The Birkhoff weak integral of functions relative to a set function in Banach spaces setting

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Abstract: In this paper, we define and study the Birkhoff weak integral in two cases: for vector functions relative to a non-negative set function and for real functions with respect to a vector set function. Some comparison results and classical integral properties are obtained: the linearity relative to the function and the measure, and the monotonicity with respect to the function, the measure and the set.

Key–Words: Birkhoff weak integral; integrable function; non-additive measure; vector integral; vector function.

1 Introduction

This paper deals with a subject in the field of non-additivity. In Measure Theory, the countable (or finite) additivity is fundamental. However, there are many aspects of the real world where the countable (or finite) additivity does not work. For example, in problems of capacity, the efficiency of a finite set of persons working together is not the sum of the efficiency of every person. The capacities and the Choquet integrals (defined by Choquet [9] in 1953-1954), as well as fuzzy measures and non-linear fuzzy integrals (defined by Sugeno [44] in 1974) have important and interesting applications in potential theory, statistical mechanics, economics, finance, the theory of transferable-utility cooperative games, artificial intelligence, data mining, decision making, computer science, subjective evaluation (e.g., [9], [22], [26], [28], [29], [33], [34], [40], [44], [45]).

By means of finite or infinite Riemann type sums, different kinds of integrals have been defined and studied for instance in [1], [2], [3], [4], [5], [6], [7], [8], [10], [11], [12], [13], [14], [16-24], [25], [27], [31], [32], [36-39], [41], [42], [43]. For example, the Birkhoff integral [2] was defined for a vector function $f : T \to X$, relative to a complete finite measure $m : \mathcal{A} \to [0, +\infty)$, using series of type $\sum\limits_{n=0}^{\infty} f(t_n)m(B_n)$, accordingly to a countable partition $\{B_n\}_{n\in\mathbb{N}}$ of T and $t_n \in B_n$, for every $n \in \mathbb{N}$.

In this paper, we define and study a new non-linear integral of Birkhoff type (named Birkhoff weak) for vector (real respectively) functions, with respect to a non-additive non-negative (vector respectively) set function, using finite Riemann type sums and countable partitions. Our definition can be placed between the Birkhoff integral [2] and the Gould integral [25]. The paper is structured in six sections. The first one is for Introduction. In Section 2, some preliminaries are presented. In Section 3, we define and study the Birkhoff weak integral of vector or real functions and establish some comparison results with Birkhoff inequality and Pettis integrability. Section 4 contains some integral properties for vector functions and Sections 5 includes some integral properties for real functions such as, the linearity relative to the function and the measure and the monotonicity with respect to the function, the measure and the set. In Section 6 we present some properties of monotonicity for the case when both the function and the set function μ have real values. Finally, in Section 7, we expose some conclusions.

2 Preliminaries

Let T be a nonempty set, $\mathcal{P}(T)$ the family of all subsets of T, $\mathcal{A}$ a σ-algebra of subsets of T and $(X, \|\cdot\|)$ a real Banach space.

Definition 2.1 *Let T be an infinite set.*

(i) A countable partition of T is a countable family of nonempty sets $P = \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset, i \neq j$ and $\bigcup_{n\in\mathbb{N}} A_n = T$.

(ii) If P and P' are two countable partitions of T, then P' is said to be finer than P, denoted by $P \leq P'$ (or, $P' \geq P$), if every set of P' is included in some set of P.

(iii) The common refinement of two countable partitions $P = \{A_i\}$ and $P' = \{B_j\}$ is the partition $P \wedge P' = \{A_i \cap B_j\}$. We denote by $\mathcal{P}$ the class of all partitions of T and if $\mathrm{A} \in \mathcal{A}$ is fixed, by $\mathcal{P}_{\mathrm{A}}$ we denote the class of all partitions of A.

Definition 2.2 *[30] Let $\mu : \mathcal{A} \to [0, +\infty)$ be a non-negative function, with $\mu(\emptyset) = 0$. μ is said to be:*

(i) monotone if $\mu(A) \leq \mu(B)$, $\forall A, B \in \mathcal{A}$, with $A \subseteq B$;

(ii) finitely additive if $\mu(A \cup B) = \mu(A) + \mu(B)$ for every disjoint $A, B \in \mathcal{A}$;

(iii) a (σ-additive) measure if $\mu(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} \mu(A_n)$, for every sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$.

Definition 2.3 *[15] Let $\mu : \mathcal{A} \to [0, +\infty)$ be a non-negative set function.*

(i) The variation $\overline{\mu}$ of μ is the set function $\overline{\mu} : \mathcal{P}(T) \to [0, +\infty]$ defined by $\overline{\mu}(E) = \sup\{\sum_{i=1}^{n} \mu(A_i)\}$, for every $E \in \mathcal{P}(T)$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=1}^{n} \subset \mathcal{A}$, with $A_i \subseteq E$, for every $i \in \{1, \ldots, n\}$. If $\mu : \mathcal{A} \to X$ is a vector set function, then in the definition of supremum, we will consider $\sum_{i=1}^{n} \|\mu(A_i)\|$.

(ii) μ is said to be of finite variation on $\mathcal{A}$ if $\overline{\mu}(T) < \infty$.

Remark 2.4 If $E \in \mathcal{A}$, then in Definition 3, we may consider the supremum over all finite partitions $\{A_i\}_{i=1}^{n} \subset \mathcal{A}$, of E.

Definition 2.5 *A property (P) about the points of T holds almost everywhere (denoted μ-ae) if there exists $A \in \mathcal{P}(T)$ so that $\widetilde{\mu}(A) = 0$ and (P) holds on $T \backslash A$.*

Definition 2.6 *[27] Let $\mu : \mathcal{A} \to [0, +\infty)$ be a non-negative set function with $\mu(\emptyset) = 0$. A function $f : T \to X$ is called Riemann-Lebesgue μ-integrable (RL-μ-integrable for short) (on T) if there exists $\alpha \in X$ having the property that for every $\varepsilon > 0$, there exists a countable partition P_ε of T in $\mathcal{A}$ such that for every other countable partition $P = \{A_n\}_{n\in\mathbb{N}}$ of T in $\mathcal{A}$, with $P \geq P_\varepsilon$ and every $t_n \in A_n$, $n \in \mathbb{N}$, the series $\sum_{n=0}^{\infty} f(t_n)\mu(A_n)$ converges unconditionally and $\| \sum_{n=0}^{\infty} f(t_n)\mu(A_n) - \alpha\| < \varepsilon$. In this case, we denote $\alpha = (RL)\int_T f d\mu$, which is called the Riemann-Lebesgue integral of f on T relative to μ.*

Remark 2.7 According to Theorem 8 of [35], if $(T, \mathcal{A}, \mu)$ is a σ-finite measure space, then Riemann-Lebesgue integrability of $f : T \to X$ is equivalent to Birkhoff [2] integrability of f.

3 The Birkhoff weak integral of functions

In this section, we define and study the Birkhoff weak integral of real or vector functions and establish some comparison results.

In the sequel, suppose $(X, \|\cdot\|)$ is a Banach space, T is infinite and $\mathcal{A}$ is a σ-algebra of subsets of T.

Definition 3.1 *Let $f : T \to X$ and $\mu : \mathcal{A} \to [0, +\infty)$ (or $f : T \to \mathbb{R}$ and $\mu : \mathcal{A} \to X$) with $\mu(\emptyset) = 0$.*

I. f is called Birkhoff weak μ-integrable (on T) (simply Bw-μ-integrable) if there exists $\alpha \in X$ so that for every $\varepsilon > 0$, there exist a countable partition P_ε of T in $\mathcal{A}$ and $n_\varepsilon \in \mathbb{N}$ such that for every other countable partition $P = \{A_n\}_{n\in\mathbb{N}}$ of T in $\mathcal{A}$, with $P \geq P_\varepsilon$, and every $t_n \in A_n$, $n \in \mathbb{N}$, it holds

$$\| \sum_{k=0}^{n} f(t_k)\mu(A_k) - \alpha\| < \varepsilon, \forall n \geq n_\varepsilon.$$

In this case, α is denoted by $(Bw)\int_T f d\mu$ and is called the Birkhoff weak integral of f on T relative to μ.

II. f is called Birkhoff weak μ-integrable on a set $E \in \mathcal{A}$ if the restriction $f|_E$ is Birkhoff weak μ-integrable on $(E, \mathcal{A}_E, \mu)$, where $\mathcal{A}_E = \{E \cap A | A \in \mathcal{A}\}$.

Remark 3.2 If it exists, the integral is unique.

Example. I. If $f(t) = 0$, for every $t \in T$, then f is Bw-μ-integrable and $(Bw)\int_T f d\mu = 0$.

II. Suppose $T = \{t_n | n \in \mathbb{N}\}$ is countable, $\{t_n\} \in \mathcal{A}$ and let $f : T \to X$ be such that the series $\sum\limits_{n=0}^{\infty} f(t_n)\mu(\{t_n\})$ is unconditionally convergent. Then f is Bw-μ-integrable and

$$(Bw)\int_T f d\mu = \sum_{n=0}^{\infty} f(t_n)\mu(\{t_n\}).$$

Remark 3.3 I. If we consider the multifunction $F = \{f\}$, where $f : T \to X$ is a vector function, and $\mu : \mathcal{A} \to [0, +\infty)$ with $\mu(\emptyset) = 0$, then by Definition 9 [10], it results that f is Bw-μ-integrable if and only if F is Bw-μ-integrable (according to [10]).

II. If we consider a real function $f : T \to \mathbb{R}$ and the set multifunction $\varphi = \{\mu\}$, where $\mu : \mathcal{A} \to X$ is a vector set function, with $\mu(\emptyset) = 0$, then by Definition 14-II [11], it follows that f is Bw-μ-integrable if and only if f is Bw-φ-integrable (according to [11]).

Theorem 3.4 *Let $f : T \to [0, +\infty)$ be a non-negative function and $\mu : \mathcal{A} \to [0, +\infty)$ a non-negative set function with $\mu(\emptyset) = 0$. If f is Bw-μ-integrable, then f is RL-μ-integrable and $(RL)\int_T f d\mu = (Bw)\int_T f d\mu$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Since f is Bw-μ-integrable, there exist $P_\varepsilon \subset \mathcal{A}$ a countable partition of T and $n_\varepsilon \in \mathbb{N}$ that satisfy the conditions of Definition 3.1. Let $P = \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ be a countable partition of T, with $P \geq P_\varepsilon$ and $t_n \in A_n$, $n \in \mathbb{N}$. Denoting $s_n = \sum\limits_{k=0}^{n} f(t_k)m(A_k)$, for every $n \in \mathbb{N}$, according to Definition 3.1, it holds

$$(*) \qquad \left| s_n - (Bw)\int_T f dm \right| < \frac{\varepsilon}{2}, \ \ \forall n \geq n_\varepsilon.$$

This shows that the series $\sum\limits_{n=0}^{\infty} f(t_n)m(A_n)$ is unconditionally convergent. By (*) it also results that $|\sum\limits_{n=0}^{\infty} f(t_n)m(A_n) - (Bw)\int_T f dm| < \varepsilon$. Consequently, f is RL-μ-integrable and $(RL)\int_T f dm = (Bw)\int_T f dm$. □

Theorem 3.5 *Suppose $(T, \mathcal{A}, \mu)$ is a σ-finite measure space and $\mu : \mathcal{A} \to [0, +\infty)$ is σ-additive. If $f : T \to X$ is Bw-μ-integrable, then f is Pettis integrable.*

Proof. Since f is Bw-μ-integrable, for every $\varepsilon > 0$, there exist a countable partition P_ε of T in $\mathcal{A}$ and $n_\varepsilon \in \mathbb{N}$ such that for every other countable partition $P = \{A_n\}_{n\in\mathbb{N}}$ of T in $\mathcal{A}$, with $P \geq P_\varepsilon$, and every $t_n \in A_n$, $n \in \mathbb{N}$, it holds

$$\| \sum_{k=0}^{n} f(t_k)\mu(A_k) - (Bw)\int_T f d\mu \| < \varepsilon, \forall n \geq n_\varepsilon.$$

Then for every $x^* \in X^*$ and every $n \geq n_\varepsilon$ it follows

$$| \sum_{k=0}^{n} (x^* \circ f)(t_k)\mu(A_k) - x^*((Bw)\int_T f d\mu)| < \varepsilon.$$

This shows that $x^* \circ f$ is Bw-μ-integrable. It also results that the series $\sum_{n=0}^{\infty}(x^* \circ f)(t_n)\mu(A_n)$ is convergent and $|\sum_{n=0}^{\infty}(x^* \circ f)(t_n)\mu(A_n) - x^*((Bw)\int_T f d\mu)| < \varepsilon$.

As in Theorem 3.14 [7], it is demonstrated that the series $\sum_{n=0}^{\infty}(x^* \circ f)(t_n)\mu(A_n)$ is unconditionally convergent. So, $x^* \circ f$ is Birkhoff μ-integrable and therefore is in L^1 for every $x^* \in X^*$. Evidently, $\int_T (x^* \circ f) d\mu = x^*((Bw)\int_T f d\mu)$, for all $x^* \in X^*$. Consequently, f is Pettis integrable. □

4 Integral properties of vector functions

In this section, we present some properties of the Birkhoff weak integral for a vector function f with respect to a non-negative set function $\mu : \mathcal{A} \to [0, +\infty)$ with $\mu(\emptyset) = 0$. According to Remark 3.3-I and the integral properties of the multifunction $F = \{f\}$ in [10], we obtain the following properties of the integral introduced in Definition 3.1.

Theorem 4.1 *If $f : T \to X$ is bounded and $f = 0$ μ-ae, then f is Bw-μ-integrable and $(Bw)\int_T f d\mu = 0$.*

Theorem 4.2 *Let $f : T \to X$ be a Bw-μ-integrable function. Then f is Bw-μ-integrable on $E \in \mathcal{A}$ if and only if $f\mathcal{X}_E$ is Bw-μ-integrable on T, where $\mathcal{X}_E$ is the characteristic function of E. In this case,*

$$(Bw)\int_E f d\mu = (Bw)\int_T f\mathcal{X}_E d\mu.$$

Theorem 4.3 *If $f, g : T \to X$ are Bw-μ-integrable, then $f + g$ is Bw-μ-integrable and*
$(Bw)\int_T (f + g) d\mu = (Bw)\int_T f d\mu + (Bw)\int_T g d\mu.$

Theorem 4.4 *Let $f, g : T \to X$ be Bw-μ-integrable vector functions. Then the following properties hold:*

(i) $\|(Bw)\int_T f d\mu - (Bw)\int_T g d\mu\| \leq \sup\limits_{t\in T} \|f(t) - g(t)\| \cdot \overline{\mu}(T)$;

(ii) $\|(Bw)\int_T f d\mu\| \leq \sup_{t\in T} \|f(t)\| \cdot \overline{\mu}(T)$.

Theorem 4.5 *Let $f : T \to X$ be a Bw-μ-integrable function and $\alpha \in \mathbb{R}$. Then:*

(i) αf is Bw-μ-integrable and

$$(Bw)\int_T \alpha f d\mu = \alpha (Bw)\int_T f d\mu;$$

(ii) f is Bw-$\alpha\mu$-integrable ($\alpha \in [0, +\infty)$) and

$$(Bw)\int_T f d(\alpha\mu) = \alpha (Bw)\int_T f d\mu.$$

Theorem 4.6 *Suppose $\mu_1, \mu_2 : \mathcal{A} \to [0, +\infty)$ are non-negative set functions with $\mu_1(\emptyset) = \mu_2(\emptyset) = 0$. If $f : T \to X$ is both Bw-μ_1-integrable and Bw-μ_2-integrable, then f is Bw-$(\mu_1 + \mu_2)$-integrable and*

$$(Bw)\int_T f d(\mu_1 + \mu_2) = (Bw)\int_T f d\mu_1 + (Bw)\int_T f d\mu_2.$$

Theorem 4.7 *Suppose $\mu : \mathcal{A} \to [0, +\infty)$ is finitely additive. Let $f, g : T \to X$ be vector functions, with $\sup\limits_{t\in T} \|f(t) - g(t)\| < +\infty$, such that f is Bw-μ-integrable and $f = g$ μ-ae. Then g is Bw-μ-integrable and $(Bw)\int_T f d\mu = (Bw)\int_T g d\mu$.*

Other properties of the integral will be presented in the sequel.

Theorem 4.8 *Let $f : T \to X$ be a Bw-μ-integrable function, such that the real function $\|f\| : T \to [0, +\infty)$ is Bw-μ-integrable. Then*

$$\|(Bw)\int_T f d\mu\| \le (Bw)\int_T \|f\| d\mu.$$

Proof. Suppose $\varepsilon > 0$ is arbitrary. Since f is Bw-μ-integrable, there exist $P^1_\varepsilon \in \mathcal{P}$ and $n^1_\varepsilon \in \mathbb{N}$ so that for every $P = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, $P \ge P^1_\varepsilon$, and every $s_n \in A_n$, $n \in \mathbb{N}$, it holds

$$\|\sum_{k=0}^{n} f(s_k)\mu(A_k) - (Bw)\int_T f d\mu\| < \frac{\varepsilon}{2},$$

for every $n \in \mathbb{N}$, $n \ge n^1_\varepsilon$.

Since $\|f\|$ is Bw-μ-integrable, there exist $P^2_\varepsilon \in \mathcal{P}$ and $n^2_\varepsilon \in \mathbb{N}$ so that for every $P = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, $P \ge P^2_\varepsilon$ and every $u_n \in B_n$, $n \in \mathbb{N}$, it holds

$$|\sum_{k=0}^{n} \|f(t_k)\|\mu(B_k) - (Bw)\int_T \|f\| d\mu| < \frac{\varepsilon}{2},$$

for every $n \in \mathbb{N}$, $n \ge n^2_\varepsilon$.

Now, we consider $P_\varepsilon = P^1_\varepsilon \wedge P^2_\varepsilon \in \mathcal{P}$ and $n_\varepsilon = \max\{n^1_\varepsilon, n^2_\varepsilon\}$. Then for every $P = \{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, $P \ge P_\varepsilon$ and every $t_n \in C_n$, $n \in \mathbb{N}$, we obtain the following inequalities

$$\|\sum_{k=0}^{n} f(t_k)\mu(C_k) - (Bw)\int_T f d\mu\| < \frac{\varepsilon}{2}, \quad (1)$$

$$|\sum_{k=0}^{n} \|f(t_k)\|\mu(C_k) - (Bw)\int_T \|f\| d\mu| < \frac{\varepsilon}{2}. \quad (2)$$

By (1) and (2) we get:

$$\|(Bw)\int_T f d\mu\| \le \|(Bw)\int_T f d\mu - \sum_{k=0}^{n} f(t_k)\mu(C_k)\| + \|\sum_{k=0}^{n} f(t_k)\mu(C_k)\|$$
$$< \frac{\varepsilon}{2} + |\sum_{k=0}^{n} \|f(t_k)\|\mu(C_k) - (Bw)\int_T \|f\| d\mu| + $$
$$+ (Bw)\int_T \|f\| d\mu < \varepsilon + (Bw)\int_T \|f\| d\mu.$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows. □

Definition 4.9 *Let $\nu : \mathcal{A} \to X$, $\nu(\emptyset) = 0$ and $\mu : \mathcal{A} \to [0, +\infty)$, $\mu(\emptyset) = 0$. It is said that ν is absolutely continuous with respect to μ (denoted by $\nu \ll \mu$) if for any $\varepsilon > 0$, there is $\delta > 0$ such that for every $E \in \mathcal{A}$,*

$$\overline{\mu}(E) < \delta \Rightarrow \|\nu(E)\| < \varepsilon.$$

Theorem 4.10 *Suppose $f : T \to X$ is bounded and Bw-μ-integrable on every set $E \in \mathcal{A}$, such that $\alpha = \sup_{t\in T} \|f(t)\| > 0$ and let $\nu : \mathcal{A} \to X$ be defined by $\nu(E) = (Bw)\int_E f d\mu$, for every $E \in \mathcal{A}$. Then $\nu \ll \mu$.*

Proof. For every $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{\alpha} > 0$ and let $E \in \mathcal{A}$ be any measurable set so that $\overline{\mu}(E) < \delta$. Then, by Theorem 4.4 - (ii), we have:

$$\|\nu(E)\| = \|(Bw)\int_E f d\mu\| \le \sup_{t\in E} \|f(t)\| \cdot \overline{\mu}(E) < \alpha\delta = \varepsilon.$$

This shows that $\nu \ll \mu$. □

Theorem 4.11 *Let $f : T \to X$ be a vector function.*

(i) If f is both Bw-μ-integrable on B and Bw-μ-integrable on C, when $B, C \in \mathcal{A}$ are disjoint measurable sets, then f is Bw-μ-integrable on $B \cup C$ and $(Bw)\int_{B\cup C} f d\mu = (Bw)\int_B f d\mu + (Bw)\int_C f d\mu$.

(ii) If f is Bw-μ-integrable on every set $E \in \mathcal{A}$, then the set function $\nu : \mathcal{A} \to X$, defined by $\nu(E) = (Bw)\int_E f d\mu, \forall E \in \mathcal{A}$, is finitely additive.

Proof. (i) Suppose $\varepsilon > 0$ is arbitrary. Since f is Bw-μ-integrable on B, there exist a partition of B, $P_B^\varepsilon = \{D_n\}_{n\in\mathbb{N}} \in \mathcal{P}_B$ and $n_\varepsilon^1 \in \mathbb{N}$ such that for every partition of B, $P = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}_B$, $P \geq P_B^\varepsilon$ and $t_n \in A_n$, $n \in \mathbb{N}$, it holds

$$\|\sum_{k=0}^{n} f(t_k)\mu(A_k) - (Bw)\int_B fd\mu\| < \frac{\varepsilon}{2}, \quad \forall n \geq n_\varepsilon^1. \tag{3}$$

Since f is Bw-μ-integrable on C, there exist a partition of C, $P_C^\varepsilon = \{E_n\}_{n\in\mathbb{N}} \in \mathcal{P}_C$ and $n_\varepsilon^2 \in \mathbb{N}$ such that for every partition of C, $P = \{U_n\}_{n\in\mathbb{N}} \in \mathcal{P}_C$, $P \geq P_C^\varepsilon$ and $s_n \in U_n$, $n \in \mathbb{N}$, it holds

$$\|\sum_{k=0}^{n} f(s_k)\mu(U_k) - (Bw)\int_C fd\mu\| < \frac{\varepsilon}{2}, \quad \forall n \geq n_\varepsilon^2. \tag{4}$$

Let $P_{B\cup C}^\varepsilon = \{D_n, E_n\}_{n\in\mathbb{N}} \in \mathcal{P}_{B\cup C}$ and $n \geq n_\varepsilon^1 + n_\varepsilon^2$. Now, for every partition of $B \cup C$, $P = \{V_n\}_{n\in\mathbb{N}} \in \mathcal{P}_{B\cup C'}$ with $P \geq P_{B\cup C}^\varepsilon$ and every $u_n \in V_n$, $n \in \mathbb{N}$, we can write $P = \{B_n'\}_{n\in\mathbb{N}} \cup \{C_n'\}_{n\in\mathbb{N}}$, where $P_B' = \{B_n'\}_{n\in\mathbb{N}} \geq P_B^\varepsilon$ and $P_C' = \{C_n'\}_{n\in\mathbb{N}} \geq P_C^\varepsilon$. Also, we can write $\sum_{k=0}^n f(u_k)\mu(V_k) = \sum_{i=0}^p f(u_k)\mu(B_k') + \sum_{j=0}^r f(u_k)\mu(C_k')$, with $p \geq n_\varepsilon^1$, $r \geq n_\varepsilon^2$. From (3) and (4) we obtain

$$\|\sum_{k=0}^{n} f(u_k)\mu(V_k) - ((Bw)\int_B fd\mu + (Bw)\int_C fd\mu)\| \leq$$
$$\leq \|\sum_{i=0}^{p} f(u_k)\mu(B_k') - (Bw)\int_B fd\mu\|$$
$$+ \|\sum_{j=0}^{r} f(u_k)\mu(C_k') - (Bw)\int_C fd\mu\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows that f is Bw-μ-integrable on $B \cup C$ and

$$(Bw)\int_{B\cup C} fd\mu = (Bw)\int_B fd\mu + (Bw)\int_C fd\mu.$$

(ii) It easily results from (i). □

5 Integral properties of real functions

In this section, we present some properties of the Birkhoff weak integral for a real function with respect to a vector set function $\mu : \mathcal{A} \to X$, with $\mu(\emptyset) = 0$. The nonnegative set function $\|\mu\| : \mathcal{A} \to [0, +\infty)$ is defined by $\|\mu\|(A) = \|\mu(A)\|$, for every $A \in \mathcal{A}$. Evidently, $\|\mu\|(\emptyset) = 0$. According to Remark 3.3-II and the properties of the integral of f with respect to the set multifunction $\varphi = \{\mu\}$ in [11], the following results are obtained.

Theorem 5.1 *If $f : T \to \mathbb{R}$ is bounded and $f = 0$ μ-ae, then f is Bw-μ-integrable and $(Bw)\int_T fd\mu = 0$.*

Theorem 5.2 *$f : T \to \mathbb{R}$ is Bw-μ-integrable on $E \in \mathcal{A}$ if and only if $f\mathcal{X}_E$ is Bw-μ-integrable on T.*

Theorem 5.3 *If $f, g : T \to \mathbb{R}$ are Bw-μ-integrable and $f(t)g(t) \geq 0$ for every $t \in T$, then $f + g$ is Bw-μ-integrable and*

$$(Bw)\int_T (f+g)d\mu = (Bw)\int_T fd\mu + (Bw)\int_T gd\mu.$$

Theorem 5.4 *Let $f, g : T \to \mathbb{R}$ be Bw-μ-integrable bounded functions. Then:*

(i) $\|(Bw)\int_T fd\mu - (Bw)\int_T gd\mu\| \leq \sup_{t\in T}|f(t) - g(t)| \cdot \overline{\mu}(T)$;

(ii) $\|(Bw)\int_T fd\mu\| \leq \sup_{t\in T}|f(t)| \cdot \overline{\mu}(T)$.

Theorem 5.5 *Let $f : T \to \mathbb{R}$ be a Bw-μ-integrable function and $\alpha \in \mathbb{R}$. Then:*

(i) αf is Bw-μ-integrable and

$$(Bw)\int_T \alpha fd\mu = \alpha(Bw)\int_T fd\mu;$$

(ii) f is Bw-$(\alpha\mu)$-integrable and

$$(Bw)\int_T fd(\alpha\mu) = \alpha(Bw)\int_T fd\mu.$$

Theorem 5.6 *Let $\mu_1, \mu_2 : \mathcal{A} \to X$ be non-negative set functions with $\mu_1(\emptyset) = \mu_2(\emptyset) = 0$. If $f : T \to \mathbb{R}$ is both Bw-μ_1-integrable and Bw-μ_2-integrable, then f is Bw-$(\mu_1 + \mu_2)$-integrable and $(Bw)\int_T fd(\mu_1 + \mu_2) = (Bw)\int_T fd\mu_1 + (Bw)\int_T fd\mu_2$.*

Theorem 5.7 *Suppose $\mu : \mathcal{A} \to X$ is finitely additive and let $A, B \in \mathcal{A}$ be disjoint sets. If $f : T \to \mathbb{R}$ is Bw-μ-integrable both on A and on B, then f is Bw-μ-integrable on $A \cup B$ and $(Bw)\int_{A\cup B} fd\mu = (Bw)\int_A fd\mu + (Bw)\int_B fd\mu$.*

Theorem 5.8 *Suppose $\mu : \mathcal{A} \to X$ is finitely additive. If $f, g : T \to \mathbb{R}$ are bounded functions such that f is Bw-μ-integrable and $f = g$ μ-ae, then g is Bw-μ-integrable and*

$$(Bw)\int_T fd\mu = (Bw)\int_T gd\mu.$$

Moreover, the following properties of the integral are obtained.

Theorem 5.9 *Suppose the nonnegative function $f : T \to [0, +\infty)$ is Bw-μ-integrable and Bw-$\|\mu\|$-integrable. Then*

$$\|(Bw)\int_T fd\mu\| \leq (Bw)\int_T fd\|\mu\|.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Since f is Bw-μ-integrable, there exist $P_\varepsilon^1 \in \mathcal{P}$ and $n_\varepsilon^1 \in \mathbb{N}$ such that for every $P = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \geq P_\varepsilon^1$, and each $s_n \in A_n$, $n \in \mathbb{N}$, it holds

$$\|\sum_{k=0}^{n} f(s_k)\mu(A_k) - (Bw)\int_T fd\mu\| < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^1.$$

Since f is Bw-$\|\mu\|$-integrable, there exist $P_\varepsilon^2 \in \mathcal{P}$ and $n_\varepsilon^2 \in \mathbb{N}$ such that for every $P = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $P \geq P_\varepsilon^2$, and each $u_n \in B_n$, $n \in \mathbb{N}$, it holds

$$|\sum_{k=0}^{n} f(u_k)\|\mu(B_k)\| - (Bw)\int_T fd\|\mu\|| < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon^2.$$

Let $P_\varepsilon = P_\varepsilon^1 \wedge P_\varepsilon^2 \in \mathcal{P}$ and $n_\varepsilon = \max\{n_\varepsilon^1, n_\varepsilon^2\}$. Then for every $P = \{C_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, $P \geq P_\varepsilon$, and every $t_n \in C_n$, $n \in \mathbb{N}$, it results

$$\|\sum_{k=0}^{n} f(t_k)\mu(C_k) - (Bw)\int_T fd\mu\| < \frac{\varepsilon}{2}, \quad (5)$$

$$|\sum_{k=0}^{n} f(t_k)\|\mu(C_k)\| - (Bw)\int_T fd\|\mu\|| < \frac{\varepsilon}{2}. \quad (6)$$

Now, by (5) and (6), it follows

$$\begin{aligned}\|(Bw)\int_T fd\mu\| &\leq \|(Bw)\int_T fd\mu - \sum_{k=0}^{n} f(t_k)\mu(C_k)\| \\ &+ \|\sum_{k=0}^{n} f(t_k)\mu(C_k)\| \\ &< \frac{\varepsilon}{2} + |\sum_{k=0}^{n} f(t_k)\|\mu(C_k)\| \\ &- (Bw)\int_T fd\|\mu\|| + (Bw)\int_T fd\|\mu\| \\ &< \varepsilon + (Bw)\int_T fd\|\mu\|, \forall \varepsilon > 0.\end{aligned}$$

Consequently, $\|(Bw)\int_T fd\mu\| \leq (Bw)\int_T fd\|\mu\|$. □

Definition 5.10 *Suppose $\nu : \mathcal{A} \to X$ and $\mu : \mathcal{A} \to X$ are vector set functions such that $\nu(\emptyset) = \mu(\emptyset) = 0$. It is said that ν is absolutely continuous with respect to μ (denoted by $\nu \ll \mu$) if for every $\varepsilon > 0$, there exists $\delta > 0$ so that:*

$$\forall E \in \mathcal{A}, \overline{\mu}(E) < \delta \Rightarrow \|\nu(E)\| < \varepsilon.$$

Theorem 5.11 *Let $f : T \to \mathbb{R}$ be a bounded function such that f is Bw-μ-integrable on every measurable set $E \in \mathcal{A}$ and $\alpha = \sup_{t\in T} |f(t)| > 0$. If we consider the vector set function $\nu : \mathcal{A} \to X$, defined by $\nu(E) = (Bw)\int_E fd\mu$, for every $E \in \mathcal{A}$, then $\nu \ll \mu$.*

Proof. Consider $\delta = \frac{\varepsilon}{\alpha} > 0$ for any $\varepsilon > 0$. Now, according to Theorem 5.4-(ii), it results:

$$\begin{aligned}\|\nu(E)\| = \|(Bw)\int_E fd\mu\| &\leq \sup_{t\in E} |f(t)| \cdot \overline{\mu}(E) \\ &< \alpha\delta = \varepsilon,\end{aligned}$$

for every measurable set $E \in \mathcal{A}$, with $\overline{\mu}(E) < \delta$. This implies $\nu \ll \mu$. □

The following theorem is demonstrated the same as Theorem 4.11.

Theorem 5.12 *Let $f : T \to \mathbb{R}$ be a real function.*

(i) If $B, C \in \mathcal{A}$ are disjoint sets and f is both Bw-μ-integrable on B and Bw-μ-integrable on C, then f is Bw-μ-integrable on $B \cup C$ and the following relation holds:

$$(Bw)\int_{B\cup C} fd\mu = (Bw)\int_B fd\mu + (Bw)\int_C fd\mu.$$

(ii) If f is Bw-μ-integrable on every set $E \in \mathcal{A}$, then the set function $\nu : \mathcal{A} \to X$, defined by $\nu(E) = (Bw)\int_E fd\mu, \forall E \in \mathcal{A}$, is finitely additive.

6 The real case

This section contains some properties of monotonicity for the case when both the function f and the set function μ have real values. These properties show that the integral is monotone with respect to the function, the measure and the set. In the sequel, suppose $\mu : \mathcal{A} \to [0, +\infty)$ is a nonnegative set function, with $\mu(\emptyset) = 0$.

Theorem 6.1 *Let $f, g : T \to \mathbb{R}$ he Bw-μ-integrable functions. Then the following properties hold:*

(i) If $f \leq g$, then $(Bw)\int_T fd\mu \leq (Bw)\int_T gd\mu$.

(ii) If $g \geq 0$, then $(Bw)\int_T gd\mu \geq 0$.

Proof. (i) Consider an arbitrary $\varepsilon > 0$. Since f is Bw-μ-integrable, then there exist $P_1 \in \mathcal{P}$ a countable partition of T and $n_\varepsilon^1 \in \mathbb{N}$ such that for every countable partition of T, $P = \{A_n\}_{n\in\mathbb{N}}$, $P \geq P_1$, and every $t_n \in A_n$, $n \in \mathbb{N}$, it holds

$$|\sum_{k=0}^{n} f(t_k)\mu(A_k) - (Bw)\int_T fd\mu| < \frac{\varepsilon}{2}, \quad (7)$$
$$\forall n \geq n_\varepsilon^1.$$

Analogously, since g is Bw-μ-integrable, there exist $P_2 \in \mathcal{P}$ a countable partition of T and $n_\varepsilon^2 \in \mathbb{N}$ such that for every countable partition of T, $P =$

$\{A_n\}_{n\in\mathbb{N}}$, $P \geq P_2$, and every $t_n \in A_n$, $n \in \mathbb{N}$, it holds

$$|\sum_{k=0}^{n} g(t_k)\mu(A_k) - (Bw)\int_T g d\mu| < \frac{\varepsilon}{2}, \quad (8)$$
$$\forall n \geq n_\varepsilon^2.$$

Now, let $P_0 = P_1 \wedge P_2$ and $n_0 = \max\{n_\varepsilon^1, n_\varepsilon^2\}$, and consider $P = \{A_n\}_{n\in\mathbb{N}}$ a countable partition of T, with $P \geq P_0$, $t_n \in A_n$ for every $n \in \mathbb{N}$ and a fixed $n \geq n_0$. Since $f \leq g$, by (7) and (8), we get

$$\begin{aligned}&(Bw)\int_T f d\mu - (Bw)\int_T g d\mu \\ &\leq |\sum_{k=0}^{n} f(t_k)\mu(A_k) - (Bw)\int_T f d\mu| + \\ &+\sum_{k=0}^{n}[f(t_k) - g(t_k)]\mu(A_k) \\ &+|\sum_{k=0}^{n} g(t_k)\mu(A_k) - (Bw)\int_T g d\mu| < \varepsilon, \ \ \forall \varepsilon > 0.\end{aligned}$$

Consequently, the inequality follows.

(ii) We apply (i) for $f = 0$. □

Theorem 6.2 *Let $\mu_1, \mu_2 : \mathcal{A} \to [0, +\infty)$ be nonnegative set functions such that $\mu_1(\emptyset) = \mu_2(\emptyset) = 0$ and $\mu_1(A) \leq \mu_2(A)$, for every $A \in \mathcal{A}$, and let $f : T \to [0, +\infty)$ be a nonnegative function which is both Bw-μ_1-integrable and Bw-μ_1-integrable. Then $(Bw)\int_T f d\mu_1 \leq (Bw)\int_T f d\mu_2$.*

Proof. Let $\varepsilon > 0$ be arbitrary. From the hypothesis, there exist $P = \{A_n\}_{n\in\mathbb{N}}$ a countable partition of T and $n_\varepsilon \in \mathbb{N}$ such that for every $t_n \in A_n$, $n \in \mathbb{N}$, it holds

$$|\sum_{k=0}^{n} f(t_k)\mu_1(A_k) - (Bw)\int_T f d\mu_1| < \frac{\varepsilon}{2}, \quad (9)$$
$$\forall n \geq n_\varepsilon$$

and

$$|\sum_{k=0}^{n} f(t_k)\mu_2(A_k) - (Bw)\int_T f d\mu_2| < \frac{\varepsilon}{2}, \quad (10)$$
$$\forall n \geq n_\varepsilon.$$

According to (9) and (10), since $\mu_1 \leq \mu_2$, we obtain

$$\begin{aligned}&(Bw)\int_T f d\mu_1 - (Bw)\int_T f d\mu_2 \\ &\leq |\sum_{k=0}^{n} f(t_k)\mu_1(A_k) - (Bw)\int_T f d\mu_1| \\ &+\sum_{k=0}^{n} f(t_k)[\mu_1(A_k) - \mu_2(A_k)] \\ &+|\sum_{k=0}^{n} f(t_k)\mu_2(A_k) - (Bw)\int_T f d\mu_2| < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows. □

Theorem 6.3 *Suppose $\mu : \mathcal{A} \to [0, +\infty)$ is monotone and $f : T \to [0, +\infty)$ is a nonnegative function. Then the following properties hold:*

(i) Let $A, B \in \mathcal{A}$ be measurable sets such that f is both Bw-μ-integrable on A and Bw-μ-integrable on B. If $A \subseteq B$, then $(Bw)\int_A f d\mu \leq (Bw)\int_B f d\mu$.

(ii) Suppose f is Bw-μ-integrable on every set $A \in \mathcal{A}$ and denote $\nu(A) = (Bw)\int_A f d\mu$, for every $A \in \mathcal{A}$. Then ν is monotone.

Proof. Let $\varepsilon > 0$ be arbitrary. Since f is Bw-μ-integrable on A, there exist $P_\varepsilon^1 = \{C_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ a partition of A and $n_\varepsilon^1 \in \mathbb{N}$ such that for each partition of A, $P = \{E_n\}_{n\in\mathbb{N}}$, with $P \geq P_\varepsilon^1$ and $t_n \in E_n$, $n \in \mathbb{N}$, it holds

$$|\sum_{k=0}^{n} f(t_k)\mu(E_k) - (Bw)\int_A f d\mu| < \frac{\varepsilon}{2}, \quad (11)$$
$$\forall n \geq n_\varepsilon^1.$$

Now, similarly in the case of B, there exist $P_\varepsilon^2 = \{D_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ a partition of B and $n_\varepsilon^2 \in \mathbb{N}$ so that for all partitions of B, $P = \{E_n\}_{n\in\mathbb{N}}$, with $P \geq P_\varepsilon^2$ and $t_n \in E_n$, $n \in \mathbb{N}$, it holds

$$|\sum_{k=0}^{n} f(t_k)\mu(E_k) - (Bw)\int_B f d\mu| < \frac{\varepsilon}{2}, \quad (12)$$
$$\forall n \geq n_\varepsilon^2.$$

Let $\widetilde{P}_\varepsilon^1 = \{C_n, B \setminus A\}$ and $P = \{E_n\}_{n\in\mathbb{N}}$ a partition of B, with $P \geq \widetilde{P}_\varepsilon^1 \wedge P_\varepsilon^2$. It results that $P_\varepsilon'' = \{E_n \cap A\}_{n\in\mathbb{N}}$ is a partition of A and $P_\varepsilon'' \geq P_\varepsilon'$. Now, consider $n_\varepsilon = \max\{n_\varepsilon^1, n_\varepsilon^2\}$ and $t_n \in E_n \cap A$,

$n \in \mathbb{N}$. According to (11) and (12) it follows:

$$(Bw)\int_A f d\mu - (Bw)\int_B f d\mu$$
$$\leq |(Bw)\int_A f d\mu - \sum_{k=0}^{n} f(t_k)\mu(E_k \cap A)|$$
$$+\sum_{k=0}^{n} f(t_k)[\mu(E_k \cap A) - \mu(E_k)]$$
$$+|\sum_{k=0}^{n} f(t_k)\mu(E_k) - (Bw)\int_B f\mu| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the inequality results.

(ii) It immediately follows from (i). □

7 Conclusion

We have defined and studied Birkhoff weak integrability of functions f relative to a set function μ. For a real Banach space $(X, \|\cdot\|)$, we have considered two cases:

(i) $f : T \to X$ and $\mu : \mathcal{A} \to [0, +\infty)$ with $\mu(\emptyset) = 0$ and

(ii) $f : T \to \mathbb{R}$ and $\mu : \mathcal{A} \to X$ with $\mu(\emptyset) = 0$.

Some comparison results and classical integral properties are obtained: linearity relative to the function and to the measure, absolutely continuity. Also, monotonicity properties for the real case are presented. As future works:

- relationships between Birkhoff weak integrability and other integrabilities (Gould, Dunford, McShane, Henstock-Kurzweil);

- properties of continuity, regularity and Birkhoff weak integrability on atoms.

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