}, D^{\beta}_{0+}$ are standard Riemann-Liouville derivative, $\varphi_p(s) = |s|^{p-2}s, p > 1$, $a \in C((0, 1), R^+)$, $J = (0, 1)$, $f \in C((0, 1) \times R^+, R^+)$, $f(t, u)$ may be singular at $t = 0$ and/or $t = 1, R^+ = [0, +\infty), h \in L^1[0, 1]$ is nonnegative and $h(t)$ may be singular at $t = 0$ and $t = 1$, $i \in [1, n - 2]$ is a fixed integer, $n - 1 < \alpha \le n, n \ge 3, 0 < \beta \le 1$ $0 < \eta \le 1, 0 \le \lambda \int_0^{\eta} h(t)t^{\alpha-1}\mathrm{d}t < \Delta$, here $\Delta = (\alpha - 1)(\alpha - 2)\cdots(\alpha - i)$.

Due to both by the intensive development of the theory of fractional calculus itself and by the wide applications such as in control, porous media, aerodynamics, electrodynamics of complex medium, polymer rheology, electromagnetic, and so on, fractional differential equations have attracted more and more researchers's much attention in recent years. We refer the readers to [1-3] for an extensive collection of such results.

In [4-12], by means of the fixed point index theory, fixed point theory together with the relevant results on u_0 bounded operator and lattice structure, the authors investigated the existence and multiplicity of positive and nontrivial solutions for fractional differential equation

$$D^{\alpha}_{0+}u(t) + a(t)f(t, u(t)) = 0,\ \ 0 < t < 1 \tag{A}$$

subject to different boundary conditions. A natural question is "How can we find the solutions when they are known to exist?". There are few results on the computation of positive solutions for fractional differential equations at present, see [8, 9, 12-15]. In [16], using the fixed point index theorem in cones, under some weak conditions concerning the first eigenvalue corresponding to the relevant linear operator, the author obtained the existence and multiplicity of positive solutions for some singular higher-order fractional differential equations. Motivated by above papers, the aim of this paper is to give the iterative procedure of positive solutions for BVP (1).

This paper has the following three new features. First, compared with [4-11], the boundary conditions contain the i-th order of the unknown function and a parameter λ. Particularly, a Lebesgue integrable function h is involved in the boundary condition. Second, compared with [16], p-Laplacian operator is involved in differential operator. This means that the problems discussed in this paper have more general form. Finally, we obtain not only the existence of positive solutions for the problems, but also establish iterative schemes for approximating the solutions. The iterative sequences begin with simple functions which is convenient in applications. Obviously, it is possible to replace the Riemann integrals in the boundary conditions by Riemann-Stieltjes integrals with minor modifications.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and lemmas. The main result is formulated in section 3 and an example is given in section 4 to illustrate how to use the main result.

2 Preliminaries and several lemmas

Let $E = C[0,1]$, $\|u\| = \max_{0\le t\le 1}|u(t)|$, then $(E, \|\cdot\|)$ is a Banach space. For the reader's convenience, we present some necessary definitions and lemmas from fractional calculus theory which can be found in the recent literature, see [1-3].

Definition 1 *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by*

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\mathrm{d}s$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2 *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,\infty) \to R$ is given by*

$$D_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)^n\int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Now, we consider the following fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t) + y(t) = 0,\ 0 < t < 1,\\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\\ u^{(i)}(1) = \lambda\displaystyle\int_0^{\eta} h(t)u(t)\mathrm{d}t. \end{cases} \tag{2}$$

Lemma 3 *[16] Assume that $\lambda\int_0^{\eta}h(t)t^{\alpha-1}\mathrm{d}t \neq \Delta$. Then for any $y \in L^1[0,1]$, the unique solution of the boundary value problems (2) can be expressed in the form*

$$u(t) = \int_0^1 G(t,s)y(s)\mathrm{d}s,\ \ t\in[0,1],$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{3}$$

$$G_1(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1-i} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, \\ \qquad\qquad\qquad 0\le s\le t\le 1,\\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1-i}}{\Gamma(\alpha)},\ \ 0\le t\le s\le 1, \end{cases}$$

$$G_2(t,s) = \frac{\lambda t^{\alpha-1}}{\Delta - \lambda\int_0^{\eta}h(t)t^{\alpha-1}\mathrm{d}t}\int_0^{\eta}h(t)G_1(t,s)\mathrm{d}t.$$

Here, $G(t,s)$ is called the Green function of BVP (2). *Obviously, $G(t,s)$ is continuous on* $[0,1]\times[0,1]$.

Lemma 4 *[16] If $0 \le \lambda\int_0^{\eta}h(t)t^{\alpha-1}\mathrm{d}t < \Delta$, then the function $G(t,s)$ defined by (3) satisfies*
$(a1)$ $G(t,s) \ge m_1 t^{\alpha-1}s(1-s)^{\alpha-1-i}, \forall\, t,s\in[0,1]$;
$(a2)$ $G(t,s) \le M_1 t^{\alpha-2}s(1-s)^{\alpha-1-i}, \forall\, t,s\in[0,1]$;
$(a3)$ $G(t,s) \le M_1 t^{\alpha-1}(1-s)^{\alpha-1-i}, \forall\, t,s\in[0,1]$;
$(a4)$ $G(t,s) > 0, \forall\, t,s\in(0,1)$;
where $m_1 = \frac{1}{\Gamma(\alpha)}\Big(1 + \frac{\lambda}{\Delta-\lambda\int_0^{\eta}h(t)t^{\alpha-1}\mathrm{d}t}\int_0^{\eta}h(t)t^{\alpha-1}\mathrm{d}t\Big)$, $M_1 = \frac{n}{\Gamma(\alpha)}\Big(1 + \frac{\lambda}{\Delta-\lambda\int_0^{\eta}h(t)t^{\alpha-1}\mathrm{d}t}\int_0^{\eta}h(t)t^{\alpha-2}\mathrm{d}t\Big)$.

Let $q > 1$ satisfy $\frac{1}{p}+\frac{1}{q} = 1$. Then, $\varphi_p^{-1}(s) = \varphi_q(s)$. To study the PFDE (1), we first consider the associated linear PFDE

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) + y(t) = 0,\ \ 0<t<1,\\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\\ D_{0+}^{\alpha}u(0) = 0,\ \ u^{(i)}(1) = \lambda\displaystyle\int_0^{\eta}h(t)u(t)\mathrm{d}t, \end{cases} \tag{4}$$

for $y\in L^1[0,1]$ and $h\ge 0$.

Lemma 5 *The unique solution for the associated linear PFDE (4) can be written by*

$$\begin{aligned} u(t) \;=\; & \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s) \\ & \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\Big)\mathrm{d}s. \end{aligned} \tag{5}$$

Proof. Let $w = D_{0+}^{\alpha}u$, $v = \varphi_p(w)$. Then, the initial value problem

$$\begin{cases} D_{0+}^{\beta}v(t) + y(t) = 0,\ t\in(0,1),\\ v(0) = 0 \end{cases} \tag{6}$$

has the solution $v(t) = c_1 t^{\beta-1} - I^{\beta}y(t),\ t\in[0,1]$. Noticing that $v(0) = 0, 0 < \beta \le 1$, we have that $c_1 = 0$. As a consequence,

$$v(t) = -I^{\beta}y(t),\ t\in[0,1]. \tag{7}$$

Considering that $D_{0+}^{\alpha}u = w, w = \varphi_p^{-1}(v)$, we have from (7) that

$$\begin{cases} D_{0+}^{\alpha}u(t) = \varphi_p^{-1}(-I^{\beta}(y(t))), \ 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \lambda \int_0^{\eta} h(t)u(t)\mathrm{d}t. \end{cases} \tag{8}$$

By Lemma 3, the solution of (8) can be expressed by (5).

We list below assumptions used in this paper for convenience.

(H_1) $h \in L^1[0,1]$ is nonnegative;

(H_2) $f(t,0) \not\equiv 0$ on $[0,1]$, $f : (0,1) \times R^+ \to R^+$ is continuous and nondecreasing on x, and there exists constant $r > 0$ such that, for any $t \in (0,1), u \in R^+$,

$$f(t, cu) \geq c^r f(t,u), \ \forall\, 0 < c \leq 1; \tag{9}$$

(H_3) $a : (0,1) \to R^+$ is continuous, $a(t) \not\equiv 0$ with $0 < \int_0^1 (1-s)^{\alpha-1-i}\varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,1)\mathrm{d}\tau\Big)a(s)\mathrm{d}s < +\infty$.

Remark 6 *If* $c \geq 1$*, then it is not difficulty to see that (9) is equivalent to*

$$f(t, cu) \leq c^r f(t,u), \ \forall\, c \geq 1. \tag{10}$$

Define a subset P in E as follows

$$P = \{u \in C([0,1], R^+) : \text{there exist two positive numbers } l_u < 1 < L_u \text{ such that } l_u t^{\alpha-1} \leq u(t) \leq L_u t^{\alpha-1}, \ t \in [0,1]\}. \tag{11}$$

Clearly, P is nonempty since $t^{\alpha-1} \in P$.

Now define an operator A as follows:

$$(Au)(t) = \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau))\mathrm{d}\tau\Big)\mathrm{d}s, \quad t \in [0,1]. \tag{12}$$

Lemma 7 *Suppose that* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold. Then* $A : P \to P$ *is completely continuous and nondecreasing.*

Proof. For any $u \in P$, there exist two positive numbers $0 < l_u < 1 < L_u$ such that

$$l_u t^{\alpha-1} \leq u(t) \leq L_u t^{\alpha-1}, \ t \in [0,1]. \tag{13}$$

Thus, we have from Lemma 4, (9), (10), $(\mathrm{H}_2), (\mathrm{H}_3)$ that

$$\begin{aligned} &(Au)(t) \\ &= \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} \int_0^1 G(t,s) \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau))\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leq \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} M_1 t^{\alpha-1} \int_0^1 (1-s)^{\alpha-1-i} \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau, L_u\tau^{\alpha-1})\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leq \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} M_1 t^{\alpha-1} \int_0^1 (1-s)^{\alpha-1-i} \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau, L_u)\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leq \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} M_1 t^{\alpha-1} \int_0^1 (1-s)^{\alpha-1-i} \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau, 1)L_u^r\mathrm{d}\tau\Big)\mathrm{d}s \\ &= \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} M_1 L_u^{r(q-1)} \int_0^1 (1-s)^{\alpha-1-i} \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau, 1)\mathrm{d}\tau\Big)\mathrm{d}s \cdot t^{\alpha-1} \\ &< +\infty, \end{aligned} \tag{14}$$

$$\begin{aligned} &(Au)(t) \\ &= \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} \int_0^1 G(t,s) \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau))\mathrm{d}\tau\Big)\mathrm{d}s \\ &\geq \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} m_1 t^{\alpha-1} \int_0^1 s(1-s)^{\alpha-1-i} \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau, l_u\tau^{\alpha-1})\mathrm{d}\tau\Big)\mathrm{d}s \\ &\geq \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} m_1 t^{\alpha-1} \int_0^1 s(1-s)^{\alpha-1-i} \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} l_u^r \tau^{r(\alpha-1)} f(\tau, 1)\mathrm{d}\tau\Big)\mathrm{d}s \\ &= \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} m_1 l_u^{r(q-1)} \int_0^1 s(1-s)^{\alpha-1-i} \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} \tau^{r(\alpha-1)} f(\tau, 1)\mathrm{d}\tau\Big)\mathrm{d}s \cdot t^{\alpha-1} \end{aligned} \tag{15}$$

which means that A is well defined, uniformly bounded and $A(P) \subset P$. By standard argument, according to the Lebesgue dominated convergence theorem and the Arzela-Ascoli theorem, it is not difficult to see that A is completely continuous. Noticing the monotonicity of f on x, we know that A is nondecreasing.

3 Main result

For notational convenience, we denote

$$\Lambda = \Big(\frac{1}{\beta\Gamma(\beta)}\Big)^{q-1} M_1 \int_0^1 (1-s)^{\alpha-1-i} s^{\beta(q-1)}\mathrm{d}s. \tag{16}$$

Theorem 8 *Suppose that conditions* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *hold. In addition, if there exists* $a>0$*, such that*

$$f(t,u)\le\varphi_p\Big(\frac{a}{\Lambda}\Big),\ (t,u)\in[0,1]\times[0,a]. \tag{17}$$

Then the BVP (1) has two positive solutions u_* *and* u^**; and there exist two positive numbers* $l_i<L_i$ $(i=1,2)$ *such that*

$$\begin{aligned}&l_1t^{\alpha-1}\le u_*(t)\le L_1t^{\alpha-1},\\ &l_2t^{\alpha-1}\le u^*(t)\le L_2t^{\alpha-1},\ t\in[0,1].\end{aligned} \tag{18}$$

Moreover, for initial values $u_0=0, v_0=at^{\alpha-1}$*, monotone sequences* $\{u_n\}_{n=1}^{\infty}$ *and* $\{v_n\}_{n=1}^{\infty}$ *satisfy* $\lim_{n\to\infty}u_n=u_*, \lim_{n\to\infty}v_n=u^*$*, where* $u_{n+1}=Au_n, v_{n+1}=Av_n, n=0,1,2,\cdots$.

Proof. Let $P_a=\{u\in P:\|u\|\le a\}$. For $u\in P_a$, we have $0\le u(s)\le\|u\|\le a, s\in[0,1]$. Thus, for $t\in[0,1]$, it follows from Lemma 4, (16) and (17) that

$$\begin{aligned}&(Au)(t)\\ &=\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1G(t,s)\\ &\quad\varphi_p^{-1}\Big(\int_0^s(s-\tau)^{\beta-1}f(\tau,u(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\ &\le\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}M_1t^{\alpha-1}\int_0^1(1-s)^{\alpha-1-i}\\ &\quad\varphi_p^{-1}\Big(\int_0^s(s-\tau)^{\beta-1}\mathrm{d}\tau\cdot\varphi_p\Big(\frac{a}{\Lambda}\Big)\Big)\mathrm{d}s\\ &=\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}M_1\int_0^1(1-s)^{\alpha-1-i}\\ &\quad\cdot\varphi_p^{-1}\Big(\frac{1}{\beta}s^{\beta}\Big)\mathrm{d}s\cdot\frac{a}{\Lambda}\\ &=\Big(\frac{1}{\beta\Gamma(\beta)}\Big)^{q-1}M_1\int_0^1(1-s)^{\alpha-1-i}\\ &\quad s^{\beta(q-1)}\mathrm{d}s\cdot\frac{a}{\Lambda}\\ &=a,\end{aligned} \tag{19}$$

which implies that $\|Au\|\le a$, i.e., $A(P_a)\subset P_a$.

Let $u_0(t)=0, t\in[0,1]$. Similar to (19), by (17) we get that $u_1=Au_0\in P_a$. Denote

$$u_{n+1}=Au_n=A^{n+1}u_0, n=1,2,\cdots.$$

It follows from $A(P_a)\subset P_a$ that $u_n\in P_a$ $(n=1,2,\cdots)$. By Lemma 7, we know that $\{u_n\}$ is a sequentially compact set.

Since $u_1=Au_0=A0\in P_a$, we have

$$u_1(t)=(Au_0)(t)=(A0)(t)\ge0=u_0(t),\ t\in[0,1].$$

By induction, we get

$$u_{n+1}(t)\ge u_n(t),\ n=1,2,\cdots.$$

Consequently, there exists $u_*\in P_a$ such that $u_n\to u_*$. Let $n\to\infty$, by the continuity of A and $u_{n+1}=Au_n$, we know that $Au_*=u_*$. This is to say that $u_*\ge0$ is a fixed point of A, i.e., u_* is a nonnegative solution for BVP (1). By (H_2), it is not difficult to see that 0 is not the solution for BVP (1). Thus, u_* is a positive solution for BVP (1).

Let $v_0(t)=at^{\alpha-1},\ t\in[0,1]$, then $v_0\in P_a$. It follows from $A(P_a)\subset P_a$ that $v_1\in P_a$. Denote

$$v_{n+1}=Av_n=A^{n+1}v_0, n=1,2,\cdots.$$

Similarly, we get that

$$v_n\in P_a,\ n=0,1,2,\cdots.$$

On the other hand, we have

$$\begin{aligned}v_1(t)&=(Av_0)(t)=\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1G(t,s)\\ &\quad\varphi_p^{-1}\Big(\int_0^s(s-\tau)^{\beta-1}f(\tau,v_0(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\ &\le\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}M_1t^{\alpha-1}\int_0^1(1-s)^{\alpha-1-i}\\ &\quad\varphi_p^{-1}\Big(\int_0^s(s-\tau)^{\beta-1}\mathrm{d}\tau\cdot\varphi_p\Big(\frac{a}{\Lambda}\Big)\Big)\mathrm{d}s\\ &\le\Big(\frac{1}{\beta\Gamma(\beta)}\Big)^{q-1}M_1t^{\alpha-1}\int_0^1(1-s)^{\alpha-1-i}\\ &\quad\cdot s^{\beta(q-1)}\mathrm{d}s\cdot\frac{a}{\Lambda}\\ &=at^{\alpha-1}=v_0(t).\end{aligned}$$

By Lemma 7, we know that $v_2=Av_1\le Av_0=v_1$. By induction, we get that

$$v_{n+1}\le v_n,\ n=0,1,2,\cdots.$$

Consequently, there exists $u^*\in P_a$ such that $v_n\to u^*$. Let $n\to\infty$, by the continuity of A and $v_{n+1}=Av_n$, we know that $Au^*=u^*$. This is to say that $u^*\ge0$ is a fixed point of A, i.e., u^* is a nonnegative solution for BVP (1). Moreover, considering that the zero function is not a solution of the problem, we have that u^* is a positive solution for BVP (1). Since $u_*,u^*\in P_a$, there exist positive numbers $l_i<L_i$ $(i=1,2)$ such that (18) holds.

Remark 9 *The iterative sequences in Theorem 8 begin with simple functions which is significant for computational purpose.*

4 An example

Consider the following singular fractional differential equations

$$\begin{cases} D_{0+}^{\frac{1}{4}}(\varphi_3(D_{0+}^{\frac{5}{2}}u(t))) + \frac{1}{\sqrt{1-t}}u^{\frac{3}{2}} = 0, 0 < t < 1, \\ u(0) = u'(0) = 0, \ \ D_{0+}^{\frac{5}{2}}u(0) = 0, \\ u'(1) = \dfrac{2}{3}\displaystyle\int_0^{\frac{3}{4}} \frac{1}{\sqrt{t}}u(t)\mathrm{d}t. \end{cases} \tag{20}$$

Let $\alpha = \frac{5}{2}, \beta = \frac{1}{4}, p = 3, q = \frac{3}{2}, \lambda = \frac{2}{3}, \eta = \frac{3}{4}, i = 1, n = 3, a(t) = \frac{1}{\sqrt{1-t}}, h(t) = \frac{1}{\sqrt{t}}, f(t,u) = u^{\frac{3}{2}}$, then $\Delta = \frac{3}{2}, 0 < \lambda \int_0^\eta h(t)t^{\alpha-1}\mathrm{d}t = \frac{3}{16} < \Delta$. Clearly, $(\mathrm{H}_1), (\mathrm{H}_2)$ hold for $h(t) = \frac{1}{\sqrt{t}}, r = \frac{3}{2}$. On the other hand, we have

$$\begin{aligned} 0 \ &< \int_0^1 (1-s)^{\alpha-1-i}\varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} \\ &\quad f(\tau,1)\mathrm{d}\tau\Big)a(s)\mathrm{d}s \\ &\leq \int_0^1 \varphi_p^{-1}\Big(\frac{1}{\beta}s^\beta\Big)\mathrm{d}s < \varphi_p^{-1}\Big(\frac{1}{\beta}\Big) < +\infty \end{aligned}$$

which implies that (H_3) also holds. By simple computation, we get

$$\begin{aligned} M_1 &= \frac{n}{\Gamma(\alpha)}\Big(1 + \frac{\lambda}{\Delta - \lambda\int_0^\eta h(t)t^{\alpha-1}\mathrm{d}t}\int_0^\eta h(t)t^{\alpha-2}\mathrm{d}t\Big) \\ &\approx 3.1166, \end{aligned}$$

$$\Lambda = \Big(\frac{4}{\Gamma(\frac{1}{4})}\Big)^{\frac{1}{2}} \times 3.1166 \times \Big(\int_0^1 (1-s)^{\frac{1}{2}}s^{\frac{1}{8}}\mathrm{d}s\Big) \approx 1.8752.$$

Take $a = 10^3$, then $f(t,u) = u^{\frac{3}{2}} \leq (10^3)^{\frac{3}{2}} = 3.1623 \times 10^4 < 2.8438 \times 10^5 = \varphi_3\Big(\frac{10^3}{1.8752}\Big)$. Thus, (17) holds. It follows from Theorem 8 that BVP (20) has the positive minimal and maximal solutions u_* and u^*; and there exist some positive numbers $l_i < L_i$ $(i = 1, 2)$ such that (18) holds.

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