General Flows and Their Adaptive Decompositions

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Abstract: Adaptive algorithms of spline-wavelet decomposition in a linear space over metrized field are proposed. The algorithms provide a priori given estimate of the deviation of the main flow from the initial one. Comparative estimates of data of the main flow under different characteristics of the irregularity of the initial flow are done. The limiting characteristics of data, when the initial flow is generated by abstract differentiable functions, are discussed.

Key–Words: signal processing, main flows, adaptive spline-wavelets, general flows

1 Introduction

Many studies have been devoted to the investigation of numerical flows (signals). There is the theory of filtration, the theory of classical wavelets, the theory of spline-wavelets (see, for example, monographs [1] – [3] and bibliography there). There exist many implementations of wavelets in different practical investigations (for instance, see [4] – [6]).

For classical wavelet decomposition (see [2] - [18]) the translation invariance of the spaces, the multiple-scale analysis, and Fourier transformer are required; that creates great difficulties for the construction of adaptive algorithms for processing numerical flows. Adaptive spline-wavelet expansions use approximate relations for constructing nested spline spaces on non-uniform grids (see [19] – [21]).

In papers [19] - [20], algorithms of adaptive spline-wavelet decomposition for numerical flows are proposed. The construction of spline-wavelet decompositions of flow of a more general nature than real numerical flow (i.e. flow of elements of linear normed space, flow of matrices or flow of p-adic numbers), encounters difficulties in implementing relevant generalizations of splines. We overcome these difficulties by a special construction: according to properties of spline-wavelet decomposition (see [19]) the construction of the main flow reduces to the trace operation over initial flow on the enlargement of the initial grid. Thus, for obtaining the adaptive main flow of spline-wavelet decomposition it is sufficient to construct adaptive approximation of the initial flow.

In this paper we propose algorithms for the construction of the main flow in adaptive spline-wavelet decomposition for flows of the elements of a linear normed space. Under condition of the same approximation we consider the ratio of the volume of the main flow mentioned above to the volume of the main flow obtained with pseudo-equidistance grid. The limit characteristics are discussed in the case of the flow generated by differentiable function.

2 Some auxiliary assertions

Here we introduce some notation used in the following.

2.1 Adaptive grid

Let (α, β) be an interval of real axis $\mathbb{R}^1$, let Ξ be a grid with rational $\xi \in (\alpha, \beta)$, $i \in \mathbf{Z}$,

$$\Xi: \quad \ldots < \xi_{-2} < \xi_{-1} < \xi_0 < \xi_1 < \xi_2 \ldots, \qquad (1)$$

$$\lim_{i \to -\infty} \xi_i = \alpha, \quad \lim_{i \to +\infty} \xi_i = \beta.$$

If $d \in \Xi$ then $d = \xi_i$ for $i \in \mathbf{Z}$; denote $d^- = \xi_{i-1}$ and $d^+ = \xi_{i+1}$.

Let us discuss $a, b \in \Xi$, $a^+ < b^-$, $a = \xi_0$, $b = \xi_M$

A set $]a, b[= \{\xi_s \mid s = 0, 1, \ldots, M\}$ is called the grid segment. Let $C\,]a, b[$ be the linear finite-dimensional space of functions $u(t)$ defined for $t \in\,]a, b[$ and $\|u\|_{C\,]a,b[} = \max_{t \in]a,b[} |u(t)|$.

Let f be a function defined on Ξ and such that

$$f(t) \geq c \quad \forall t \in]a, b[, \qquad c = const > 0. \qquad (2)$$

By definition, put

$$\varepsilon^* = \max_{\xi \in]a,b[} \max_{t \in \{\xi, \xi^+\}} f(t)(\xi^+ - \xi), \qquad (3)$$

$$\varepsilon^{**} = (b - a)\|f\|_{C\,]a,b[}. \qquad (4)$$

Lemma 1 *If $\varepsilon \in (\varepsilon^*, \varepsilon^{**})$ and conditions (2) – (4) are fulfilled, then there exists the unique natural integer $K = K(f, \varepsilon, \Xi)$ and the grid $\widetilde{X} \subset]a, b[$,*

$$\widetilde{X} = \widetilde{X}(f, \varepsilon, \Xi) :$$

$$a = \widetilde{x}_0 < \widetilde{x}_1 < \ldots < \widetilde{x}_K \le \widetilde{x}_{K+1} = b \tag{5}$$

such that

$$\max_{t \in]\widetilde{x}_s,\, \widetilde{x}_{s+1}[} f(t)(\widetilde{x}_{s+1} - \widetilde{x}_s) \le \varepsilon <$$

$$< \max_{t \in]\widetilde{x}_s,\, \widetilde{x}^+_{s+1}[} f(t)(\widetilde{x}^+_{s+1} - \widetilde{x}_s) \tag{6}$$

$$\forall s \in \{0, 1, \ldots, K-1\},$$

$$\max_{t \in]\widetilde{x}_K,\, b[} f(t)(b - \widetilde{x}_K) \le \varepsilon, \quad \widetilde{X} \subset \Xi. \tag{7}$$

The proof of Lemma 1 is given by mathematical induction as to parameter s; the induction is the source of the algorithm for the construction of grid (5) with properties (6) – (7) (see [19], see also an illustrative example);the grid is called *the adaptive grid.*

Summation of relations (6) leads to inequality

$$\sum_{s=0}^{K-1} \max_{t \in]\widetilde{x}_s,\, \widetilde{x}_{s+1}[} f(t)(\widetilde{x}_{s+1} - \widetilde{x}_s) \le K\varepsilon <$$

$$< \sum_{s=0}^{K-1} \max_{t \in]\widetilde{x}_s,\, \widetilde{x}^+_{s+1}[} f(t)(\widetilde{x}^+_{s+1} - \widetilde{x}_s). \tag{8}$$

2.2 Pseudo-equidistant grid

By definition, put

$$\overline{\varepsilon}^* = \max_{\xi \in]a, b^-[} (\xi^+ - \xi) \|f\|_{C]a,b[}. \tag{9}$$

Under the condition of

$$\varepsilon \in (\overline{\varepsilon}^*, \varepsilon^{**}) \tag{10}$$

we find values[1]

$$N = N(f, \varepsilon, \Xi) = \lfloor \varepsilon^{**}/\varepsilon \rfloor - 1, \tag{11}$$

and

$$h = h(f, \varepsilon, \Xi) = \frac{b-a}{N+1}. \tag{12}$$

On grid segment $]a, b[$ we discuss a set

$$\overline{X} = \overline{X}(f, \varepsilon, \Xi) :$$

$$a = \overline{x}_0 < \overline{x}_1 < \ldots < \overline{x}_N = b, \ \overline{X} \subset \Xi, \tag{13}$$

where

$$\overline{x}_{s+1} - \overline{x}_s \le h < \overline{x}^+_{s+1} - \overline{x}_s, \tag{14}$$
$$s \in \{0, 1, \ldots, N-1\}.$$

In the next we add a knot $\overline{x}_{N+1} \in \Xi$ to the grid $\overline{X}$, where $\overline{x}_{N+1} > \overline{x}_N$ and

$$\overline{x}_{N+1} - \overline{x}_N \le h. \tag{15}$$

Therefore

$$(b-a)\|f\|_{C]a,b[} - 2\varepsilon < N\varepsilon \le$$

$$\le (b-a)\|f\|_{C]a,b[} - \varepsilon. \tag{16}$$

Using (2.2), we get $\frac{b-a}{N+1}\|f\|_{C]a,b[} < \varepsilon$; thus by (12), follows inequality $h\|f\|_{C]a,b[} < \varepsilon$. Taking into account the right side of inequality (2.14) and inequality (15), we obtain

$$\max_{t \in]\overline{x}_s,\, \overline{x}_{s+1}[} f(t)\,(\overline{x}_{s+1} - \overline{x}_s) \le \varepsilon,$$

$$s \in \{0, 1, \ldots, N\}. \tag{17}$$

Grid (13) with properties (14) – (15) is named *pseudo-equidistant grid with mesh width* h (see [19]). Taking into account (4), (9) – (11), we have the following assertion

Lemma 2 *If $\varepsilon \in (\overline{\varepsilon}^*, \varepsilon^{**})$, then grid (13) with properties (14) – (15) exists.*

2.3 Relative quantity of knots

Let's suppose that function $f(t)$ is continuous on segment $[a, b]$, and

$$f(t) \ge c > 0 \quad \forall t \in [a, b]. \tag{18}$$

Consider the sequence of grids $\Xi(\lambda)$,

$$\Xi(\lambda) : \ldots < \xi_{-1}(\lambda) < \xi_0(\lambda) < \xi_1(\lambda) < \ldots, \tag{19}$$

depending on parameter $\lambda > 0$ such that $a, b \in \Xi(\lambda)$.

By definition, put

$$]a, b[_\lambda = \Xi(\lambda) \cap [a, b], \quad h_\lambda = \max_{\xi \in]a, b^-[} (\xi^+ - \xi).$$

Theorem 3 *If function $f(t)$ is continuous and satisfies condition (18), and the sequence of grids (19) such that*

$$\lim_{\lambda \to +0} h_\lambda = 0, \tag{20}$$

then the relation

$$\lim_{\varepsilon \to +0} \lim_{\lambda \to +0} \frac{N}{K} = \frac{\|f\|_{C[a,b]}}{\frac{1}{b-a}\int_a^b f(t)dt}$$

is true.

[1]For value r the epression $\lfloor r \rfloor$ is integer number k with property $0 \le r - k < 1$.

3 Approximation of flow

Let $\mathcal{F}$ be a metrized field[2]; the appropriate metric is denoted by $|\cdot|$ and it has the following properties: a) $|f| \geq 0\ \forall f \in \mathcal{F}$, and $|f| = 0 \iff f = 0$, b) the relations $|f+g| \leq |f| + |g|$ and c) $|fg| = |f||g|$ are right $\forall f, g \in \mathcal{F}$.

Consider linear normed space $\mathcal{M}$ over field $\mathcal{F}$; let $\|\cdot\|$ be a norm in the space.

Denote by $\mathbf{C}_{\mathcal{M}}\,]a,b\,[$ the linear finite-dimensional space of abstract functions $U(t)$, $t \in\,]a,b\,[$, with values of the functions[3] in space $\mathcal{M}$. Let

$$\|U\|_{\mathbf{C}_{\mathcal{M}}\,]a,b\,[} = \max_{t\in\,]a,b\,[} \|U(t)\|$$

be a norm in space $\mathbf{C}_{\mathcal{M}}\,]a,b\,[$. The element $U(t)$ of space $\mathbf{C}_{\mathcal{M}}\,]a,b\,[$ is called the *general flow*. Later we need to use abstract functions defined on segment $[c,d]$ of the real axis such that their range of values is situated in $\mathcal{M}$; for them the differentiation is introduced in the usual way. Therefore we also discuss the linear spaces $\mathbf{C}_{\mathcal{M}}[c,d]$, $\mathbf{C}^1_{\mathcal{M}}[c,d]$ of continuous and of continuously differentiated abstract functions accordingly.

Let $U(t)$ be a function defined on grid (1). By definition, put

$$D_\Xi U(\xi) = \frac{U(\xi^+) - U(\xi)}{\xi^+ - \xi}.$$

Let $\widehat{X}$ be subset of grid Ξ such that

$$\widehat{X}: \quad a = \widehat{x}_0 < \widehat{x}_1 < \widehat{x}_2 < \ldots < \widehat{x}_{\widehat{K}} < \widehat{x}_{\widehat{K}+1} = b.$$

Let

$$\widetilde{U}(t) = U(\widehat{x}_j) + \frac{U(\widehat{x}_{j+1}) - U(\widehat{x}_j)}{\widehat{x}_{j+1} - \widehat{x}_j}(t - \widehat{x}_j)$$

$$\forall t \in [\widehat{x}_j, \widehat{x}_{j+1}),\ j \in \{0, 1, \ldots, \widehat{K}\}$$

be a piecewise linear interpolation of function $U(t)$, defined on segment $]a,b\,[$.

It's evident that

$$\|U(t) - \widetilde{U}(t)\| \leq$$

$$\leq (\widehat{x}_{j+1} - \widehat{x}_j) \max_{\xi\in\,]\widehat{x}_j, \widehat{x}^-_{j+1}\,[} \|D_\Xi U(\xi)\|, \qquad (21)$$

[2]The field of real numbers, the field of complex numbers and the field of p-adic numbers are metrized fields (i.e. fields with evaluation).

[3]The expression "abstract function" is often replaced by the word "function"; that doesn't lead to confusion because in all cases when we discuss an abstract function with values in the space $\mathcal{M}$, we denote it with capital letter or semiboldface type.

4 On number of grid knots

4.1 A grid of adaptive type

Theorem 4 *Suppose that*

$$\|D_\Xi U(t)\| \geq c > 0 \qquad \forall t \in \Xi. \qquad (22)$$

If $\eta > 0$, and grid $\widehat{X}$ coincides with grid $\widetilde{X}(\|D_\Xi U(t)\|, \eta, \Xi)$, then

1) the quantity of knots $K'_{U,\Xi}(\eta) = K(\|D_\Xi U(t)\|, \eta, \Xi)$ of the grid satisfy relations

$$\sum_{s=0}^{K-1} \max_{t\in\,]\widetilde{x}_s,\, \widetilde{x}_{s+1}\,[} \|D_\Xi U(t)\|(\widetilde{x}_{s+1} - \widetilde{x}_s)/\eta \leq$$

$$\leq K'_{U,\Xi}(\eta) <$$

$$< \sum_{s=0}^{K-1} \max_{t\in\,]\widetilde{x}_s,\, \widetilde{x}^+_{s+1}\,[} \|D_\Xi U(t)\|(\widetilde{x}^+_{s+1} - \widetilde{x}_s)/\eta, \quad (23)$$

2) inequality

$$\|U(t) - \widetilde{U}(t)\| \leq \eta \qquad \forall t \in\,]a, b, [\qquad (24)$$

is true,

3) if there are sequences of grids (19) with condition (20) and $U \in \mathbf{C}^1_{\mathcal{M}}[a,b]$, for which $\|U'(t)\| \geq c > 0\ \forall t \in [a,b]$, then relation

$$\lim_{\eta'\to+0} \lim_{\lambda\to+0} K'_{U,\Xi(\lambda)}(\eta')\eta' = \int_a^b \|U'(t)\| dt \qquad (25)$$

is fulfilled.

Proof: Formula (23) follows from relation (8), where it needs to put $f(t) = \|D_\Xi U(t)\|$. Under condition (22) the inequality (24) follows from (3) and (6), where $f(t) = \|D_\Xi U(t)\|$, $\varepsilon = \eta$. Finally, formula (25) follows from (23) by passing to the limit.

4.2 Pseudo-equidistant grid

Theorem 5 *If grid $\widehat{X}$ coincides with grid $\overline{X}(\|D_\Xi U\|, \eta, \Xi)$, then*

1) the number $N'_{U,\Xi}(\eta) = N(\|D_\Xi U\|, \eta, \Xi)$ of inner knots of the grid satisfies the relation

$$(b-a)\|D_\Xi U\|_{\mathbf{C}_{\mathcal{M}}\,]a,b\,[}/\eta - 2 < N'_{U,\Xi}(\eta) \leq$$

$$\leq (b-a)\|D_\Xi U\|_{\mathbf{C}_{\mathcal{M}}\,]a,b\,[}/\eta, \qquad (26)$$

2) inequality

$$\|U(t) - \widetilde{U}(t)\| \leq \eta \qquad \forall t \in\,]a, b\,[\qquad (27)$$

is right.

Proof: Considering grid $\widehat{X} = \overline{X}(\|D_\Xi U\|, \eta, \Xi)$, we apply formula (2.2); as a result we get the relation (26). The inequality (27) follows from relations (3) and (2.2) if $f(t) = \|D_\Xi U(t)\|$ and $\varepsilon = \eta$.

4.3 Comparative characteristics of the quantity of knots under the condition of the same approximation

Theorem 6 *Consider the family of grids (19) – (20). Let $U(t)$, $t \in [a,b]$, be a continuously differentiable function with property*

$$\|U'\|_{\mathbf{C}_{\mathcal{M}}[a,b]} \neq 0; \tag{28}$$

then

$$\lim_{\eta\to+0}\lim_{\lambda\to+0}\frac{K'_{U,\Xi}(\eta)}{N'_{U,\Xi}(\eta)} = \frac{\frac{1}{b-a}\int_a^b \|U'(t)\|dt}{\|U'\|_{\mathbf{C}_{\mathcal{M}}[a,b]}}. \tag{29}$$

Proof: Under the conditions of (28) we can discuss a ratio $\frac{K'_{U,\Xi}(\eta)}{N'_{U,\Xi}(\eta)}$, taking into account relations (23) – (24) and (26) – (27); the passing to the limit gives the correlation (29).

5 Wavelet support

The construction of embedded grids and evaluation of approximations we considered before. In this section we suppose that embedded grids have been constructed; here we discuss calibration relations, which are wavelet support in the next discussion.

5.1 Embedded grid

Let m be a natural number; by definition, put

$$J_m = \{0,1,\dots,m\}, \qquad J'_m = \{-1,0,1,\dots,m\}.$$

Consider the functions $\{\omega_j(t)\}_{j\in J'_{M-1}}$ as elements of the space $C]a,b[$:

$$\omega_j(\xi_s) = \delta_{s,j+1}, \qquad s \in J_M.$$

Let $g^{(i)}$, $i \in J'_{M-1}$ be the linear functionals defined by relations

$$\langle g^{(i)}, u\rangle = u(\xi_{i+1}) \qquad \forall u \in C]a,b[. \tag{30}$$

The system $\{\omega_j\}_{j\in J'_{M-1}}$ is the basis of the space $C]a,b[$; we have

$$\langle g^{(i)}, \omega_j\rangle = \delta_{i,j} \qquad \forall\, i,j \in J'_{M-1}.$$

In further we discuss a set $]c,d[$ as an empty set if $c > d$.

Suppose $5 \le K < M$. Consider an injective map κ of the set J_K to the set J_M such that

$$\kappa(0)=0, \quad \kappa(i)<\kappa(i+1), \quad \kappa(K)=M. \tag{31}$$

Let $J^* \subset J_M$ be the set defined by the formula

$$J^* = \kappa J_K. \tag{32}$$

In view of (31) – (32) the revised map κ^{-1} defined on the set J^* uniquely: $\forall r \in J^* \quad \kappa^{-1} : r \longrightarrow s, \quad s \in J_K, \qquad J_K = \kappa^{-1}J^*$.

Let

$$\widehat{X}: \quad a = \widehat{x}_0 < \widehat{x}_1 < \ldots < \widehat{x}_K = b$$

be a new grid with knots $\widehat{x}_i = \xi_{\kappa(i)}$, $i \in J_K$.

Sometimes we discuss additional knots ξ_{-1} and $\widehat{x}_{-1}$ with property $\xi_{-1} = \widehat{x}_{-1} < a$.

5.2 Calibration relations

Consider functions $\widehat{\omega}_j(t)$, $j \in J'_{K-1}$ defined by relations

$$\widehat{\omega}_i(t) = (t-\xi_{\kappa(i)})(\xi_{\kappa(i+1)}-\xi_{\kappa(i)})^{-1}$$

$$\text{for}\quad t \in]\xi^{+}_{\kappa(i)}, \xi_{\kappa(i+1)}[,\ i \in J_{K-1}, \tag{33}$$

$$\widehat{\omega}_i(t) = (\xi_{\kappa(i+2)}-t)(\xi_{\kappa(i+2)}-\xi_{\kappa(i+1)})^{-1}$$

$$\text{for}\quad t \in]\xi_{\kappa(i+1)}, \xi^{-}_{\kappa(i+2)}[,\ i \in J'_{K-2}; \tag{34}$$

$$\widehat{\omega}_i(t) = 0 \quad \text{for} \quad t \in]a,b[\setminus]\xi^{+}_{\kappa(i)}, \xi^{-}_{\kappa(i+2)}[. \tag{35}$$

It is clear to see that

$$\widehat{\omega}_i(\xi_{\kappa(i+1)}) = 1 \quad \forall i \in J'_{K-1}. \tag{36}$$

In the following we use the notation

$$\operatorname{supp}\widehat{\omega}_i =]\widehat{x}_i, \widehat{x}_{i+2}[.$$

Splines $\widehat{\omega}_i$ could be written as linear combinations of splines ω_j:

$$\widehat{\omega}_i(t) = \sum_{j\in J'_{M-1}} p_{i,j}\omega_j(t) \quad \forall t \in]a,b[, \quad i \in J'_{K-1}; \tag{37}$$

formulas (37) are called *calibration relations*.

Applying the functionals $g^{(j)}$ to (37) and taking into account relations (30), we have

$$p_{-1,j} = \widehat{\omega}_{-1}(\xi_{j+1})$$

$$\forall j \in \{\kappa(0)-1, \kappa(0), \dots, \kappa(1)-2\}, \tag{38}$$

$$p_{i,j} = \widehat{\omega}_i(\xi_{j+1})$$

$$\forall j \in \{\kappa(i), \dots, \kappa(i+2)-2\} \quad \forall i \in J_{K-2}, \tag{39}$$

$$p_{K-1,j} = \widehat{\omega}_{K-1}(\xi_{j+1})$$

$$\forall j \in \{\kappa(K-1), \dots, \kappa(K)-1\}; \tag{40}$$

the numbers $p_{r,s}$, $r \in J'_{K-1}$, $s \in J'_{M-1}$, which are absent in these formulas, are equal to zero.

Consider functionals

$$\langle \widehat{g}^{(i)}, u\rangle = u(\widehat{x}_{i+1})\ \forall u \in C]a,b[,\ i \in J'_{K-1}. \tag{41}$$

5.3 Matrix of restriction

Discuss matrix $P = (p_{i,j})_{i\in J'_{K-1}, j\in J'_{M-1}}$; here $p_{i,j} = \langle g^{(j)}, \widehat{\omega}_i\rangle$. The matrix P is called a restriction matrix. We introduce the ascending ordered subsets of set **Z**:

$$J^0 = \{-1, \dots, \kappa(1) - 2\},$$

$$J^1(r) = \{\kappa(r), \dots, \kappa(r+1) - 1\} \qquad \forall r \in J_{K-1},$$

$$J^2(r) = \{\kappa(r+1), \dots, \kappa(r+2) - 2\} \qquad \forall r \in J_{K-2},$$

$$J(r) = J^1(r) \bigcup J^2(r) \quad \forall r \in J_{K-2},$$

$$J(K-1) = J^1(K-1).$$

The ascending ordered set will be discussed as empty if its first element is more then last one.

Theorem 7 *The calibration relations*

$$\widehat{\omega}_r(t) = \sum_{q\in J'_{M-1}} p_{r,q}\omega_q(t) \ \forall t \in]a,b[, \ r \in J'_{K-1}, \tag{42}$$

are right; here

$$p_{-1,q} = \frac{\xi_{\kappa(1)} - \xi_{q+1}}{\xi_{\kappa(1)} - \xi_{\kappa(0)}} \qquad q \in J^0, \tag{43}$$

$$p_{r,q} = \frac{\xi_{q+1} - \xi_{\kappa(r)}}{\xi_{\kappa(r+1)} - \xi_{\kappa(r)}}$$
$$q \in J^1(r), \quad r \in J_{K-1}, \tag{44}$$

$$p_{r,q} = \frac{\xi_{\kappa(r+2)} - \xi_{q+1}}{\xi_{\kappa(r+2)} - \xi_{\kappa(r+1)}}$$
$$q \in J^2(r), \quad r \in J_{K-2}, \tag{45}$$

with elements $p_{r,q}$ unmentioned in formulas (43) – (7) equal to zero.

Proof. First of all we note that relations (7) and (7) aren't converse to each other, because for data r the sets $J^1(r)$ and $J^2(r)$ aren't intersects. It's clear to see that formulas (42) – (7) follow from relations (38) – (40) by correlations (33) – (36).

5.4 Matrix of prolongation

Consider matrix $Q = (q_{s,j})_{s\in J'_{K-1}, j\in J'_{M-1}}$ with elements

$$q_{s,j} = \langle \widehat{g}^{(s)}, \omega_j\rangle; \tag{46}$$

the matrix Q is called *the matrix of prolongation*.

Taking into account the formulas (41), (46), we obtain the next assertions (see also [20]).

Theorem 8 *In the matrix Q*

1) if $j+1 \notin J^*$, then the column $q^{(j)} = (q_{sj})_{s\in J'_{K-1}}$ is zero column;

2) if $j+1 \in J^*$, then the column $q^{(j)}$ contains the unit on the s_0-th place, where $\kappa(s_0+1) = j+1$; the other elements of the column are equal to zero.

6 General flows and their reconstruction

Consider linear spaces

$$\mathcal{S} = \mathcal{S}(X, \varphi, \mathcal{M}) = \{\mathbf{u} \mid \mathbf{u}(t) =$$
$$= \sum_{j\in J'_{M-1}} \mathbf{C}_j\omega_j(t) \ \forall \mathbf{C}_s \in \mathcal{M} \ \forall j \in J'_{M-1}, t \in]a,b[\},$$

$$\widehat{\mathcal{S}} = \mathcal{S}(\widehat{X}, \varphi, \mathcal{M}) = \{\mathbf{u} \mid \mathbf{u}(t) =$$
$$= \sum_{i\in J'_{K-1}} \mathbf{A}_i\widehat{\omega}_i(t) \quad \forall \mathbf{A}_s \in \mathcal{M} \ \forall s \in J'_{K-1}, t \in]a,b[\}.$$

Taking into account the calibration relations (42), we have $\widehat{\mathcal{S}} \subset \mathcal{S} \subset \mathbf{C}_{\mathcal{M}}]\mathbf{a},\mathbf{b}[$.

Suppose there is the next equivalence

$$\sum_{j\in J'_{M-1}} \mathbf{C}_j\omega_j(t) \equiv \mathbf{0} \quad \forall t \in]a,b[\Longleftrightarrow$$

$$\Longleftrightarrow \ \mathbf{C}_j = \mathbf{0} \quad \forall j \in J'_{M-1}. \tag{47}$$

If (47) is fulfilled then we say that *the system $\{\omega_j\}_{j\in J'_{M-1}}$ is linear independent over the space $\mathcal{S}$.*

Let $\mathcal{P}$ be an operation of projection for the space $\mathcal{S}$ on the space $\widehat{\mathcal{S}}$ defined by formula

$$\mathcal{P}\mathbf{u} = \sum_{s\in J'_{K-1}} \sum_{j\in J'_{M-1}} \mathbf{C}_j\langle \widehat{g}^{(s)}, \omega_j\rangle\widehat{\omega}_s$$

$$\forall \mathbf{u} = \sum_{j\in J'_{M-1}} \mathbf{C}_j\omega_j \in \mathcal{S}. \tag{48}$$

By definition, put

$$\langle \widehat{g}^{(s)}, \mathbf{u}\rangle = \sum_{j\in J'_{M-1}} \mathbf{C}_j\langle \widehat{g}^{(s)}, \omega_j\rangle;$$

by (48) we have

$$\mathcal{P}\mathbf{u}(t) = \langle \widehat{g}^{(k-1)}, \mathbf{u}\rangle\widehat{\omega}_{k-1}(t) + \langle \widehat{g}^{(k)}, \mathbf{u}\rangle\widehat{\omega}_k(t)$$

$$\forall t \in t \in]\widehat{x}_k, \widehat{x}_{k+1}[, \quad k \in J_{K-1}.$$

The operation $\mathcal{P}$ defines wavelet decomposition

$$\mathcal{S} = \mathcal{S} + \mathcal{W} \tag{49}$$

of space $\mathcal{S}$, which is named *the initial space*, on space $\widehat{\mathcal{S}}$ (the last one is named *the main space*) and space $\mathcal{W}$, which is named *the wavelet space*.

Let $\mathbf{C} = (\mathbf{C}_{-1}, \mathbf{C}_0, \mathbf{C}_1, \dots, \mathbf{C}_{M-1})^T$ be the initial flow of elements from space $\mathcal{M}$. By definition, put

$$\mathbf{u} = \sum_{s\in J'_{M-1}} \mathbf{C}_s\omega_s. \tag{50}$$

Using the relation (49), we get the second representation of the element $\mathbf{u}$:

$$\mathbf{u} = \widehat{\mathbf{u}} + \mathbf{w}, \tag{51}$$

where

$$\widehat{\mathbf{u}} = \sum_{i\in {J'}_{K-1}} \mathbf{A}_i\widehat{\omega}_i, \qquad \mathbf{w} = \sum_{j\in {J'}_{M-1}} \mathbf{B}_j\omega_j,$$

$$\mathbf{B}_j,\ \mathbf{C}_s\ \in \mathcal{M} \quad \forall j,s \in {J'}_{M-1},$$

$$\mathbf{A}_i = \langle \widehat{g}^{(i)}, \mathbf{u}\rangle \quad \forall i \in {J'}_{K-1}. \tag{52}$$

By (50) – (51) we have

$$\sum_{j\in {J'}_{M-1}} \mathbf{C}_j\omega_j = \sum_{i\in {J'}_{K-1}} \mathbf{A}_i \sum_{j\in {J'}_{M-1}} p_{i,j}\omega_j +$$

$$+ \sum_{j\in {J'}_{M-1}} \mathbf{B}_j\omega_j,$$

whence taking into account the linear independence of the system $\{\omega_j\}_{j\in {J'}_{M-1}}$ over the space $\mathcal{S}$, we get *the formulas of reconstruction*

$$\mathbf{C}_j = \sum_{i\in {J'}_{K-1}} p_{i,j}\mathbf{A}_i + \mathbf{B}_j \qquad \forall j \in {J'}_{M-1}. \tag{53}$$

7 Formulas of decomposition

Using the representation (52), we rewrite formulas (53) in the form

$$\mathbf{C}_j = \sum_{i\in {J'}_{K-1}} \langle \widehat{g}^{(i)}, \mathbf{u}\rangle p_{i,j} + \mathbf{B}_j \qquad \forall j \in {J'}_{M-1}$$

and taking into account (50), we have

$$\mathbf{C}_j = \sum_{i\in {J'}_{K-1}} \sum_{s\in {J'}_{M-1}} \mathbf{C}_s \langle \widehat{g}^{(i)}, \omega_s\rangle p_{i,j} + \mathbf{B}_j$$

$$\forall j \in {J'}_{M-1};$$

now we get

$$\mathbf{B}_j = \mathbf{C}_j - \sum_{s\in {J'}_{M-1}} \Big(\sum_{i\in {J'}_{K-1}} q_{i,s}p_{i,j}\Big)\mathbf{C}_s. \tag{54}$$

Substituting (50) in (52), we have

$$\mathbf{A}_i = \langle \widehat{g}^{(i)}, \sum_{s\in {J'}_{M-1}} \mathbf{C}_s\omega_s\rangle \qquad \forall i \in {J'}_{K-1};$$

therefore

$$\mathbf{A}_i = \sum_{s\in {J'}_{M-1}} q_{i,s}\mathbf{C}_s \qquad \forall i \in {J'}_{K-1}. \tag{55}$$

The formulas (54) – (55) are called *the formulas of decomposition*.

Using the vectors

$$\mathbf{A} = (\mathbf{A}_{-1}, \mathbf{A}_0, \ldots, \mathbf{A}_{K-1})^T,$$

$$\mathbf{B} = (\mathbf{B}_{-1}, \mathbf{B}_0, \ldots, \mathbf{B}_{M-1})^T,$$

we rewrite formulas (53) and (54) – (55) in matrix form: the formulas of decomposition (54) – (55) take the form

$$\mathbf{A} = Q\mathbf{C}, \qquad \mathbf{B} = \mathbf{C} - P^TQ\mathbf{C},$$

and the formulas of reconstruction (53) are represented as

$$\mathbf{C} = P^T\mathbf{A} + \mathbf{B}.$$

Using obtained assertions (see Theorems 7 and 8) for the elements of matrices P and Q, we get the following propositions.

Theorem 9 *The formulas of decomposition have the following properties*

$$\mathbf{A}_i = \mathbf{C}_{\kappa(i+1)-1} \qquad \forall i \in {J'}_{K-1}, \tag{56}$$

$$\mathbf{B}_q = 0 \qquad \forall q+1 \in J^*, \tag{57}$$

$$\mathbf{B}_q = \mathbf{C}_q - \sum_{j\in {J'}_{K-1}} \langle g^{(q)}, \widehat{\omega}_j\rangle \mathbf{C}_{\kappa(j+1)-1}$$

$$\forall q+1 \in J_M \backslash J^*. \tag{58}$$

Theorem 10 *The wavelet flow satisfies the next relations: for $q+1 \in J_M\backslash J^*$ the equalities*

$$\mathbf{B}_q = \mathbf{C}_q - (\widehat{x}_{i+1} - \widehat{x}_i)^{-1}\Big[(\widehat{x}_{i+1} - \xi_{q+1})\mathbf{C}_{\kappa(i)-1} +$$

$$+ (\xi_{q+1} - \widehat{x}_i)\mathbf{C}_{\kappa(i+1)-1}\Big]$$

are fulfilled; here

$$\widehat{x}_i < \xi_{q+1} < \widehat{x}_{i+1}. \tag{59}$$

The formula (58) can be written in the form

$$\mathbf{B}_q = \mathbf{C}_q - p_{i-1,q}\mathbf{C}_{\kappa(i)-1} - p_{i,q}\mathbf{C}_{\kappa(i+1)-1},$$

where i satisfies to relation (59).

The formulas (57) – (58) demonstrate that the space of wavelet flows $\mathcal{B}$ is

$$\mathcal{B} = \{\mathbf{B} \mid \mathbf{B} = (\mathbf{B}_{-1}, \mathbf{B}_0, \ldots, \mathbf{B}_{M-1})$$

$$\forall \mathbf{B}_{j-1} \in \mathcal{M},\ j \in J_M\backslash J^*; \mathbf{B}_{i-1} = 0\ \forall i \in J^*\}.$$

The relation (56) indicates that the construction of the main flow is reduced to values of initial flow on the embedded grid. If the embedded grid is adaptive, then the deviation of the main flow from the initial flow is defined by Theorem 4, and if the constructed grid is the pseudo-equidistant grid, then the mentioned deviation is given by Theorem 5.

8 Conclusion

The results give us the opportunity to obtain the main flow in wavelet decomposition for flows of elements from linear normed spaces; sometimes it is very important to have decomposition of flows of matrices or flows of p-adic numbers. The results also demonstrate a large economy of computer memory in the case of usage of adaptive algorithms for construction of the main flow. Now it is simple to obtain formulas of decomposition and reconstruction; we hope to represent the application of them to flow of matrices in an extended version of this paper.

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