A Comparison of Approximations with left, right and middle Integro-Differential Polynomial Splines of the Fifth Order

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Abstract: - This paper deals with the construction of integro-differential polynomial splines of the fifth order on a uniform grid of nodes. It is supposed that values of function in nodes and the values of integrals over intervals are known. The properties of the left, the right and the middle integro-differential polynomial splines are investigated. The approximation with these splines is constructed on every grid interval separately. The results of numerical approximation by the left, the right, and the middle integro-differential splines show that the middle splines are preferable. Errors of approximation of the left, the right and the middle integro-differential polynomial splines of one variable of the fifth order are given. The approximation of functions of two variables is constructed using the tensor product. Numerical examples are presented.

Key-Words: - integro-differential polynomial splines, approximation, tensor product

1 Introduction

A variety of splines with different properties are used in calculations in many engineering projects [1,3]. Among them are analysis-suitable T-splines of arbitrary degree, which are useful for modeling cracks in plane problems, and for solving boundary-value problems, cubic, bicubic and biquadratic B-splines, trigonometric, orthogonal splines. These splines are applied to the construction of curves and surfaces, to the designing of ship hulls, to the transformation of a sound signal's frequency and to many others [1–13]. In this paper we discuss the construction of the polynomial splines which use three integrals over subintervals in addition to the values of the function in the nodes. As in previous papers, we construct the approximation separately for each subinterval. As usual, local spline approximation uses values of the approximated function and, sometimes, values of its derivatives.

2 Approximation of the function

Suppose that n, m are natural numbers, while $a,\ b,\ c,\ d, h$ are real numbers, $h=(b-a)/n$. Let function $u(x)$ be such that $u \in C^5[a-3h, b]$.

We have the grid of interpolation nodes $\{x_j\}$ such that $x_{-k} = a - kh,\ k = 3,2,1,\ x_0 = a,$ $x_{j+1} = x_j + h,\ j=0,\dots,n,\ x_n = b.$

Suppose that $u(x_j),\ j = 0,1,\dots,n,$ and

$$\int_{x_{j-1}}^{x_j} u(\xi)d\xi,\ \int_{x_{j-2}}^{x_j} u(\xi)d\xi,\ \int_{x_{j-3}}^{x_j} u(\xi)d\xi\ ,\ j = 0,\dots,n,$$

are known. We denote $\tilde{\tilde{u}}(x)$ as an approximation of the function $u(x)$ in the interval $[x_j, x_{j+1}] \subset [a,b]$:

$$\tilde{\tilde{u}}(x) = u(x_j)w_j(x) + u(x_{j+1})w_{j+1}(x) + \int_{x_{j-1}}^{x_j} u(\xi)d\xi\ w_j^{<-1,0>}(x) + \int_{x_{j-2}}^{x_j} u(\xi)d\xi\ w_j^{<-2,0>}(x) + \int_{x_{j-3}}^{x_j} u(\xi)d\xi\ w_j^{<-3,0>}(x).$$

We obtain basic splines $w_j(x),\ w_{j+1}(x),$ $w_j^{<-1,0>}(x),\ w_j^{<-2,0>}(x),\ w_j^{<-3,0>}(x)$ from the system:

$\tilde{\tilde{u}}(x) \equiv u(x), \quad u(x) = x^{j-1}, \quad i = 1,2,3,4,5.$

If $x = x_j + th, \ t \in [0,1],$ then the basic splines can be written in the form:

$$w_j(x_j + th) = \frac{(1-t)(125t^3 + 577t^2 + 736t + 222)}{222},$$

$$w_{j+1}(x_j + th) = \frac{t(12 + 33t + 24t^2 + 5t^3)}{74},$$

$$w_j^{<-1,0>}(x_j + th) = \frac{t(t-1)(155t^2 + 603t + 516)}{148h},$$

$$w_j^{<-2,0>}(x_j + th) = \frac{t(1-t)(55t^2 + 171t + 90)}{148h},$$

$$w_j^{<-3,0>}(x_j + th) = \frac{t(t-1)(85t^2 + 197t + 92)}{1332h}.$$

We can also construct the approximation in this form:

$$V(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x) + \int_{x_{j-1}}^{x_j} u(\xi)d\xi \, \omega_j^{<-1,0>}(x) + \int_{x_{j-2}}^{x_{j-1}} u(\xi)d\xi \, \omega_j^{<-2,-1>}(x) + \int_{x_{j-3}}^{x_{j-2}} u(\xi)d\xi \, \omega_j^{<-3,-2>}(x), \ x \in [x_j, x_{j+1}]. \quad (1)$$

We obtain basic splines $\omega_{j,0}(x), \ \omega_{j+1,0}(x),$

$\omega_j^{<s,s+1>}(x), \ s = -1,-2,-3$, from the system:

$$V(x) = u(x), \ u(x) = x^{j-1}, \ j = 1,2,3,4,5. \quad (2)$$

If $x = x_j + th, \ t \in [0,1]$, then the basic splines can be written in the following form:

$$\omega_j(x_j + th) = \frac{(1-t)(125t^3 + 577t^2 + 736t + 222)}{222},$$

$$\omega_{j+1}(x_j + th) = \frac{t(12 + 33t + 24t^2 + 5t^3)}{74},$$

$$\omega_j^{<-1,0>}(x_j + th) = \frac{t(t-1)(985t^2 + 4085t + 3926)}{1332h},$$

$$\omega_j^{<-2,-1>}(x_j + th) = -\frac{t(t-1)(205t^2 + 671t + 359)}{666h}.$$

$$\omega_j^{<-3,-2>}(x_j + th) = \frac{t(t-1)(85t^2 + 197t + 92)}{1332h},$$

It can be shown that the next relations are fulfilled:

1)$V(x_j) = u(x_j)$, 2) $V(x_{j+1}) = u(x_{j+1})$,

3) $\int_{x_{j-1}}^{x_j} V(x)dx = \int_{x_{j-1}}^{x_j} u(x)dx,$

4) $\int_{x_{j-2}}^{x_{j-1}} V(x)dx = \int_{x_{j-2}}^{x_{j-1}} u(x)dx,$

5) $\int_{x_{j-3}}^{x_{j-2}} V(x)dx = \int_{x_{j-3}}^{x_{j-2}} u(x)dx.$

Firstly, let us notice that statements 1)-2) follow from the next relations:

$$\omega_j(x_j) = 1, \quad \omega_j(x_{j+1}) = 0,$$
$$\omega_{j+1}(x_j) = 0, \quad \omega_{j+1}(x_{j+1}) = 1,$$
$$\omega_j^{<-1,0>}(x_j) = 0, \quad \omega_j^{<-1,0>}(x_{j+1}) = 0,$$
$$\omega_j^{<-2,-1>}(x_j) = 0, \quad \omega_j^{<-2,-1>}(x_{j+1}) = 0,$$
$$\omega_j^{<-3,-2>}(x_j) = 0, \quad \omega_j^{<-3,-2>}(x_{j+1}) = 0.$$

Similarly, statements 3)-5) follow from the next relations:

$$\int_{x_{j-1}}^{x_j} \omega_j(x)dx = 0, \quad \int_{x_{j-1}}^{x_j} \omega_{j+1,0}(x)dx = 0,$$

$$\int_{x_{j-1}}^{x_j} \omega_j^{<-1,0>}(x)dx = 1, \ \int_{x_{j-1}}^{x_j} \omega_j^{<-2,-1>}dx = 0,$$

$$\int_{x_{j-1}}^{x_j} \omega_j^{<-3,-2>}(x)dx = 0, \ \int_{x_{j-2}}^{x_{j-1}} \omega_j(x)dx = 0,$$

$$\int_{x_{j-2}}^{x_{j-1}} \omega_{j+1,0}(x)dx = 0, \quad \int_{x_{j-2}}^{x_{j-1}} \omega_j^{<-1,0>}(x)dx = 0,$$

$$\int_{x_{j-2}}^{x_{j-1}} \omega_j^{<-2,-1>}(x)dx = 1, \quad \int_{x_{j-2}}^{x_{j-1}} \omega_j^{<-3,-2>}(x)dx = 0,$$

$$\int_{x_{j-3}}^{x_{j-2}} \omega_j(x)dx = 0, \qquad \int_{x_{j-3}}^{x_{j-2}} \omega_{j+1,0}(x)dx = 0,$$

$$\int_{x_{j-3}}^{x_{j-2}} \omega_j^{<-1,0>}(x)dx = 0, \ \int_{x_{j-3}}^{x_{j-2}} \omega_j^{<-2,-1>}(x)dx = 0,$$

$$\int_{x_{j-3}}^{x_{j-2}} \omega_j^{<-3,-2>}(x)dx = 1.$$

Our aim is to determine if $V(x) \overset{\approx}{=} u(x)$.

Lemma 1. Let function $u \in C^5[a-3h,b]$.
The next statement is valid:

$$V(x) \overset{\approx}{=} u(x), \ \ x \in [x_j, x_{j+1}], \ \ j = 0,1,...,n-1.$$

Proof. It can be shown that the next relations are valid:

$$I_1 = w_j^{<-1,0>}\left(x_j + th\right) + w_j^{<-2,0>}\left(x_j + th\right) + w_j^{<-3,0>}\left(x_j + th\right) = \omega_j^{<-1,0>}\left(x_j + th\right),$$

$$I_2 = w_j^{<-2,0>}(x_j + th) + w_j^{<-3,0>}(x_j + th) = \omega_j^{<-2,-1>}(x_j + th),$$

$$I_3 = w_j^{<-3,0>}(x_j + th) = \omega_j^{<-3,-2>}(x_j + th).$$

Therefore, we obtain:

$$\overset{\approx}{u}\left(x_j + th\right) = u\left(x_j\right)\omega_{j,0}\left(x\right) + u\left(x_{j+1}\right)\omega_{j+1,0}\left(x\right) + \int_{x_{j-1}}^{x_j} u(\xi)d\xi \, I_1 + \int_{x_{j-2}}^{x_{j-1}} u(\xi)d\xi \, I_2 + \int_{x_{j-3}}^{x_{j-2}} u(\xi)d\xi \, I_3 =$$

$$= u\left(x_j\right)\omega_{j,0}\left(x\right) + u\left(x_{j+1}\right)\omega_{j+1,0}\left(x\right) + \int_{x_{j-1}}^{x_j} u(\xi)d\xi \, \omega_j^{<-1,0>}\left(x_j + th\right) + \int_{x-2}^{x_{j-1}} u(\xi)d\xi \, \omega_j^{<-2,-1>}\left(x_j + th\right) + \int_{x_{j-3}}^{x_{j-2}} u(\xi)d\xi \, \omega_j^{<-3,-2>}\left(x_j + th\right) = V\left(x_j + th\right).$$

The proof is complete.

Lemma 2. Let the function be such that

$u \in C^5[a-3h,b]$.

The next statement is valid:

$$|V(x) - u(x)| \le K_1 h^5 \left\|u^{(5)}\right\|_{[x_{j-3},x_{j+1}]},$$

$$x \in [x_j, x_{j+1}], \ \ K_1 = 0.085.$$

Proof. Using relations (1), (2) and the Taylor formula with remainder in the integral form

$$u(x) = \sum_{k=0}^{4} \frac{u^{(k)}(x_j)}{k!}(x - x_j)^k + \frac{1}{4!}\int_0^1 u^{(5)}((x-x_j)y + x_j)(x-x_j)^5(1-y)^4 dy.$$

we get $|V(x) - u(x)| \le K_1 h^5 \max\limits_{[x_{j-3},x_{j+1}]} |u^{(5)}|$,

where $K_1 = 0.085$.

Table 1 shows actual and theoretical errors of approximation of functions constructed with formula (1) when $[a,b] = [-1,1]$, $h = 0.1$. Calculations were done in Maple with Digits = 15.

Table 1. Actual and theoretical errors of approximation constructed with formula (1)

$u(x)$	$\max\limits_{[-1,1]}\lvert u-V\rvert$ actual errors	$\max\limits_{[-1,1]}\lvert u-V\rvert$ theoretical errors
$\sin(3x)\cos(5x)$	$0.26 \cdot 10^{-2}$	$0.139 \cdot 10^{-1}$
$x^5/5!$	$0.180 \cdot 10^{-6}$	$0.85 \cdot 10^{-6}$
$1/(1 + 25x^2)$	$0.25 \cdot 10^{-1}$	0.27

Figure 1 shows the errors of approximation of function $\sin(3x)\cos(5x)$ with splines (1) when $h = 0.1$, $a = -1$, $b = 1$.
Figure 2 shows the errors of approximation of function $1/(1 + 25x^2)$ with splines (1) when $h = 0.1$, $a = -1$, $b = 1$.

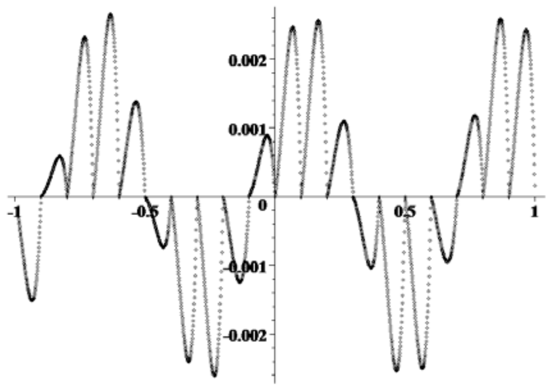

Fig.1. Errors of approximation of function $\sin(3x)\cos(5x)$ with splines (1).

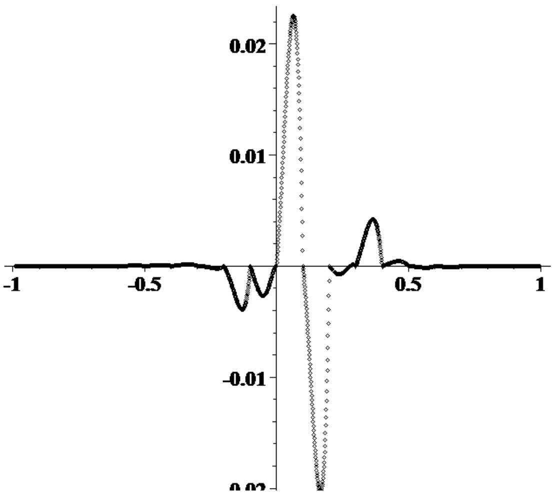

Fig.2. Errors of approximation of function $1/\left(1+25x^2\right)$ with splines (1).

3 Comparison with Lagrange type splines

Suppose we know the values of function $u \in C^5[a-3h,b]$ in the points x_j.

We consider the interpolation with Lagrange type splines

$$W(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x) + u(x_{j-1})\omega_{j-1}(x) + u(x_{j-2})\omega_{j-2}(x) + u(x_{j-3})\omega_{j-3}(x),\ x \in [x_j, x_{j+1}]. \quad (3)$$

It can be found that

$$\omega_{j+1}(x) = (x-x_j)(x-x_{j-1})(x-x_{j-2})\times (x-x_{j-3})/Z_{j+1},$$

$$Z_{j+1} = (x_{j+1}-x_j)(x_{j+1}-x_{j-1})(x_{j+1}-x_{j-2})\times (x_{j+1}-x_{j-3}),$$

$$\omega_j(x) = (x-x_{j+1})(x-x_{j-1})(x-x_{j-2})\times \left(x-x_{j-3}\right)/Z_j,$$

$$Z_j = \left(x_j-x_{j+1}\right)\left(x_j-x_{j-1}\right)\left(x_j-x_{j-2}\right)\times \left(x_j-x_{j-3}\right),$$

$$\omega_{j-1}(x) = \left(x-x_{j+1}\right)\left(x-x_j\right)\left(x-x_{j-2}\right)\times \left(x-x_{j-3}\right)/Z_{j-1},$$

$$Z_{j-1} = \left(x_{j-1}-x_{j+1}\right)\left(x_{j-1}-x_j\right)\left(x_{j-1}-x_{j-2}\right)\times \left(x_{j-1}-x_{j-3}\right),$$

$$\omega_{j-2}(x) = \left(x-x_{j+1}\right)\left(x-x_j\right)\left(x-x_{j-1}\right)\times \left(x-x_{j-3}\right)/Z_{j-2},$$

$$Z_{j-2} = \left(x_{j-2}-x_{j+1}\right)\left(x_{j-2}-x_j\right)\left(x_{j-2}-x_{j-1}\right)\times \left(x_{j-2}-x_{j-3}\right),$$

$$\omega_{j-3}(x) = \left(x-x_{j+1}\right)\left(x-x_j\right)\left(x-x_{j-1}\right)\times \left(x-x_{j-2}\right)/Z_{j-3},$$

$$Z_{j-3} = \left(x_{j-3}-x_{j+1}\right)\left(x_{j-3}-x_j\right)\left(x_{j-3}-x_{j-1}\right)\times \left(x_{j-3}-x_{j-2}\right).$$

Lemma 3. Suppose $u \in C^5[a-3h,b]$. There is a point $\eta \in [x_{j-3}, x_{j+1}]$, such that

$$u(x)-W(x) = \frac{u^{(5)}(\eta)}{5!}(x-x_j)(x-x_{j+1})\times (x-x_{j-1})\left(x-x_{j-2}\right)\left(x-x_{j-3}\right),$$

$$x \in \left[x_j, x_{j+1}\right].$$

Proof. The points $x_{j-i},\ i=-1,0,1,2,3$ are the points of interpolation. Using the formula of the remainder term of Lagrange interpolation we obtain the formula.

Corollary. If $M = \max_{x\in[a-3h,b]} \left|u^{(5)}(x)\right|$ and we put $x = x_j + th,\ t \in [0,1]$, then

$$|W(x_j+th)-u(x_j+th)| \le 3.63\,\frac{Mh^5}{5!}.$$

Proof. Obviously,

$$|W(x_j+th)-u(x_j+th)| \le \frac{Mh^5}{5!}\times t(t-1)(t+1)(t+2)(t+3).$$

It can be obtained, that

$$\max_{t\in[0,1]} |t(t-1)(t+1)(t+2)(t+3)| = 3.63,$$

when $t \approx 0.6444$.

Table 2 shows actual and theoretical errors of approximation of functions constructed with formula (3) when $[a,b] = [-1,1]$, $h = 0.1$. Calculations were done in Maple with Digits = 15

Table 2. Actual and theoretical errors of approximation constructed with formula (3)

$u(x)$	$\max_{[-1,1]}\lvert u-W\rvert$ *actual errors*	$\max_{[-1,1]}\lvert u-W\rvert$ *theoretical errors*
$\sin(3x)\cos(5x)$	$0.45{\cdot}10^{-2}$	$0.50{\cdot}10^{-2}$
$x^5/5!$	$0.30259{\cdot}10^{-6}$	$0.30262{\cdot}10^{-6}$
$1/(1+25x^2)$	$0.34{\cdot}10^{-1}$	$0.95{\cdot}10^{-1}$

Figure 3 shows the errors of approximation of function $\sin(3x)\cos(5x)$ with splines (3) when $h=0.1,\ a=-1,\ b=1.$

Figure 4 shows the errors of approximation of function $1/(1+25x^2)$ with splines (3) when $h=0.1,\ a=-1,\ b=1.$

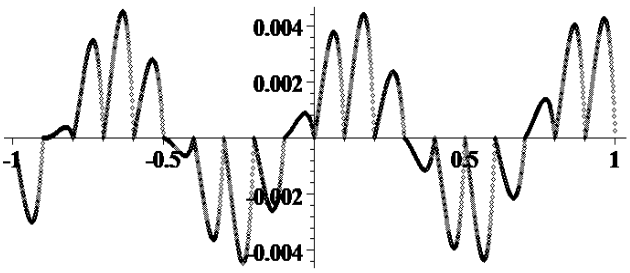

Fig.3. Errors of approximation of function $\sin(3x)\cos(5x)$ with splines (3).

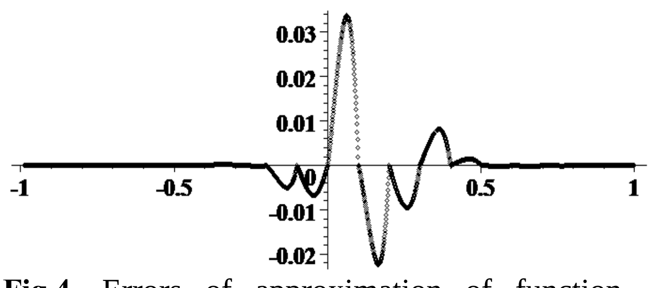

Fig.4. Errors of approximation of function $1/(1+25x^2)$ with splines (3).

4 Other splines. Part 2

Let the function $u(x)$ be such that $u\in C^5[a-2h,b]$. Suppose $h=(b-a)/n.$ Let the grid of interpolation nodes x_j be such that

$$x_{-2}=a-2h,\quad x_{-1}=a-h,\quad x_{j+1}=x_j+h,$$
$$j=0,\dots,n,\quad x_{n+1}=b+h.$$

Suppose that $u(x_j)$, $j=0,\dots,n,$ and $\left(\int_{x_{j-2}}^{x_j}u(x)dx\right)$, $\left(\int_{x_{j-1}}^{x_j}u(x)dx\right)$, $\left(\int_{x_j}^{x_{j+1}}u(x)dx\right)$, $j=0,\dots,n,$ are known. We denote $\overset{\approx}{u}_1(x)$ as an approximation of the function $u(x)$ in the interval $[x_j,x_{j+1}]\subset[a,b]$:

$$\overset{\approx}{u}_1(x)=u(x_j)w_j(x)+u(x_{j+1})w_{j+1}(x)+$$
$$+\int_{x_{j-2}}^{x_j}u(\xi)d\xi\ w_j^{<-2,0>}+\int_{x_{j-1}}^{x_j}u(\xi)d\xi\ w_j^{<-1,0>}(x)+$$
$$+\int_{x_j}^{x_{j+1}}u(\xi)d\xi\ w_j^{<0,1>}(x).$$

We obtain the basic splines $w_j^{<0,1>}(x)$, $w_j(x)$, $w_{j+1}(x)$, $w_j^{<s,0>}(x)$, $s=-2,-1,$ from the system:

$$\overset{\approx}{u}_1(x)=u(x),\ u=x^i,\ i=0,1,2,3,4.$$

If $x\in[x_j,x_{j+1}],\ x=x_j+th,\ t\in[0,1],$ then the basic splines can be written in the form:

$$w_j(x_j+th)=1-t-\frac{9}{2}t^2+2t^3+\frac{5}{2}t^4,$$

$$w_{j+1}(x_j+th)=\frac{t(5t^3-4-3t+8t^2)}{6},$$

$$w_j^{<-2,0>}(x_j+th)=-\frac{t(5t^3+2-3t+4t^2)}{36h},$$

$$w_j^{<-1,0>}(x_j+th)=-\frac{t(4-9t+5t^3)}{4h},$$

$$w_j^{<0,1>}(x_j+th)=-\frac{t(85t^3-92-105t+112t^2)}{36h}.$$

As in the previous section, we can also take the approximation in the form:

$$V_1(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x) + \\ + \int_{x_{j-2}}^{x_{j-1}} u(\xi)d\xi\ \omega_j^{<-2,-1>}(x) + \int_{x_{j-1}}^{x_j} u(\xi)d\xi\ \omega_j^{<-1,0>}(x) + \\ + \int_{x_j}^{x_{j+1}} u(\xi)d\xi\ \omega_j^{<0,1>}(x). \qquad (4)$$

We obtain the basic splines $\omega_j(x), \omega_{j+1}(x)$, $\omega_j^{<s,s+1>}(x),\ s=-2,-1,0,$ then the basic splines can be written in the form:

$$\omega_j(x_j+th) = 1 - t - \frac{9}{2}t^2 + 2t^3 + \frac{5}{2}t^4,$$

$$\omega_{j+1}(x_j+th) = \frac{t(5t^3 - 4 - 3t + 8t^2)}{6},$$

$$\omega_j^{<-2,-1>}(x_j+th) = \frac{t(5t^3 + 2 - 3t - 4t^2)}{36h},$$

$$\omega_j^{<-1,0>}(x_j+th) = -\frac{t(17 - 39t + 2t^2 + 20t^3)}{18h},$$

$$\omega_j^{<0,1>}(x_j+th) = -\frac{t(85t^3 - 92 - 105t + 112t^2)}{36h}.$$

The following Lemma is valid.

Lemma 4. Let function $u \in C^5[a-2h, b]$ The next statement is valid: $V_1(x) \overset{\approx}{=} u(x)$.

Proof.
The proof is similar to the proof of Lemma 1.

Lemma 5. Let the function be such that $u \in C^5[a-2h, b]$.

The next statement is valid:

$$|V_1(x) - u(x)| \le K_2\, h^5 \left\| u^{(5)} \right\|_{[x_{j-2}, x_{j+1}]},$$

$$x \in [x_j, x_{j+1}],\ K_2 = 0.0199.$$

Proof.
The proof is similar to the proof of Lemma 2.

Figure 5 shows the errors of approximation of function $sin(3x)cos(5x)$ with splines (4) when $h = 0.1,\ a = -1,\ b = 1.$

Figure 6 shows the errors of approximation of function $1/(1 + 25x^2)$ with splines (4) when $h = 0.1,\ a = -1,\ b = 1.$

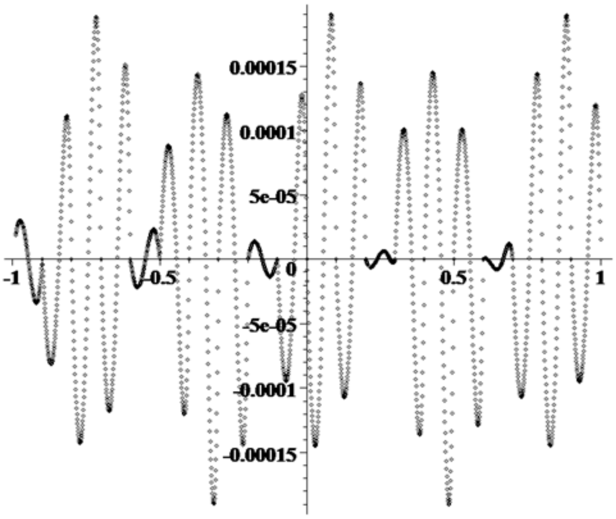

Fig.5. Errors of approximation of function $sin(3x)cos(5x)$ with splines (4).

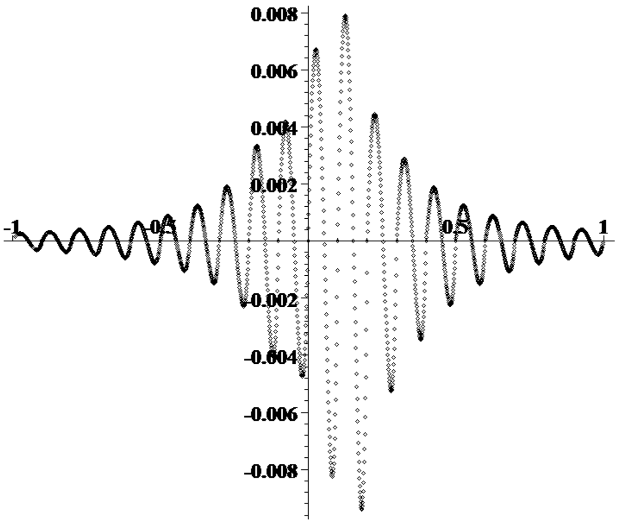

Fig.6. Errors of approximation of function $1/(1 + 25x^2)$ with splines (4).

5 Other splines. Part 3

Let function $u(x)$ be such that $u \in C^5[a-h, b+h]$. We have the grid of interpolation nodes x_j so that

$x_{-1} = a-h,\ x_0 = a,\ x_{j+1} = x_j + h,$

$j = 0,...,n, \quad x_{n+1} = b+h,\ x_{n+2} = b+2h.$

Suppose that $u(x_j),\ j = 0,1,...n,$ and

$\left(\int_{x_{j-1}}^{x_j} u(x)dx\right), \left(\int_{x_j}^{x_{j+1}} u(x)dx\right), \left(\int_{x_j}^{x_{j+2}} u(x)dx\right)$ are known.

We denote $\approx{u}_2(x)$ as an approximation of function $u(x)$ in interval $[x_j, x_{j+1}] \subset [a,b]$:

$$\overset{\approx}{u}_2(x) = u(x_j)w_j(x) + u(x_{j+1})w_{j+1}(x) + \int_{x_{j-1}}^{x_j} u(\xi)d\xi(x)\, w_j^{<-1,0>}(x) + \int_{x_j}^{x_{j+1}} u(\xi)d\xi(x)\, w_j^{<0,1>}(x) + \int_{x_j}^{x_{j+2}} u(\xi)d\xi(x)\, w_j^{<0,2>}(x).$$

We obtain the basic splines $w_j(x),\ w_{j+1}(x),$ $w_j^{<-1,0>}(x), w_j^{<0,s>}(x),\ s = 1,2$ from this system:

$$\overset{\approx}{u}_2(x) = u(x),\ u = x^i,\ i = 0,1,2,3,4.$$

If $x = x_j + th,\ t \in [0,1].$ then the basic splines can be written in the form:

$$w_j(x_j + th) = 1 - 3t - \frac{3}{2}t^2 + 6t^3 - \frac{5}{2}t^4,$$

$$w_{j+1}(x_j + th) = -\frac{t(4 - 3t - 8t^2 + 5t^3)}{2},$$

$$w_j^{<-1,0>}(x_j + th) = \frac{t(-4 + 15t - 16t^2 + 5t^3)}{12h},$$

$$w_j^{<0,1>}(x_j + th) = \frac{t(20 - 3t - 32t^2 + 15t^3)}{4h},$$

$$w_j^{<0,2>}(x_j + th) = \frac{t(2 - 3t - 4t^2 + 5t^3)}{12h}.$$

Similar to the previous section, we can also make an approximation in this form:

$$V_2(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x) + \int_{x_{j-1}}^{x_j} u(\xi)d\xi\ \omega_j^{<-1,0>}(x) + \int_{x_j}^{x_{j+1}} u(\xi)d\xi\ \omega_j^{<0,1>}(x) + \int_{x_{j+1}}^{x_{j+2}} u(\xi)d\xi\ \omega_j^{<1,2>}(x). \qquad (5)$$

We obtain the basic splines $\omega_j(x),\ \omega_{j+1}(x),$ $\omega_j^{<s,s+1>}(x),\ s = -1,0,1$, from this system:

$V_2(x) \equiv u(x),\ u = x^i,\ i = 0,1,2,3,4.$

If $x \in [x_j, x_{j+1}],\ x = x_j + th,\ t \in [0,1],$ then the basic splines can be written in the form:

$$\omega_j(x_j + th) = 1 - 3t - \frac{3}{2}t^2 + 6t^3 - \frac{5}{2}t^4,$$

$$\omega_{j+1}(x_j + th) = -\frac{t(4 - 3t - 8t^2 + 5t^3)}{2},$$

$$\omega_j^{<-1,0>}(x_j + th) = \frac{t(-4 + 15t - 16t^2 + 5t^3)}{12h},$$

$$\omega_j^{<0,1>}(x_j + th) = \frac{t(31 - 6t - 50t^2 + 25t^3)}{6h},$$

$$\omega_j^{<1,2>}(x_j + th) = \frac{t(2 - 3t - 4t^2 + 5t^3)}{12h}.$$

Our aim is to determine if $V_2(x) = \overset{\approx}{u}_2(x)$.

Lemma 6 Let function $u \in C^5[a-h, b+h]$. The next statement is valid: $V_2(x) = \overset{\approx}{u}_2(x)$.

Proof. The proof is similar to the proof of Lemma 1.

Lemma 7. Let the function be such that $u \in C^5[a-h, b+h]$.

The next statement is valid:

$|V_2(x) - u(x)| \le K_3\, h^5 \left\|u^{(5)}\right\|_{[x_{j-1}, x_{j+1}]},$

$x \in [x_j, x_{j+1}],\ K_3 = 0.0175.$

Proof. The proof is similar to the proof of Lemma 2.

Tables 3 and 4 show actual and theoretical errors of approximation of functions constructed with formulas (4), (5) when [*a,b*] = [-1,1], *h* = 0.1. Calculations were done in Maple with *Digits* = 15.

Comparison of the numerical and theoretical approximation errors, given in Tables 1, 2, 3, show that among the integro-differential splines, the middle ones are preferable. In addition, interpolation with of integro-differential splines is preferable to interpolation with Lagrangian splines.

Table 3. Actual errors of approximation functions constructed with formulas (4), (5)

$u(x)$	$\max\limits_{[-1,1]} \lvert u - V_1 \rvert$	$\max\limits_{[-1,1]} \lvert u - V_2 \rvert$
$\sin(3x)\times\cos(5x)$	$0.1902\cdot10^{-3}$	$0.9353\cdot10^{-4}$
$\frac{x^5}{5!}$	$0.1211\cdot10^{-7}$	$0.6026\cdot10^{-8}$
$\frac{1}{(1+25x^2)}$	$0.9393\cdot10^{-2}$	$0.1242\cdot10^{-2}$

Table 4. Theoretical errors of approximation functions constructed with formulas (4), (5)

$u(x)$	$\max\limits_{[-1,1]} \lvert u - V_1 \rvert$	$\max\limits_{[-1,1]} \lvert u - V_2 \rvert$
$\sin(3x)\times\cos(5x)$	$0.31\cdot10^{-2}$	$0.28\cdot10^{-2}$
$\frac{x^5}{5!}$	$0.199\cdot10^{-7}$	$0.175\cdot10^{-6}$
$\frac{1}{(1+25x^2)}$	$0.625\cdot10^{-1}$	$0.55\cdot10^{-1}$

Figure 7 shows the errors of approximation of function $sin(3x)cos(5x)$ with splines (5) when $h = 0.1,\ a = -1,\ b = 1.$

Figure 8 shows the errors of approximation of function $1/(1 + 25x^2)$ with splines (5) when $h = 0.1,\ a = -1,\ b = 1.$

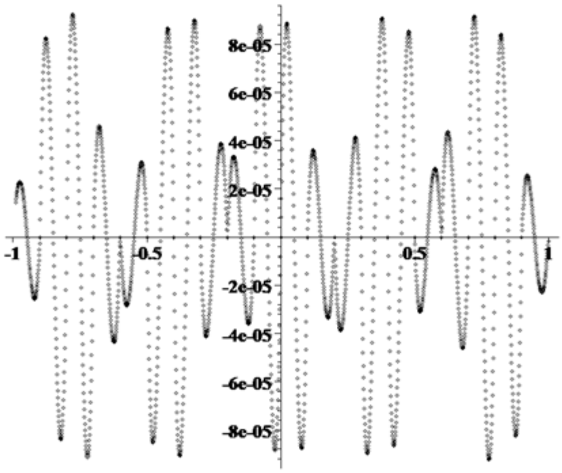

Fig.7. Errors of approximation of function $sin(3x)cos(5x)$ with splines (5).

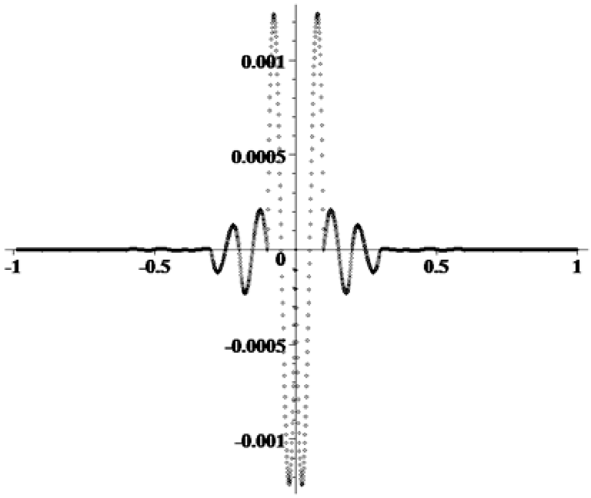

Fig.8. Errors of approximation of function $1/(1+25x^2)$ with splines (5).

6 Approximation functions of two variables

Suppose we construct a set of lines parallel to *x*-axis and *y*-axis which are drawn in a rectangular domain *D* on a plane with a constant step h along the *x*-axis and *y*-axis. For the approximation of a function of two variables we can use the next formula using the tensor product in every elementary rectangular in *D*:

$$V_4(x,y)=\sum_{s=1}^{3}\sum_{i=1}^{3}\int_{y_k}^{y_{k+s}} u(x,y)dxdy\ \omega_j^{<i>}(x)\ \omega_k^{<s>}(y)+$$
$$+\sum_{i=1}^{3}\sum_{s=0}^{3}\int_{x_j}^{x_{j+i}} u(x,y_{k+s})dx\ \omega_{k+s}(y)\ \omega_j^{<i>}(x)+$$
$$+\sum_{s=1}^{3}\sum_{i=0}^{3}\int_{y_k}^{y_{k+s}} u(x_{j+i},y)dy\ \omega_{j+i}(x)\ \omega_k^{<s>}(y)+$$
$$+\sum_{i=0}^{3}\sum_{s=0}^{3} u(x_{j+i},y_{k+s})\ \omega_{k+s}(y)\ \omega_{j+i}(x).$$

Figure 8, 9 show plots of function $u(x,y)=sin(2x-2y)\cdot cos(3x-3y)$ and the error of its approximation constructed with tensor product of splines (1), step $h=0.2$ in $[-1,-1]\times[-1,-1]$.

Figure 10 shows plot of the error of approximation function $u(x,y)=(x-y)^3$, constructed with tensor product of splines (1), step $h=0.2$ in $[-1,-1]\times[-1,-1]$.

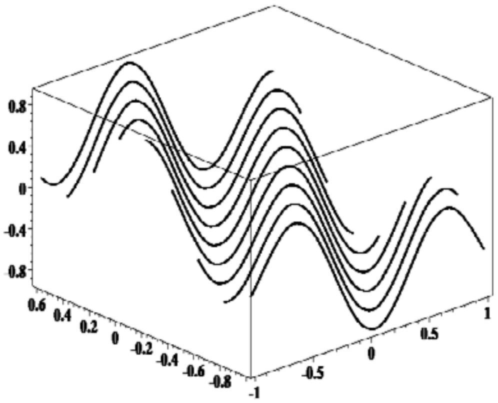

Fig.8. Approximation of function $u(x,y)=sin(2x-2y)\cdot cos(3x-3y)$ with tensor product of splines (1).

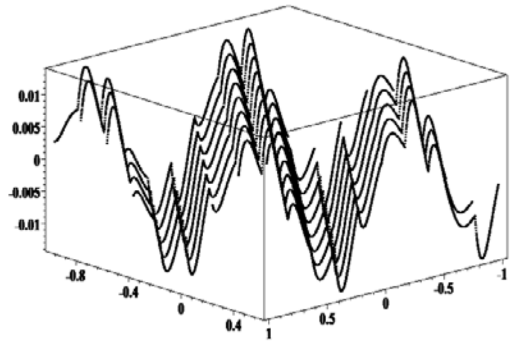

Fig.9. Errors of approximation of function $u(x,y)=sin(2x-2y)\cdot cos(3x-3y)$, $h=0.2$.

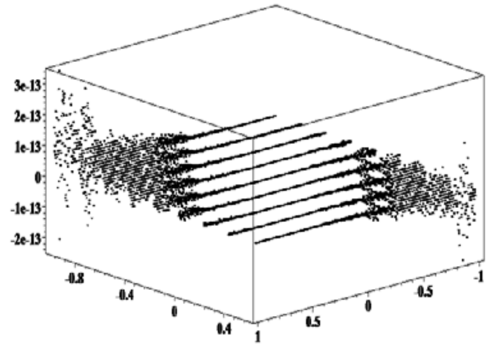

Fig.10. Errors of approximation of function $u(x,y)=(x-y)^3$, $h=0.2$.

7 Conclusion

Here we investigated approximation using the values of integrals of the function over the subintervals immediately to the left of subinterval $\left[x_j,x_{j+1}\right]$, immediately to the right of this subinterval, and immediately to the left and to the right of this subinterval. The results of numerical approximation by the left, the right, and the middle integro-differential splines and theoretical errors of approximation show that the middle integro-differential splines are preferable. In addition, interpolation with of integro-differential splines is preferable to interpolation with Lagrangian splines. If the values of the integral of the function are unknown, we can use quadrature formulae with the fifth order of approximation. For the approximation of a function of two variables we can use a formula using the tensor product in every elementary rectangular.

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