A Doubly Nonlinear Parabolic Equations with a $\sigma$-Finite Measure as its Initial Value

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Abstract: A class of equations

$$u_t - \text{div}(\sigma(u, \nabla u)) = f(x, t), (x, t) \in S_T = \mathbb{R}^N \times (0, T),$$

is considered. These equations arise in the study of turbulent filtration of gas or liquid through porous media. If the initial value is a $\sigma$-finite measure, the existence and and no existence of the solutions of the equation are researched.

Key–Words: Nonlinear parabolic equation, Cauchy problem, Existence, $\sigma$-Finite measure.

1 Introduction

Let $S_T = \mathbb{R}^N \times (0, T)$. Consider an equation of the type

$$u_t - \text{div}(\sigma(u, \nabla u)) = f(x, t), (x, t) \in S_T, \quad (1)$$

where $\nabla u = (u_{x_1}, \ldots, u_{x_N})$, $\sigma = (\sigma^1, \ldots, \sigma^N)$, $f = f(x, t)$ is a bounded function on $S_T$, the functions $\sigma^i(u, \zeta)$ are continuous on $\mathbb{R} \times \mathbb{R}^N$ and for any $u \in \mathbb{R}, \zeta \in \mathbb{R}^N, \sigma$ satisfies

$$\sigma(u, \zeta) \cdot \zeta \geq \nu_0 |u|^{(m-1)(p-1)}|\zeta|^p - \Phi_0(u), \quad (2)$$

$$|\sigma(u, \zeta)| \leq \mu_1 |u|^{(m-1)(p-1)}|\zeta|^{p-1} + \Phi_0(u), \quad (3)$$

with the constants $p \in (1, 2), m > 1, \nu_0 > 0, \mu_1 > 0$ and $\Phi_i(u) \geq 0$. Equation (1) with the conditions (2)-(3) are the particular cases of the so-called doubly nonlinear parabolic equations (DNPE), which arise particularly in the study of turbulent filtration of gas or liquid through porous media. A typical example is the polytropic filtration equation

$$u_t = \text{div}(\sigma(u, \nabla u^m) |\nabla u^m|^{p-2} \nabla u^m), \quad (4)$$

which includes when $m = 1$, the well-known $p$-diffusion equation, when $p = 2$, the well-known porous medium equation. Whether $m = 1$ or $p = 2$, if the initial value $u(x, 0)$ is suitably regular, the existence of the weak solutions had been profoundly probed in Wu-Zhao [1], Gmira [2], Yuan-Zhao [3], Zhao [4, 10, 15], Zhao-Yuan [5], Li-Xia [11], DiBenedetto [12], Ivanov [13, 30-35], DiBenedetto-Herrero [16], Zhao-Xu [19], Yuan [22], Berins [23], Filo [24], Ishige [25], Aronson-Caffarelli [26], Aronson-Peletier [27], Dahlberg-Kenig [28], Vazquez [30-31], Zhan [6, 22, 36-38] and references therein. Recently, the author [36] had prove the existence of the solution to (4) with

$$u(x, 0) = \mu, \quad x \in \mathbb{R}^N, \quad (5)$$

where the initial value $\mu$ is a nonnegative $\sigma$-finite measure.

In particular, Ivanov [35] had proved that there is an unique strong solution of the initial boundary value problem of equation (1), and the strong solution is Hölder continuous, provided that the following conditions are true.

(0) The functions $\sigma^i(u, \zeta), u^{-\alpha} \sigma^i(u, \zeta)$, are continuous on $\mathbb{R}^+ \times \mathbb{R}^N$, where $\alpha = \frac{(m-1)(p-1)}{p}$.

(1)(Growth conditions) For any $u \in \mathbb{R}, \zeta \in \mathbb{R}^N,$

$$\sigma(u, \zeta) \cdot \zeta \geq \nu_0 |u|^{(m-1)(p-1)}|\zeta|^p - \Phi_0(u), \quad (6)$$

$$|\sigma(u, \zeta)| \leq \mu_1 |u|^{(m-1)(p-1)}|\zeta|^{p-1} + \Phi_0(u), \quad (7)$$

where $\mu(u) \geq 1$ is nondecreasing, $\nu_0 > 0, \mu_0 > 0, \mu_1 > 0$ are constants.

(2)(Monotonicity condition) There exists $\nu_1 > 0$ and a continuous vector function $\tilde{b}(u) \in \mathbb{R}^N$ such that
for any \( u \in \mathbb{R}, \zeta_1, \zeta_2 \in \mathbb{R}^N \),
\[
[a(u, \zeta_1) - a(v, \zeta_2)] \cdot (\zeta_1 - \zeta_2) \\
\geq \nu_1 |u|^{(m-1)(p-1)}|\zeta_1 - \zeta_2|^2 + |\zeta_1 - \zeta_2|^{1-\frac{2}{p}}.
\] (8)

(Local Lipschitz condition) For any \( u, v \in [\varepsilon, M], \varepsilon > 0, M > \varepsilon \), and any \( \zeta \in \mathbb{R}^N \),
\[
|a(u, \zeta) - a(v, \zeta)| \leq \Lambda |u - v| (|\zeta|^{p-1} + 1),
\] (9)

where \( \Lambda = \Lambda(\varepsilon, M) \geq 0 \) is constant.

Certainly, we must point out that many papers by Dibenedetto, Gianazza, Vespri etc. had studied the Harnack inequality and the regularity of the equation with the type of (1), for examples, one can refer to their reviewing article [39]. Very recently, Fornaro S., Sosio M. and Vespri V. had published a paper to get their reviewing article [39]. Very recently, Fornaro S., Sosio M. and Vespri V. had published a paper to get the \( L^r_{loc} - L^\infty_{loc} \) estimates of the solutions to the equation with the type of (1).

In the paper, by the above conditions (0)-(3), we shall research the existence of Cauchy problem of equation (1) with the measure initial value (5) when \( 1 < m < \frac{1}{p-1} \). For simplicity, we assume that when \( u \geq 1 \),
\[
\mu(u) = u^{\frac{p-1}{p}},
\] (10)

while \( u \leq 1, \mu(u) \equiv 1 \).

We define that

**Definition 1** A measurable nonnegative function \( u \) is said to be a weak solution of Cauchy problem of equation (1) with the initial value (5) if \( u \) satisfies

\[
u \in C(0, T; L^1_{loc}(\mathbb{R}^N))
\] (11)

\[
u^m \in L^p(0, T; W^{1,p}_{loc}(\mathbb{R}^N))
\] (12)

\[
\nabla \nu^m \in L^p_{loc}(S_T),
\] (13)

and for any \( t \in (0, T) \), denoting \( S_t = \mathbb{R}^N \times (0, t) \),
\[
\int_{\mathbb{R}^N} u(x, t) \varphi(x, t) dx \\
+ \int_{S_t} (-u \varphi_t + a(u, \nabla u) \nabla \varphi) dx dt \\
= \int_{\mathbb{R}^N} \varphi(x, 0) dx + \int_{S_t} f(x, t) \varphi dx dt
\] (14)

where \( \varphi \in C^1(S_t) \) and \( \varphi = 0 \) if \( |x| \) is large enough.

Let us introduce a basic iteration lemma. One can refer to [8, page 161, Lemma 3.1; or 1, page 141, Lemma 1.6] for its proof.

**Lemma 2** Let \( f(t) \) be a nonnegative bounded function defined in \([r_0, r_1]\). If for \( r_0 \leq t < s \leq r_1 \),
\[
g(t) \leq \theta g(s) + (A(s-t)^{-\alpha} + B),
\] (15)

where \( A, B, \alpha, \theta \) are nonnegative constants, \( 0 \leq \theta < 1 \). Then for \( r_0 \leq \rho < r \leq r_1 \),
\[
g(\rho) \leq c(A(r-\rho)^{-\alpha} + B),
\] (16)

where \( c \) is a constant depending on \( \alpha, \theta \).

By this lemma, using the standard Moser’s iteration method, we will prove the following theorems.

**Theorem 3** Suppose \( f(x, t) \leq 0 \), and
\[
2 > p > \frac{(m+1)N}{mN+1}, 1 < m < \frac{1}{p-1}, N \geq 2.
\] (17)

Assume that (10), and the conditions (0),(1), (3) are true. Then there is a weak solution \( u \) for equation (1) with (5), which satisfies
\[
\sup_{0 < \tau < t} \int_{B_{2R}} u(x, \tau) dx \leq c \int_{B_{2R}} |d\mu| + cR^{M_1},
\] (18)

where
\[
M_1 = \frac{N}{1 - m(p-1)} \left[ t^{\frac{1}{p}(m+1)(p-1)} + \frac{1}{t^{1-m(p-1)}} \right] \\
+ t^{\frac{p}{p-m(p-1)} + \frac{1}{p-1}} + t^{\frac{p}{p-m(p-1)} - \frac{1}{p-1}}.
\]

\[
\sup_{x \in B_{BR}} |u(x, t)| \leq c(N, p, R_0)t^{-\frac{N}{p}} \left( \int_{B_{4R}} u_0 dx \right)^{\frac{p}{N}}
\] (19)

where
\[
M_2 = \frac{Np}{k[1-m(p-1)]t^{\frac{1}{p}(m+1)(p-1)}} \\
+ t^{\frac{1}{p-m(p-1)} + \frac{1}{p-1}} + t^{\frac{p}{p-m(p-1)}},
\]

\( k = p - N[1-m(p-1)], R_0 \geq 1 \) is a constant.

**Theorem 4** Suppose \( f(x, t) \leq 0 \),
\[
1 < p \leq \frac{(m+1)N}{mN+1}, 1 < m < \frac{1}{p-1},
\] (20)

and
\[
\frac{1}{m} < \min \{ 1 + \frac{2}{m} - p, 1 - m(p-1) \},
\] (21)

then there is not solution of Cauchy problem of equation (1) with the following initial value
\[
u(x, 0) = \delta(x),
\] (22)

where \( \delta \) is classical Dirac function.

We use some ideas of [15] and [32] in our paper. By the way, if we only consider equation (4), condition (21) is unnecessary.

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2 Proof of Theorem 3

In order to prove the theorem, we quote a lemma.

Lemma 4 Let \( Q_l (l = 0, 1, 2, \ldots) \) be the sequences of the bounded open domains in \( S_T \), \( Q_{l+1} \subset Q_l \). If for all \( q \geq 1, v \in L^q(Q_0) \), and there are constants \( \alpha_0 \geq 0, \lambda, C_0 > 0, K > 1 \) such that

\[
\int \int_{Q_{l+1}} |v|^{\alpha_0 + \lambda K^{l+1}} \, dx \, dt \\
\leq \left( C_0 C_1^l \int \int_{Q_l} |v|^{\alpha_0 + \lambda K^l} \, dx \, dt \right)^K,
\]

then

\[
\text{ess sup}_{\infty} |v| \\
\leq \left( \frac{K^{-1} C_1^l}{C_0} \int \int_{Q_0} |v|^{\alpha_0 + \lambda K^0} \, dx \, dt \right)^{\frac{1}{\lambda K^0}},
\]

where \( C_1 = C_{1}^K \), \( K_1 = \sum_{l=0}^{\infty} lK^{-(l-l_0)} \), and \( l_0 \geq 0 \) is any nonnegative integer.

The proof of Theorem 3 Let \( u \) be the solution of the regularized equation of (1)

\[
u_t - \text{div}(\vec{a}(u, \nabla u)) - \varepsilon \Delta u = f(x, t), \tag{23}
\]

with nonnegative initial value

\[
u(x, 0) = u_0 \in C_0^\infty(\mathbb{R}^N). \tag{24}
\]

Then, \( \vec{a}_1(u, \nabla u) = \vec{a}(u, \nabla u) + \varepsilon \nabla u \) satisfies the assumptions of (0)-(3), from [35], we know that the problem (23)-(24) has a unique strong solution \( u_\varepsilon \). Let \( \varepsilon \to 0 \). By a similar discussion as [5], we are able to show that \( u_\varepsilon \to u \) is the solution of equation (1) with initial value (24), \( u \) is Hölder continuous and satisfies

\[
\int_{\mathbb{R}^N} u(x, t) \varphi(x, t) \, dx \\
+ \int \int_{S_t} (-u \varphi_t + \vec{a}(u, \nabla u) \nabla \varphi) \, dx \, dt \\
= \int_{\mathbb{R}^N} \varphi(x, 0) u_0(x) \, dx + \int \int_{S_t} f(x, t) \varphi \, dx \, dt, \tag{25}
\]

Moreover, similar as in [32], it is easily to know that \( u^m \in L^\infty(0, T; L^1(\mathbb{R}^N)) \cap L^p(0, T; W^{1,p}(\mathbb{R}^N)). \)

Now, let \( u_{0n} \in C_0^\infty(\mathbb{R}^N) \) be nonnegative such that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} u_{0n} \varphi \, dx = \int_{\mathbb{R}^N} \varphi \, d\mu.
\]

Assume that \( u_n \) is the solution of equation (1) with initial value \( u_{0n} \), by the following lemma 6-lemma 10, \( \{u_n\} \) is uniformly bounded on every compact set \( K \subset S_T \). Hence by [13] or [34-35], \( \{u_n\} \) is equicontinuous on every compact set \( K \subset S_T \). Then there exists a subsequence such that \( u_n \to u \) as \( n \to \infty \) in \( C(K), \nabla u_{n}^m \to \nabla u^m \) in \( L^p(0, T; L^p(\mathbb{R}^N)) \), and by a standard limiting process (referring to [1],[5], [6],[13], [34]), the properties (18), (19) of \( u \) are true, then Theorem 3 is proved.

In what follows, we will give lemma 6-lemma 10 and the proofs to complete the proof of Theorem 3. First of all, due to that \( u_n \) is Hölder continuous, \( D_1 = \{(x, t) \in (0, t) \times B_{2R} : u_n \geq 1\} \) and \( D_2 = \{(x, t) \in (0, t) \times B_{2R} : 0 \leq u_n \leq 1\} \) are well defined, then we have the additive property of the integral domains, for example, we have

\[
\int_0^t \int_{B_{2R}} \xi^p u_n^{m-1} (u_n^{m+1} + 1) \, dx \, d\tau \\
= \int \int_{D_1 \cup D_2} \xi^p u_n^{m-1} (u_n^{m+1} + 1) \, dx \, d\tau,
\]

However, if \( 0 \leq u_n \leq 1 \), the following estimates is true clearly, so we only need to deal with the domain \( D_1 \) instead of the whole domain \((0, t) \times B_{2R} \). Without loss generality, in the proof of what follows, we may assume that \( u_n \geq 1 \) in the whole domain \((0, t) \times B_{2R} \) instead of \( D_2 \). For simplism, we denote \( u_n \) as \( u \) if it is clear from the context.

Lemma 6 Suppose that (6) and (7) are true. For any given \( R_0 > 0 \), if \( R > R_0 \), then the nonnegative solution of equation (1) with the initial value (24) satisfies

\[
\sup_{x \in B_R} |u(x, t)| \\
\leq c(N, p, R_0) t^{-N+p} \left( \int_0^t \int_{B_{2R}} |u(x, t)| \, dx \, d\tau \right)^{\frac{p}{N}},
\]

where \( k = p + N(p - 1) \).

Proof As usual, let \( B_R(x_0) = \{x : |x - x_0| < R\} \), and if \( x_0 = 0 \), simply denoted as \( B_R \). Let \( \xi \) be the cut function of \( B_{2R} \times (0, T) \) and \( \alpha > 0 \) be a constant to be chosen later. Choosing the testing function in (25) as \( \xi^p u_n^{m-1} \), then one can obtain

\[
\int_0^t \int_{B_{2R}} \vec{a}(u, \nabla u) \nabla \varphi \, dx \, d\tau \\
= \int_0^t \int_{B_{2R}} \vec{a}(u, \nabla u) \, M_3 \, dx \, d\tau,
\]
\[
\begin{align*}
&= p \int_0^t \int_{B_{2R}} \bar{a}(u, \nabla u) \xi^{p-1} u^{\alpha m} \nabla \xi \, dx \, d\tau \\
&+ m \alpha \int_0^t \int_{B_{2R}} \bar{a}(u, \nabla u) \xi^{p} u^{\alpha m-1} \nabla u \, dx \, d\tau.
\end{align*}
\]

where \( M_3 = (p \xi^{p-1} u^{\alpha m} \nabla \xi + m \alpha \xi^{p} u^{\alpha m-1} \nabla u) \).

According to the assumptions (6) and (7),

\[
\begin{align*}
&\quad m \alpha \int_0^t \int_{B_{2R}} \bar{a}(u, \nabla u) \xi^{p} u^{\alpha m-1} \nabla u \, dx \, d\tau \\
&\geq m \alpha \mu_0 \int_0^t \int_{B_{2R}} \xi^{p} u^{\alpha m-1+(m-1)(p-1)} |\nabla u|^{p} \, dx \, d\tau \\
&\quad - m \alpha \mu_0 \int_0^t \int_{B_{2R}} \xi^{p} u^{\alpha m-1} (u^{m(p-1)+1}) |\nabla u|^{p} \, dx \, d\tau \\
&\geq m \alpha \mu_0 \int_0^t \int_{B_{2R}} \xi^{p} u^{\alpha m-1+(m-1)(p-1)} |\nabla u|^{p} \, dx \, d\tau \\
&\quad - 2 m \alpha \mu_0 \int_0^t \int_{B_{2R}} \xi^{p} u^{\alpha m+1} \, dx \, d\tau.
\end{align*}
\]

By Young inequality,

\[
\xi^{p-1} u^{\alpha m} |\nabla u^{m}|^{p-1} \leq \varepsilon \xi u^{m(\alpha-1)} |\nabla u^{m}|^{p} + c(\varepsilon) u^{m(\alpha-1a+p)},
\]

then

\[
\begin{align*}
&\int_0^t \int_{B_{2R}} \xi^{p-1} u^{\alpha m} |\nabla u^{m}|^{p-1} |\nabla \xi| \, dx \, d\tau \\
&\leq \varepsilon \xi u^{m(\alpha-1)} |\nabla u^{m}|^{p} + c(\varepsilon) u^{m(\alpha-1a+p)}, \quad (27)
\end{align*}
\]

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\[
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\]

\[
\begin{align*}
&\quad \int_0^t \int_{B_{2R}} \xi^{p-1} u^{\alpha m} |\nabla u^{m}|^{p-1} |\nabla \xi| \, dx \, d\tau \\
&\leq \varepsilon \int_0^t \int_{B_{2R}} \xi u^{m(\alpha-1)} |\nabla u^{m}|^{p} \, dx \, d\tau \\
&+ c(\varepsilon) \int_0^t \int_{B_{2R}} u^{m(\alpha-1a+p)} |\nabla \xi|^{p} \, dx \, d\tau. \quad (28)
\end{align*}
\]

By (26)-(28), since

\[
\begin{align*}
&\varepsilon \int_0^t \int_{B_{2R}} \xi u^{m(\alpha-1)} |\nabla u^{m}|^{p} \, dx \, d\tau
\end{align*}
\]
can be compensated by
\[ \alpha \int_0^T \int_{B_{2R}} \xi^p |\nabla u^m|^p \, dx \, dt, \]
and noticing that \( 0 \leq \xi \leq 1, m(p-1) < 1 \),
\[ \xi^p \geq \frac{(\alpha m+1)p}{m(\alpha+1)p} \]
we have
\[
\begin{align*}
\sup_{0 < t < T} & \int_{B_{2R}} (\xi u^m)^{\alpha+1/p} \, dx \\
 & + \int_0^T \int_{B_{2R}} |\nabla (\xi u^m)|^p \, dx \, dt \\
 & \leq c \int_0^T \int_{B_{2R}} |\nabla \xi|^p u^m \, dx \, dt \\
 & + c \int_0^T \int_{B_{2R}} |\xi| u^{\alpha+1} \, dx \, dt \\
 & + c \int_0^T \int_{B_{2R}} |\nabla \xi|^p u^{\alpha+1} \, dx \, dt \\
 & + c \int_0^T \int_{B_{2R}} u^{\alpha+1} \, dx \, dt. \quad (29)
\end{align*}
\]

Let \( \nu = \xi u^{m(\alpha+1)/p} \). By Sobolev inequality (referred to [7], page 62), we have
\[
\int_0^T \int_{B_{2R}} \nu^p \, dx \, dt \\
\leq c \left( \sup_{0 < t < T} \int_{B_{2R}} \nu^{\alpha+1} dx \right) M_6 \\
\int_0^T \left( \int_{B_{2R}} |\nabla \nu|^p \, dx \right)^{\frac{\delta r}{p}} \, dt, \quad (30)
\]
where
\[ M_6 = \frac{m(\alpha-1+p)}{(\alpha m+1)p} \left( 1 - \delta r \right), \]
\[ \delta = \frac{m(\alpha-1+p)}{\alpha m+1} - 1 \left( \frac{1}{N} - \frac{1}{p} + \frac{m(\alpha-1+p)}{\alpha m+1} \right)^{-1}. \]

Thus, if we choose
\[ r = p \left( 1 + \frac{1}{N} \frac{m+1}{m(\alpha-1+p)} \right), \]
noticing that for any \( a \geq 0, b \geq 0, \)
\[ a^\frac{p}{N} b \leq (a + b)^{1 + \frac{p}{N}} \]
is always true, then from (29), (30)
\[
\begin{align*}
\int_0^T \int_{B_{2R}} & \xi^p |\nabla u^m|^p \, dx \, dt \\
& \leq c \left( \sup_{0 < t < T} \int_{B_{2R}} \xi^{m(\alpha+1)p} u^{m+1} \, dx \right)^{\frac{p}{N}} \\
& \int_0^T \int_{B_{2R}} |\nabla (\xi u^m)|^p \, dx \, dt \\
& \leq c \left( \sup_{0 < t < T} \int_{B_{2R}} \xi^{m(\alpha+1)p} u^{m+1} \, dx \right)^{\frac{p}{N}} \\
& + \int_0^T \int_{B_{2R}} \left( |\nabla \xi|^p u^{m+1} \right) \, dx \, dt + \frac{1}{m+1} \int_0^T \left( \int_{B_{2R}} u^{m+1} \, dx \right) \, dt. \quad (31)
\end{align*}
\]

Now for \( s \in \left[ \frac{1}{2}, 1 \right], \) let \( R_l = 2R(s + \frac{1}{2}) \),
\[ T_l = \frac{T}{2} - \frac{T}{2} (s + \frac{1}{2}) \], \( Q_{R_l} = B_{R_l} \times (T_l, T) \), \( l = 0, 1, 2, \ldots \). Suppose that \( \xi_l \) is the cut functions on \( Q_{R_l} \) which satisfy
\[ \xi_l = 1, \text{ in } Q_{R_{l+1}}; \quad \xi_l = 0, \text{ in } S_{l+1} \setminus Q_{R_l}. \]
\[ |\nabla \xi_l| \leq \frac{2l+1}{(1-s)R}, \quad 0 \leq |\xi_l| \leq \frac{2l+1}{(1-s)T}. \]

Denote \( \gamma = 1 + \frac{p}{N} \) and choose \( \alpha \) such that
\[ m\alpha + 1 = \frac{N}{p} (1 - m(p-1)) + \gamma', \]
which implies
\[ m(\alpha-1+p) + \frac{(\alpha m+1)p}{N} = \frac{N}{p} (1 - m(p-1)) + \gamma'. \]

Applying (31) to \( Q_{R_l} \), one obtains,
\[
\begin{align*}
\int & \int_{Q_{R_l}} u^{\frac{\gamma'}{m(\alpha-1+p)+1}} \, dx \, dt \\
& \leq \left[ c^l ((1-s)R)^{-\gamma'} \left( \int_{Q_{R_l}} M_7 \, dx \right)^{\frac{m(\alpha-1+p)}{m+1}} \right].
\end{align*}
\]
\begin{equation}
+c^l(1-s)^{-1}T^{-1}\int_{Q_{R_1}} M_7 dxdt \gamma
\end{equation}
\begin{equation}
+c\int_{Q_{R_1}} (1 + |\nabla \xi|) u^{\alpha m+1} dxdt
\end{equation}
\begin{equation}
\leq c(1-s)^{-1}R^{-1}\int_{Q_{R_1}} M_7 dxdt. \quad (32)
\end{equation}

where $M_7 = u^N(1-m(p-1)) + \gamma^l$.

Hence, from (32), by the standard Moser’s iteration, using Lemma 5, we have

\begin{equation}
\sup_{Q_{2R}} u \leq \left[ c(1-s)^{-\frac{N+\mu}{\mu}} T^{-\frac{N+\mu}{\mu}} \int_{Q_{2R}} M_8 dxdt \right]^{\frac{1}{\gamma}}
\end{equation}
\begin{equation}
\leq (\sup u)^{1-\frac{k}{p\gamma}} \left( c \left( (1-s)T \right)^{-\frac{N+\mu}{\mu}} \int_{Q_{2R}} u dxdt \right)^{\frac{1}{\gamma}}
\end{equation}
\begin{equation}
\leq \frac{1}{4} \sup_{Q_{2R}} u + c((1-s)T)^{-\frac{N+\mu}{\mu}} \left( \int_{Q_{2R}} u dxdt \right)^{\frac{k}{\gamma}}.
\end{equation}

where $M_8 = u^N(m(p-1)+1) + \gamma$.

Hence by lemma 2, we obtain the conclusion.

\textbf{Lemma 7} Suppose $1 < p < 2$, $m(p-1) < 1$, $f(x,t) \leq 0$, $R \geq 1$, then the solution $u$ of Cauchy problem of equation (1) with the initial value (24) satisfies

\begin{equation}
\sup_{0 < \tau < t} \int_{B_R} u(x,\tau)dx \leq c \int_{B_{2R}} u(x,0)dx + cR^{1-m(p-1)} M_9. \quad (33)
\end{equation}

where $M_9 = \left[ t^{\frac{1}{p-(m+1)(p-1)}} + \frac{1}{1-m(p-1)} + t^{\frac{1}{p-1}} \right]$.

\textbf{Proof} Let $\xi$ be the cut function on $B_{2R}$, and $\xi$ satisfy that $\xi = 1$ on $B_{2R}$, $|\nabla \xi| \leq (1-s)^{-1}R^{-1}$, $s \in \left[ \frac{1}{2}, 1 \right)$. For any $t > 0$, we have

\begin{equation}
\int_{B_{2R}} u(x,t)dx \leq \int_{B_{2R}} u_0dx
\end{equation}
\begin{equation}
+ \frac{c}{(1-s)R} \int_{0}^{t} \int_{B_{2R}} |\nabla u|^\mu \xi^{p-1} dxdt
\end{equation}
\begin{equation}
+ \frac{c}{(1-s)R} \int_{0}^{t} \int_{B_{2R}} u^{(m-1)(p-1)} \xi^{p-1} dxdt
\end{equation}
\begin{equation}
+ \int_{0}^{t} \int_{B_{2R}} f(x,t) \xi^p dxdt, \quad (34)
\end{equation}

where the constant $c$ depends on $\mu_1, m$.

We choose testing function

\begin{equation}
\varphi = \xi^{p-1} \frac{\partial \varphi}{\partial \tau} u^m (1-\frac{\alpha p}{\mu p - 1})
\end{equation}

in (25), where $\alpha, \beta$ are constants to be chosen later, then

\begin{equation}
\int_{0}^{t} \int_{B_{2R}} \xi^{p-1} \frac{\partial \varphi}{\partial \tau} u^m (1-\frac{\alpha p}{\mu p - 1}) dxdt
\end{equation}
\begin{equation}
= M_{10} \int_{0}^{t} \int_{B_{2R}} \xi^{p-1} \frac{\partial \varphi}{\partial \tau} u^m (1-\frac{\alpha p}{\mu p - 1})+1 dxdt
\end{equation}
\begin{equation}
= M_{10} \int_{B_{2R}} \xi^{p-1} \frac{\partial \varphi}{\partial \tau} u^m (1-\frac{\alpha p}{\mu p - 1})+1 dx
\end{equation}
\begin{equation}
- M_{10} \frac{\beta p}{p-1} \int_{0}^{t} \int_{B_{2R}} \xi^{p-1} \frac{\partial \varphi}{\partial \tau} u^m (1-\frac{\alpha p}{\mu p - 1})+1 dxdt,
\end{equation}

where $M_{10} = [m(1-\frac{\alpha p}{\mu p - 1})+1]^{-1}$. According to the assumptions (6) and (7),

\begin{equation}
\int_{0}^{t} \int_{B_{2R}} \xi^{p-1} \frac{\partial \varphi}{\partial \tau} u^m (1-\frac{\alpha p}{\mu p - 1}) dxdt
\end{equation}
\begin{equation}
= - \int_{0}^{t} \int_{B_{2R}} \frac{\partial \varphi}{\partial \tau} \nabla (\xi^p u^m (1-\frac{\alpha p}{\mu p - 1})) \cdot \nabla u \ n dxdt
\end{equation}
\begin{equation}
= -p \int_{0}^{t} \int_{B_{2R}} \frac{\partial \varphi}{\partial \tau} \xi^{p-1} \frac{\partial \varphi}{\partial \tau} u^m (1-\frac{\alpha p}{\mu p - 1}) \nabla \xi \cdot \nabla u \ n dxdt
\end{equation}
\begin{equation}
+ M_{11} \int_{0}^{t} \int_{B_{2R}} \frac{\partial \varphi}{\partial \tau} \xi^{p-1} \frac{\partial \varphi}{\partial \tau} u^{\alpha m} \cdot \nabla u \ n dxdt
\end{equation}
\begin{equation}
= - I_1 + I_2
\end{equation}

where $M_{11} = (\frac{\alpha p}{\mu p - 1} - 1)$.

\begin{equation}
I_1 \leq p \int_{0}^{t} \int_{B_{2R}} M_{12} |\nabla \xi||M_{13} dxdt
\end{equation}
\begin{equation}
= \frac{\mu_1 p}{m p - 1} \int_{0}^{t} \int_{B_{2R}} M_{12} |\nabla \xi||\nabla u^m|^{p-1} dxdt
\end{equation}
\begin{equation}
+ p \int_{0}^{t} \int_{B_{2R}} \xi^{p-1} \frac{\partial \varphi}{\partial \tau} u^m (1-\frac{\alpha p}{\mu p - 1})+1 |\nabla \xi| dxdt
\end{equation}
\begin{equation}
\leq \int_{0}^{t} \int_{B_{2R}} M_{14} \varepsilon |\nabla u^m|^p + c(\varepsilon) u^{np} |\nabla \xi|^p dxdt
\end{equation}
\begin{equation}
+ M_{15} \int_{0}^{t} \int_{B_{2R}} \frac{\partial \varphi}{\partial \tau} u^m (1-\frac{\alpha p}{\mu p - 1})+1 (\frac{m-1}{p} (\frac{m}{p} - 1)) dxdt.
\end{equation}

where

\begin{equation}
M_{12} = \xi^{p-1} \frac{\partial \varphi}{\partial \tau} u^m (1-\frac{\alpha p}{\mu p - 1})
\end{equation}
\[ M_{13} = (\mu_1 u^{(m-1)(p-1)}|\nabla u|^{p-1} + u), \]
\[ M_{14} = \xi^p \tau^{p-1} u^{-\frac{\alpha m}{p-1}}, \]
\[ M_{15} = \frac{p}{(1-s)R}. \]

while \( I_2 \) is equal to
\[ M_{16} \int_0^t \int_{B_{2R}} \tau^{p-1} \xi^p u^{-\frac{\alpha m}{p-1}} \nabla u \cdot \tilde{a}(u, \nabla u) dx d\tau \]
\[ \geq M_{16} \nu_0 \int_0^t \int_{B_{2R}} \tau^{p-1} \xi^p M_{17} |\nabla u|^p dx d\tau \]
\[ -M_{16} \mu_0 \int_0^t \int_{B_{2R}} \tau^{p-1} \xi^p u^{-\frac{\alpha m}{p-1} + m - 1} M_{18} dx d\tau \]
\[ \geq \frac{\nu_0}{m^{p-2}} M_{16} \int_0^t \int_{B_{2R}} \tau^{p-1} \xi^p u^{-\frac{\alpha m}{p-1}} |\nabla u|^m dx d\tau \]
\[ -2m \mu_0 \left( \frac{\alpha p}{p - 1} - 1 \right) \int_0^t \int_{B_{2R}} \tau^{p-1} u^{-\frac{\alpha m}{p-1} + mp} dx d\tau \]
where \( M_{16} = m \left( \frac{\alpha p}{p - 1} \right) \),
\[ M_{17} = u^{-\frac{\alpha m}{p-1} + m - 1 + (m-1)(p-1)}, \]
\[ M_{18} = (u^{(m-1)(p-1)} + 1). \]

By the above inequalities, we have
\[ \left[ \frac{\nu_0}{m^{p-1}} \left( \frac{\alpha p}{p - 1} - 1 \right) - \varepsilon \right] \int_0^t \int_{B_{2R}} M_{19} |\nabla u|^m dx d\tau \]
\[ + c \int_0^t \int_{B_{2R}} \xi^p \tau^{p-1} u^{m \left( 1 - \frac{\alpha m}{p-1} \right) + 1} dx d\tau \]
\[ = c \int_{B_{2R}} \xi^p \tau^{p-1} u^{m \left( 1 - \frac{\alpha m}{p-1} \right) + 1} dx \]
\[ + \frac{c(\varepsilon)}{(1-s)R} \int_0^t \int_{B_{2R}} \xi^p \tau^{p-1} u^{-\frac{\alpha m}{p-1}} \nabla u^m dx d\tau \]
\[ + \frac{p}{(1-s)R} \int_0^t \int_{B_{2R}} \tau^{p-1} u^{m \left( 1 - \frac{\alpha m}{p-1} \right) + 1} dx d\tau \]
\[ + 2m \mu_0 \left( \frac{\alpha p}{p - 1} - 1 \right) \int_0^t \int_{B_{2R}} \tau^{p-1} u^{-\frac{\alpha m}{p-1} + mp} dx d\tau \]
\[ + \int_0^t \int_{B_{2R}} \xi^p \tau^{p-1} u^{m \left( 1 - \frac{\alpha m}{p-1} \right)} f(x, t) dx d\tau. \quad (35) \]

where \( M_{19} = \tau^{p-1} \xi^p u^{-\frac{\alpha m}{p-1}} \).

The later choice of \( \varepsilon \) can guarantee that we can choose small \( \varepsilon > 0 \) such that
\[ \frac{\nu_0}{m^{p-1}} \left( \frac{\alpha p}{p - 1} - 1 \right) - \varepsilon > 0, \quad (36) \]
then, noticing that \( 0 \geq f(x, t) \in L^\infty(S_T), \ R \geq 1, \) we have
\[ \int_0^t \int_{B_{2R}} \xi^p \tau^{p-1} u^{-\frac{\alpha m}{p-1}} \nabla u^m dx d\tau \]
\[ \leq c \int_0^t \int_{B_{2R}} \tau^{p-1} u^{m \left( 1 - \frac{\alpha m}{p-1} \right) + 1} dx \]
\[ + \int_0^t \int_{B_{2R}} \tau^{p-1} u^{-\frac{\alpha m}{p-1}} \nabla u^m dx d\tau \]
\[ + \int_0^t \int_{B_{2R}} \tau^{p-1} u^{m \left( 1 - \frac{\alpha m}{p-1} \right) + 1} dx d\tau. \quad (37) \]

At the same time, by Hölder inequality
\[ \int_0^t \int_{B_{2R}} |\nabla u|^p \xi^p \tau^{p-1} dx d\tau \]
\[ = \int_0^t \int_{B_{2R}} |\nabla u|^p \tau^{p-1} \beta u^{\alpha m} dx d\tau \]
\[ \leq \left( \int_0^t \int_{B_{2R}} \xi^p \tau^{p-1} |\nabla u|^m u^{-\frac{\alpha m}{p-1}} dx d\tau \right)^{\frac{p-1}{p}} \]
\[ \left( \int_0^t \int_{B_{2R}} \tau^{-p} u^{\alpha m} dx d\tau \right)^{\frac{1}{p}}. \quad (38) \]

By (37), (38),
\[ \int_0^t \int_{B_{2R}} |\nabla u|^p \xi^p \tau^{p-1} dx d\tau \]
\[ \leq c \int_0^t \int_{B_{2R}} \tau^{p-1} u^{m \left( 1 - \frac{\alpha m}{p-1} \right) + 1} dx \]
\[ + \int_0^t \int_{B_{2R}} \tau^{p-1} u^{-\frac{\alpha m}{p-1}} \nabla u^m dx d\tau \]
\[ + \int_0^t \int_{B_{2R}} \tau^{p-1} u^{m \left( 1 - \frac{\alpha m}{p-1} \right) + 1} dx d\tau \]
\[ \left( \int_0^t \int_{B_{2R}} \tau^{-p} u^{\alpha m} dx d\tau \right)^{\frac{1}{p}}. \quad (39) \]

It is able to make more explicitly estimates to the above inequality by considering the two cases which follows.

1. If \( \frac{1}{mp} > p - 1 \), we choose \( \alpha = \frac{1}{mp}, \beta = \frac{1}{2p} \) in (39). Then
\[ \int_0^t \int_{B_{2R}} |\nabla u|^p \xi^p \tau^{p-1} dx d\tau \]
\[ \leq c \int_0^t \int_{B_{2R}} \tau^{\frac{1}{(p-1)}} u^{mp-\frac{1}{p-1}} dx d\tau \]
\[ + \int_{B_{2R}} t^{\frac{1}{2(p-1)}} u^{m(1-\frac{1}{m(p-1)}+1) dx} + \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}-1} u^{m(1-\frac{1}{m(p-1)}+1) dx dt} \leq \left( \int_0^t \int_{B_{2R}} \tau^{-\frac{1}{2}} u dx dt \right)^{\frac{1}{p}}. \tag{40} \]

Since
\[ \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}-1} u^{m(1-\frac{1}{m(p-1)}+1) dx dt} = \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}-1\frac{1}{2}}(m(1-\frac{1}{m(p-1)}+1)) M_{20} dx dt \leq \left( \int_0^t \int_{B_{2R}} u \tau^{-\frac{1}{2}} dx dt \right)^{\frac{1}{p-1-m}} \left( \int_0^t \int_{B_{2R}} M_{21} dx dt \right)^{\frac{1}{p-1-m}} \leq \left( \int_0^t \int_{B_{2R}} u \tau^{-\frac{1}{2}} dx dt \right)^{\frac{1}{p-1-m}} \left( \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2}} M_{21} dx dt \right)^{\frac{1}{p-1-m}} \leq \left( \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2}} \right)^{\frac{1}{p-1-m}} M_{20}^{\frac{1}{p-1-m}} M_{21}^{\frac{1}{p-1-m}} \]

然后
\[ \left[ \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}-1} u^{m(1-\frac{1}{m(p-1)}+1) dx dt} \right]^{\frac{1}{p}} \leq \left( \int_0^t \int_{B_{2R}} u \tau^{-\frac{1}{2}} dx dt \right)^{\frac{m(p-1)+p-2}{p}} \left[ \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2}} \right]^{\frac{1}{p-1-m}} M_{20}^{\frac{1}{p-1-m}} M_{21}^{\frac{1}{p-1-m}} \leq c \left[ \int_0^t \int_{B_{2R}} u \tau^{-\frac{1}{2}} dx dt \right]^{\frac{m(p-1)+p-2}{p}} R^{\frac{N(1-m(p-1))}{p-1}} t^{\frac{2-(m+1)(p-1)}{2p}} \]

where \( M_{21} = R^{\frac{N(1-m(p-1))}{p-1}} t^{\frac{1}{2p}} \),

\[ \left[ \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}-1} u^{m(1-\frac{1}{m(p-1)}+1) dx dt} \right]^{\frac{1}{p}} \leq \left( \int_0^t \int_{B_{2R}} u \tau^{-\frac{1}{2}} dx dt \right)^{\frac{m(p-1)+p-2}{p}} R^{\frac{N(1-m(p-1))}{p-1}} t^{\frac{1}{2p}} \]

where \( M_{22} = (\tau^{\frac{1}{2(p-1)}-1} u^{m(1-\frac{1}{m(p-1)}+1}) \right)^{\frac{1}{p-1-m}} \]

\[ \left( \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2}} u dx dt \right)^{\frac{1}{p}} \leq c \left[ \int_0^t \int_{B_{2R}} u \tau^{-\frac{1}{2}} dx dt \right]^{\frac{m(p-1)+p-2}{p}} R^{\frac{N(1-m(p-1))}{p-1}} t^{\frac{1}{2p}} \]

we have
\[ \int_0^t \int_{B_{2R}} |\nabla u|^{m(p-1)} \zeta |x|^{p-1} dx dt \leq c \left( \int_0^t \int_{B_{2R}} u^{m(p-1)} \right)^{\frac{1}{p-1-m}} + \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2}} u \left( \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2}} \right)^{\frac{1}{p-1-m}} M_{20}^{\frac{1}{p-1-m}} M_{21}^{\frac{1}{p-1-m}} \leq \left( \int_0^t \int_{B_{2R}} u \tau^{-\frac{1}{2}} dx dt \right)^{\frac{1}{p}} \left( \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2}} \right)^{\frac{1}{p-1-m}} M_{20}^{\frac{1}{p-1-m}} M_{21}^{\frac{1}{p-1-m}} \leq c \left( \int_0^t \int_{B_{2R}} u \tau^{-\frac{1}{2}} dx dt \right)^{\frac{m(p-1)+p-2}{p}} R^{\frac{N(1-m(p-1))}{p-1}} t^{\frac{1}{2p}} \]
\[
\begin{align*}
&\leq cR^{N(1-m(p-1))}t^{2-m(p-1)}
\left(\int_{0}^{t} \int_{B_{2R}} \tau^{-\frac{1}{2}} u dx d\tau\right)^{m(p-1)}
+ cR^\frac{N(m-mp+1)p}{p} t^{2-(m+1)(p-1)}
\left(\int_{0}^{t} \int_{B_{2R}} \tau^{-\frac{1}{2}} u dx d\tau\right)^{\frac{(m+1)(p-1)}{p}}
+ c\left(\int_{B_{2R}} u dx\right)^{\frac{m(p-1)+p-2}{p}} R^{1-m(p-1)} N t^{\frac{1}{2p}}
\left(\int_{0}^{t} \int_{B_{2R}} \tau^{-\frac{1}{2}} u dx d\tau\right)^{\frac{1}{p}}.
\end{align*}
\]

By (34), (41),

\[
\sup_{0<\tau<t} \int_{B_{2R}} u(x, \tau) dx
\leq \int_{B_{2R}} u(x, 0) dx + c \int_{0}^{t} \int_{B_{2R}} |\nabla u|^p \xi^{-1} dx d\tau
+ c \int_{0}^{t} \int_{B_{2R}} u dx d\tau
\leq c(R^{N(1-m(p-1))} t^{2-m(p-1)}
\left(\int_{0}^{t} \int_{B_{2R}} \tau^{-\frac{1}{2}} u dx d\tau\right)^{m(p-1)}
+ cR^\frac{N(m-mp+1)p}{p} t^{2-(m+1)(p-1)}
\left(\int_{0}^{t} \int_{B_{2R}} \tau^{-\frac{1}{2}} u dx d\tau\right)^{\frac{(m+1)(p-1)}{p}}
+ c\left(\int_{B_{2R}} u dx\right)^{\frac{m(p-1)+p-2}{p}} R^{1-m(p-1)} N t^{\frac{1}{2p}}
\left(\int_{0}^{t} \int_{B_{2R}} \tau^{-\frac{1}{2}} u dx d\tau\right)^{\frac{1}{p}}.
\]

\[
\begin{align*}
&\leq c(\varepsilon) R^{N(t^{1-m(p-1)} + t^{p-m(p-1)} + t^{p-m(p-1)})}
\left(\int_{0}^{t} \int_{B_{2R}} \tau^{-\frac{1}{2}} u dx d\tau\right)^{\frac{1}{p}}.
\end{align*}
\]

where \(M_{24} = R^{N(1-m(p-1))} t^{p-m(p-1)}\).

(2) If \(\frac{1}{mp} \leq p-1\), choose \(\alpha = p-1, \beta = \frac{p-1}{2}\) in (39). Then, using the fact \(R \geq 1\),

\[
\begin{align*}
&\int_{0}^{t} \int_{B_{2R}} |\nabla u|^p \xi^{-1} dx d\tau
\leq c\left(\int_{0}^{t} \int_{B_{2R}} t^\frac{p}{2} u^{1-m(p-1)} dx + M_{25}\right)^{\frac{p-1}{p}}
\times \left(\int_{0}^{t} \int_{B_{2R}} u^{mp(p-1)} \tau^{\frac{(p-1)(p-1)}{2}} dx d\tau\right)^{\frac{1}{p}}.
\end{align*}
\]

Since \(\left(\int_{0}^{t} \int_{B_{2R}} t^\frac{p}{2} u^{1-m(p-1)} dx dt\right)^{\frac{p-1}{p}} \leq \left(\int_{0}^{t} \int_{B_{2R}} M_{26} dx d\tau\right)^{\frac{(m+1)(mp-1)}{p}}\)

\[
\times \left(\int_{0}^{t} \int_{B_{2R}} M_{27} dx d\tau\right)^{\frac{m(p-1)^2}{p}}
\leq c\left(\int_{0}^{t} \int_{B_{2R}} u^{\frac{1}{2}} dx d\tau\right)^{\frac{(m+1)(mp-1)}{p}} M_{28},
\]

where

\[
M_{26} = \left(u^{m+1-mp} \tau^{\frac{m+1-mp}{2}}\right)^{\frac{1}{m+1-mp}},
\]

\[
M_{27} = \left(\tau^{\frac{p-2}{2} + \frac{m+1-mp}{2}}\right)^{\frac{1}{m+1-mp}},
\]

\[
M_{28} = t^{\frac{(p-1)(m+1)}{p}} R^{\frac{\alpha}{p}}.
\]

\[
\left(\int_{0}^{t} \int_{B_{2R}} u^{mp(p-1)} \tau^{\frac{(p-1)(p-1)}{2}} dx d\tau\right)^{\frac{1}{p}}
\leq c\left(\int_{0}^{t} \int_{B_{2R}} u^{\frac{1}{2}} dx d\tau\right)^{\frac{(m+1)(mp-1)}{p}} M_{28},
\]
we have
\[
\int_0^t \int_{B_{2R}} |\nabla u|^p |\xi|^{p-1} dxd\tau \leq c\left( \int_0^t \int_{B_{2R}} u^{\frac{1}{p}} dx \right)^{m(p-1)} R \left( \int_{B_{2R}} u dx \right)^{\frac{1}{m} \left(1-m(p-1)\right)} \left( \int_{B_{2R}} u^{\frac{1}{p}} dx \right)^{\frac{1}{m} \left(1-m(p-1)\right)} + c(t) \left( \int_0^t \int_{B_{2R}} u^{\frac{1}{p}} dx \right)^{m(p-1)} R \left( \int_{B_{2R}} u dx \right)^{\frac{1}{m} \left(1-m(p-1)\right)} \left( \int_{B_{2R}} u^{\frac{1}{p}} dx \right)^{\frac{1}{m} \left(1-m(p-1)\right)}.
\]  
(43)
By (34), (43),
\[
\int_{B_{2R}} u(x, t) dx \leq \int_{B_{2R}} u_0 dx + \int_0^t \int_{B_{2R}} |\nabla u|^p |\xi|^{p-1} dxd\tau + c\left( \int_0^t \int_{B_{2R}} u^{\frac{1}{p}} dx \right)^{m(p-1)} R \left( \int_{B_{2R}} u dx \right)^{\frac{1}{m} \left(1-m(p-1)\right)} \left( \int_{B_{2R}} u^{\frac{1}{p}} dx \right)^{\frac{1}{m} \left(1-m(p-1)\right)} + c(t) \left( \int_0^t \int_{B_{2R}} u^{\frac{1}{p}} dx \right)^{m(p-1)} R \left( \int_{B_{2R}} u dx \right)^{\frac{1}{m} \left(1-m(p-1)\right)} \left( \int_{B_{2R}} u^{\frac{1}{p}} dx \right)^{\frac{1}{m} \left(1-m(p-1)\right)}.
\]
\[
\leq \int_{B_{2R}} u(x, 0) dx + \varepsilon \sup_{0<\tau<t} \int_{B_{2R}} u dx \leq \int_{B_{2R}} u(x, 0) dx + \varepsilon \sup_{0<\tau<t} \int_{B_{2R}} u dx + c(\varepsilon) \left[ t^{\frac{1}{1-m(p-1)} \frac{R}{p}} + \frac{1}{t^{\frac{1}{p}} R} \frac{N}{N+1-p} \frac{R^{N}}{R^{(m+1)(p-1)}} \right] + \frac{1}{t^{\frac{1}{p}} R} \frac{N}{N+1-p} \frac{R^{N}}{R^{(m+1)(p-1)}}.
\]
From (42), (44), noticing that $R \geq 1$, we have
\[
\sup_{0<\tau<t} \int_{B_{2R}} u(x, \tau) dx \leq c \int_{B_{2R}} u(x, 0) dx + c R \left[ t^{\frac{1}{1-m(p-1)} \frac{R}{p}} + \frac{1}{t^{\frac{1}{p}} R} \frac{N}{N+1-p} \frac{R^{N}}{R^{(m+1)(p-1)}} \right] + \frac{1}{t^{\frac{1}{p}} R} \frac{N}{N+1-p} \frac{R^{N}}{R^{(m+1)(p-1)}}.
\]
We know that (33) is true.

By Lemma 6, Lemma 7, it is easy to deduce the following Lemma 8, Lemma 9.

**Lemma 8** Let $1 < p < 2$, $m(p-1) < 1$ and $R > R_0 \geq 1$, $f(x, t) \leq 0$. Then the solution $u$ of Cauchy problem (1)-(2) satisfies
\[
\sup_{x \in B_R} |u(x, t)| \leq c(N, p, R_0) t^{-\frac{N}{p}} \left( \int_{B_{4R}} u_0 dx \right)^{\frac{1}{p}} + R \left[ t^{\frac{1}{p}} \frac{N}{N+1-p} \frac{R^{N}}{R^{(m+1)(p-1)}} \right] + \frac{1}{t^{\frac{1}{p}} R} \frac{N}{N+1-p} \frac{R^{N}}{R^{(m+1)(p-1)}}.
\]

**Lemma 9** Let $1 < p < 2$, $m(p-1) < 1$. There exists a constant $c$ such that
\[
\int_0^t \int_{B_{2R}} |\nabla u|^p |\xi|^{p-1} dxd\tau \leq c R^{N(1-m(p-1))} \left( \sup_{0<\tau<t} \int_{B_{2R}} u dx \right)^{m(p-1)} \left[ t^{\frac{1}{p}} R^N \left( \sup_{0<\tau<t} \int_{B_{2R}} u dx \right)^{\frac{1}{p}} \frac{N}{N+1-p} \frac{R^{N}}{R^{(m+1)(p-1)}} \right] + c R^{N(1-m(p-1))} \left( \sup_{0<\tau<t} \int_{B_{2R}} u dx \right)^{m(p-1)} \left[ t^{\frac{1}{p}} R^N \left( \sup_{0<\tau<t} \int_{B_{2R}} u dx \right)^{\frac{1}{p}} \frac{N}{N+1-p} \frac{R^{N}}{R^{(m+1)(p-1)}} \right].
\]
(45)

**Lemma 10** Let $\frac{N(m+1)}{N+1} < p < 2$, $1 < m < \frac{1}{p-1}$. E-ISSN: 2224-2880
Then the solution $u$ of Cauchy problem of equation (1) with the initial value (24) satisfies

$$\nabla u^m \in L^p_{\text{loc}}(S_T).$$

It implies that

$$|\nabla u^m|^{p-2}\nabla u^m \in L^{\frac{p}{p-1}}_{\text{loc}}(S_T). \quad (46)$$

**Proof** Let $\eta$ be the smooth cut function in $B_{4R} \times (\frac{t_1}{8}, t)$ which satisfies $\eta = 1$ on $B_{2R} \times (\frac{t_1}{4}, t)$ and

$$|\nabla \eta| \leq \frac{4}{R}, \quad 0 \leq \eta_t \leq \frac{4}{t_1}.$$

Multiplying (1) by $u^m \eta^p$ and integrating by parts,

$$\int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} \operatorname{div} \tilde{a}(u, \nabla u) u^m \eta^p dxd\tau$$

$$= -\int_{\frac{t_1}{4}}^{t} \int_{B_{2R}} \tilde{a}(u, \nabla u) M_{29} dxd\tau$$

$$m \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} \tilde{a}(u, \nabla u) u^{m-1} \eta^p \nabla u dx d\tau$$

$$\geq m \nu_0 \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} \eta^p u^{m-1} u^{(m-1)(p-1)} \nabla u^p dxd\tau$$

$$-m \mu_0 \int_{\frac{t_1}{4}}^{t} \int_{B_{2R}} u^{m-1} \eta^p (u^{m(p-1)} + 1) dxd\tau$$

$$\geq m^{1-p} \nu_0 \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} \eta^p |\nabla u|^p dxd\tau$$

$$-m^{1-p} \mu_0 \int_{\frac{t_1}{4}}^{t} \int_{B_{2R}} u^{m-1} \eta^p (u^{m(p-1)} + 1) dxd\tau,$$

$$\geq m^{1-p} \nu_0 \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} \eta^p |\nabla u|^p dxd\tau$$

$$-2m \mu_0 \int_{\frac{t_1}{4}}^{t} \int_{B_{2R}} u^{mp} dxd\tau,$$

where $M_{29} = (mu^{m-1} \eta^p \nabla u + p \eta^{p-1} u^m \nabla \eta)$.

Using Young inequality,

$$p \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} u^m \eta^{p-1} \tilde{a}(u, \nabla u) \cdot \nabla \eta dxd\tau$$

$$\leq \int_{\frac{t_1}{4}}^{t} \int_{B_{2R}} u^m \eta^{p-1} M_{30} dxd\tau$$

$$= \frac{p \mu_1}{m^{p-1}} \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} u^m \eta^{p-1} |\nabla \eta| |\nabla u^m|^{p-1} dxd\tau$$

$$+ p \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} u^{m+\frac{(m-1)p}{p}} \eta^{p-1} |\nabla \eta| dxd\tau$$

$$+ \varepsilon \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} \eta^{p} |\nabla u^m|^{p} dxd\tau$$

$$+ \frac{c(\varepsilon)}{R^p} \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} u^{mp} dxd\tau$$

$$+ \frac{c(\varepsilon)}{R^p} \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} u^{mp} dxd\tau$$

$$+ \frac{c}{R} \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} u^{m+1} dxd\tau,$$

where

$$M_{30} = (\mu_1 u^{(m-1)(p-1)} |\nabla u|^{p-1} + u^{m(p-1)}).$$

By the above inequalities, we have

$$\int_{\frac{t_1}{4}}^{t} \int_{B_{2R}} |\nabla u^m|^{p} dxd\tau$$

$$\leq c \sup_{x \in B_{2R}} |u(x, t)|^{mp} R^N,$$

$$\leq c t R^N \left( \int_{B_{2R}} u_0 dx \right)^{\frac{m}{k}}$$

$$+ R^N \left( 1 - \frac{mNp}{k} \right) \left( \int_{B_{2R}} u_0 dx \right)^{\frac{m^2p^2}{k}}$$

$$+ R^{N+1 - \frac{mNp}{k}} \left( \int_{B_{2R}} u_0 dx \right)^{\frac{m^3p^3}{k}}, \quad (48)$$

By Lemma 8, for $R > 1$,

$$c \left( \frac{1}{R^p} + 1 \right) \int_{\frac{t_1}{4}}^{t} \int_{B_{4R}} u^{mp} dxd\tau$$

$$\leq c \sup_{x \in B_{2R}} |u(x, t)|^{mp} R^N,$$

$$\leq c t R^N \left( \int_{B_{2R}} u_0 dx \right)^{\frac{m}{k}}$$

$$+ R^N \left( 1 - \frac{mNp}{k} \right) \left( \int_{B_{2R}} u_0 dx \right)^{\frac{m^2p^2}{k}}$$

$$+ R^{N+1 - \frac{mNp}{k}} \left( \int_{B_{2R}} u_0 dx \right)^{\frac{m^3p^3}{k}}.$$
In this section, we will prove Theorem 4.

3 Proof of Theorem 4

In this section, we will prove Theorem 4.

Lemma 11 If $1 < p \leq \frac{(m+1)N}{mN+1}$, $m(p-1) < 1$, $f(x, t) \leq 0$ and the constant $\alpha$ satisfies

$$1 - \frac{1}{m} < \alpha < \max\{1 + \frac{2}{m} - p, 1 - m(p-1)\},$$

then the solution of Cauchy problem of equation (1) with the initial value (22) has the following properties.

(1) For any given $R > 0$,

$$\int_0^T \int_{B_R} \frac{u^{m(\alpha-1)}}{1 + u^{m\alpha}} |\nabla u^m| \, dx \, dt \leq c,$$

(51)

(2)

$$\int_0^T \int_{B_R} u^{m(p-1)+\frac{1}{p}-\alpha} \, dx \, dt < c.$$

Proof: (1) By Definition 1, for any

$$\psi(x) \in C_0^\infty(\mathbb{R}^N), \varepsilon \in (0, T),$$

we have

$$\int_{\mathbb{R}^N} \int_0^T \frac{u^{m(\alpha-1)}}{1 + u^{m\alpha}} \, dx \, dt \leq c,$$

$$\int_0^T \int_{B_R} \frac{u^{m(\alpha-1)}}{1 + u^{m\alpha}} |\nabla u^m| \, dx \, dt \leq c,$$

$$\int_0^T \int_{B_R} u^{m(p-1)+\frac{1}{p}-\alpha} \, dx \, dt < c.$$

Noticing that

$$\int_\varepsilon^T \int_{\mathbb{R}^N} \frac{u^{m(\alpha-1)}}{1 + u^{m\alpha}} \, dx \, dt \leq c,$$

$$\int_0^T \int_{\mathbb{R}^N} u^{m(p-1)+\frac{1}{p}-\alpha} \, dx \, dt < c.$$
then, we have

\[
\begin{align*}
&\sup_{0<t<T} \int_{\mathbb{R}^N} u(x,t)\psi(x)^p \, dx \\
&+ \int_0^T \int_{\mathbb{R}^N} \frac{u^m (\alpha-1)}{(1 + u^m)2} |\nabla u|^p \psi \, dx \, dt \\
&= c[1 + \int_0^T \int_{\mathbb{R}^N} \frac{u^m (\alpha-1)}{(1 + u^m)2} |\nabla \psi|^p \, dx \, dt \\
&+ \int_0^T \int_{\mathbb{R}^N} \frac{u^m - (1 + u^m)}{(1 + u^m)^2} |\nabla \psi|^p \, dx \, dt \\
&+ \int_0^T \int_{\mathbb{R}^N} \frac{u^m (\alpha-1) + 1}{(1 + u^m)2} |\nabla \psi|^p \, dx \, dt]
\end{align*}
\]

By (50), using Young inequality, we have

\[
\sup_{0<t<T} \int_{\mathbb{R}^N} u(x,t)\psi(x)^p \, dx \\
\leq c \left[ 1 + \int_0^T \int_{\mathbb{R}^N} \frac{u^m (\alpha-1)}{(1 + u^m)2} |\nabla \psi|^p \, dx \, dt \right],
\]

using Young inequality again,

\[
\sup_{0<t<T} \int_{\mathbb{R}^N} u(x,t)\psi(x)^p \, dx \\
\leq c + \epsilon \int_0^T \int_{\mathbb{R}^N} u^p \psi \, dx \, dt \\
+ c(\epsilon) \int_0^T \int_{\mathbb{R}^N} \left( \frac{|\nabla \psi|}{\psi^{1-m(p-1)-\alpha}} \right)^{\frac{p}{1-m(p-1)-\alpha}} \, dx \, dt
\]

\[
\leq c + \epsilon \sup_{0<t<T} \int_{\mathbb{R}^N} u(x,t)\psi(x)^p \, dx \\
+ c(\epsilon) \int_0^T \int_{\mathbb{R}^N} \left( \frac{|\nabla \psi|}{\psi^{1-m(p-1)-\alpha}} \right)^{\frac{p}{1-m(p-1)-\alpha}} \, dx \, dt.
\]

Now, choosing \( \alpha < 1 - m(p-1) \) such that

\[
\frac{p}{1-m(p-1)-\alpha} \geq K \geq N + 1,
\]

where \( K \) is large enough such that

\[
1 - m(p-1) - \frac{p}{K} > 0.
\]

Let \( X \in C^\infty_0(B_{2R}), X \mid_{B_R} = 1, h \geq 2 - m(p-1) - \alpha, \psi = X^R \). By the above formula, we have

\[
\sup_{0<t<T} \int_{\mathbb{R}^N} u(x,t)\psi(x)^p \, dx \leq c. \tag{55}
\]

Combining (54) with (55), we get (51), i.e.

\[
\int_0^T \int_{B_{2R}} \frac{u^m(\alpha-1)}{(1 + u^m)2} |\nabla u|^p \, dx \, dt \leq c.
\]

Now, let

\[
w = u^{\frac{m(p-1-\alpha)}{p}}.
\]

By Sobolev inequality,

\[
\left( \int_{\mathbb{R}^N} |\nabla \psi|^p \, dx \right)^{\frac{1}{p}} \leq c \left( \int_{\mathbb{R}^N} |\nabla \psi|^{p}\, dx \right)^{\frac{\theta}{p}} \tag{56}
\]

where,

\[
\theta = \left( \frac{p-1}{p} - \frac{1}{\gamma} \right) \left( \frac{1}{N-1} - \frac{p-1-\alpha}{p} \right) - 1.
\]

For \( \gamma = \frac{p(p-1+\frac{\alpha}{N-1})}{p-1} \), by (56), we have

\[
\int_0^T \int_{\mathbb{R}^N} \psi^p w^r \, dx \, dt \\
\leq \int_0^T \int_{\mathbb{R}^N} |\nabla (\psi w)|^p \, dx \, dt \\
+ c(\epsilon) \int_0^T \int_{\mathbb{R}^N} \left( \left( \int_{B_{2R}} w^{p-1-\alpha} \, dx \right)^{\frac{\gamma}{N-1}} \tag{57}
\]

Hence, by (55), (51), we have

\[
\int_0^T \int_{\mathbb{R}^N} \psi^p u^m(\alpha-1) \, dx \, dt \\
\leq c \left( 1 + \int_0^T \int_{\mathbb{R}^N} |\nabla \psi|^p u^m(\alpha-1) \, dx \, dt \right).
\]

We can prove (52) in a similar way.

Lemma 12 If \( 1 < p \leq \frac{m+1}{mN+N+1} \), then the solution of Cauchy problem (1)-(22) satisfies

\[
\int \int_{\Omega_T} [u_t - \vec{a}(u, \nabla u) \cdot \nabla \xi + f(x,t)\xi] \, dx \, dt = 0. \tag{57}
\]
where \( \xi \in C_{0}^{\infty}(R^{N} \times (-T, T)) \).

Proof Let
\[
\psi_{k}(x, t) = \eta_{k}(x, t) = \eta_{k}(|x|^{2})\xi(x, t),
\]
where \( \xi \in C_{0}^{\infty}(R^{N} \times (-T, T)), \eta \in c^{\infty}(R) \);
\( \eta(s) = 1 \) when \( s \geq 2; \eta(s) = 0 \) when \( s \leq 1 \).

Let \( \eta_{k}(x) = \eta(k|x|^{2}) \). By the definition of weak solution,
\[
\int_{0}^{T} \int_{R^{N}} [u(\xi_{k})_{t} - \bar{a}(u, \nabla u) \cdot \nabla(\xi_{k}) + f(x, t)\xi_{k}] \, dx \, dt = 0.
\]
To prove the lemma, it is enough to prove that
\[
\lim_{k \to \infty} \int_{S_{T}} \bar{a}(u, \nabla u) \cdot \nabla \eta_{k} \xi \, dx \, dt = 0. \tag{58}
\]
Denoting \( D_{k} = \{ x : k^{-1} < |x|^{2} < 2k^{-1} \} \), clearly mes \( D_{k} \leq ck^{-N} \). Hence, by Hölder inequality and Lemma 11, we have
\[
k^{\frac{1}{2}} \int_{0}^{T} \int_{D_{k}} |\nabla u^{m}|^{p-1} \, dx \, dt
\leq k^{\frac{1}{2}} \left( \int_{0}^{T} \int_{D_{k}} \frac{u^{m(\alpha-1)}}{(1 + u^{\alpha})^{2}} |\nabla u^{m}|^{p} \, dx \, dt \right)^{\frac{p-1}{p}}
\times \left( \int_{0}^{T} \int_{D_{k}} (1 + u^{\alpha})^{2(p-1)} u^{m(1-\alpha)} \, dx \, dt \right)^{\frac{1}{p}}
\leq ck^{\frac{1}{2}} \left( \int_{0}^{T} \int_{D_{k}} u^{m(p-1)(1-\alpha)} \, dx \, dt \right)^{\frac{1}{p}}
\leq c_{1} \left( \int_{0}^{T} \int_{D_{k}} u^{m(p-1) + \frac{p}{2} - \alpha} \, dx \, dt \right)^{\frac{m(p-1)(1-\alpha)}{(m(p-1) + \frac{p}{2} - \alpha)p}}
\times k^{\frac{1}{2}} \frac{1}{2} \frac{p - N\alpha - \alpha N(p-1)}{2p(m(p-1) + \frac{p}{2} - \alpha)} , \tag{59}
\]
where \( u_{1} = \max\{u, 1\} \). Since \( 1 < p \leq \frac{(m+1)N}{mN+1} \), if \( p < \frac{(m+1)N}{mN+1} \), we have
\[
1 - \frac{p - N\alpha - \alpha N(p-1)}{2p(m(p-1) + \frac{p}{2} - \alpha)} < 0.
\]
Thus the right hand side of the inequality (59) tends to zero as \( k \to \infty \). At the same time,
\[
\left| \int_{S_{T}} \bar{a}(u, \nabla u) \nabla \eta_{k} \xi \, dx \, dt \right|
\leq \int_{S_{T}} [\mu_{1}u^{(m-1)(p-1)}|\nabla u|^{p-1} + \mu(u^{m-1}(p-1)}] |\nabla \eta_{k}| \xi \, dx \, dt , \tag{60}
\]
its right hand side also tends to the zero as \( k \to \infty \). By (7), one knows that (58) is true.

If \( p = \frac{(m+1)N}{mN+1} \), (58) is obtained by (41), (43). Thus we get lemma 12.

The proof Theorem 4 Suppose to the contrary that Cauchy problem of equation (1) with the initial value (22) has a solution. Then by lemma 12, we have
\[
\int_{S_{T}} [u_{t} - \bar{a}(u, \nabla u) \cdot \nabla \xi + f \xi] \, dx \, dt = 0, \tag{61}
\]
where \( \xi \in C_{0}^{\infty}(R^{N} \times (-T, T)) \).

Let \( \eta_{h}(t) = 1 - \int_{t-h}^{t} j_{h}(s) \, ds \), where
\( j_{h} \in C_{0}^{1}(-2h, 2h) \), \( j_{h} \geq 0 \), \( \int_{R} j_{h}(s) \, ds = 1 \),
\( \tau \in (0, T) \), \( 2h < T - \tau \).

Clearly, \( \eta_{h} \in C^{\infty}(R) \). If \( t < \tau + h \), \( 0 \leq \eta_{h} \leq 1 \); if \( t < T \), \( \lim_{h \to 0} \eta_{h}(t) = 0 \).

For any \( \forall \chi \in C_{0}^{\infty}(R^{N}) \), we choose \( \xi = \chi(x)\eta_{h}(t) \) in (57), then
\[
- \int_{0}^{T} \int_{R^{N}} j_{h}(t - \tau - 2h) u \chi dx \, dt
- \int_{0}^{T} \int_{R^{N}} \bar{a}(u, \nabla u) \cdot \nabla \chi \eta_{h} - f \xi \, dx \, dt = 0.
\]
Let \( h \to 0^{+} \). We have
\[
\int_{R^{N}} u(x, \tau) \chi(x) \, dx
= - \int_{0}^{T} \int_{R^{N}} \bar{a}(u, \nabla u) \cdot \nabla \chi \eta_{h} - f \xi \, dx \, ds,
\]
which implies that, for \( \forall \chi \in C_{0}^{\infty}(R^{N}) \),
\[
\lim_{\tau \to 0} \int_{R^{N}} u(x, \tau) \chi(x) \, dx = 0.
\]
It contradicts (22). So, there is not the solution for the Cauchy problem of equation (1) with the initial value (22).

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