Reliability analysis of hypercube networks and folded hypercube networks

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Abstract: - A network is often modeled by a graph $G=(V,E)$ with the vertices representing nodes such as processors or stations, and the edges representing links between the nodes. One fundamental consideration in the design of networks is reliability. Let G be a connected graph and P be graph-theoretic property. The conditional connectivity $\lambda(G,P)$ or $\kappa(G,P)$ is the minimum cardinality of a set of edges or vertices, if it exists, whose deletion disconnects G and each remaining component has property P. Let F be a vertex set or edge set of G and P be the property of $G-F$ with at least r components. Then we have r-component connectivity $c\kappa_r(G)$ and the r-component edge connectivity $c\lambda_r(G)$. In this paper, we determine the r-component edge connectivity of hypercubes and folded hypercubes.

Key-Words: - Reliability; Conditional connectivity; Cut; Networks; Component; Graph

1 Introduction

A network is often modeled by a graph $G=(V,E)$ with the vertices representing nodes such as processors or stations, and the edges representing links between the nodes. One fundamental consideration in the design of networks is reliability [2,9]. Let $G=(V,E)$ be a connected graph, $N_G(v)$ the neighbors of a vertex v in G (simply $N(v)$), $E(v)$ the edges incident to v. Moreover, for $S\subset V$, $G[S]$ is the subgraph induced by S, $N_G(S)=\cup_{v\in S}N(v)-S, N_G[S]=N_G(S)\cup S$, and $G-S$ denotes the subgraph of G induced by the vertex set of $V\setminus S$. If $u,v\in V$, $d(u,v)$ denotes the length of a shortest (u,v)-path. For $X,Y\subset V$, denote by $[X,Y]$ the set of edges of G with one end in X and the other in Y. For graph-theoretical terminology and notation not defined here we follow [1]. All graphs considered in this paper are simple, finite and undirected.

A r-component cut of G is a set of vertices whose deletion results in a graph with at least r components. r-component connectivity $c\kappa_r(G)$ of G is the size of the smallest r-component cut. The r-component edge connectivity $c\lambda_r(G)$ can be defined correspondingly. We can see that $c\kappa_{r+1}(G)\geq c\kappa_r(G)$ for each positive integer r. The connectivity $\kappa(G)$ is the 2-component connectivity $c\kappa_2(G)$. The r-component (edge) connectivity was introduced in [3] and [11] independently. Fabrega and Fiol introduced extraconnectivity in [5]. Let $F\subseteq V$ be a vertex set, F is called extra-cut, if $G-F$ is not connected and each component of $G-F$ has more than k vertices. The extraconnectivity $\kappa_k(G)$ is the cardinality of the minimum extra-cuts.

The hypercube $Q_n=(V,E)$ with $|V|=2^n$ and $|E|=n2^{n-1}$.Every vertex can be represent by an n-bit binary string. Two vertices are adjacent if and only if their binary string representation differs in only one bit position. The n-dimensional folded hypercube FQ_n is proposed by El-Amawy and Latifi [4]. FQ_n is obtained from Q_n by adding 2^{n-1} edges, called complementary edges, each of them is between vertices

$$x=(x_1,\cdots,x_n) \text{ and } \overline{x}=(\overline{x}_1,\cdots,\overline{x}_n),$$

where $\overline{x}_i=1-x_i$. Obviously, FQ_n is obtained from Q_n by adding a perfect matching M where $M=\{(x,\overline{x}):x\in V(Q_n)\}$. Because Q_n can be

expressed as $Q_{n-1}^0 \odot Q_{n-1}^1$, where Q_{n-1}^0 and Q_{n-1}^1 are two $n-1$-dimensional hypercubes with the prefix 0 and 1 of each vertex, respectively. Furthermore, Q_n can be viewed as $G(Q_{n-1}^0, Q_{n-1}^1, M_0)$, where $M_0 = \{(0u, 1u) : 0u \in V(Q_{n-1}^0), 1u \in V(Q_{n-1}^1)\}$.Similarly, FQ_n can be viewed as $G(Q_{n-1}^0, Q_{n-1}^1, M_0 + \bar{M})$, where
$M_0 = \{(0u, 1u) : 0u \in V(Q_{n-1}^0), 1u \in V(Q_{n-1}^1)\}$
and
$\bar{M} = \{(0u, 1\bar{u}) : 0u \in V(Q_{n-1}^0), 1\bar{u} \in V(Q_{n-1}^1)\}$.

FQ_n is $(n+1)$-regular and $(n+1)$-connected. Moreover, FQ_n is a Cayley graph. It has diameter $\lceil n/2 \rceil$, about a half of the diameter of Q_n [4]. Thus, the folded hypercube FQ_n is an enhancement on the hypercube Q_n.

The fault tolerance analysis of hypercubes and folded hypercubes has recently attracted the attention of many researchers [6,7,10,12,13,17,18, 20,21]. In [8], Hsu et al. determined the r-component connectivity of the hypercube Q_n for $r = 2, 3, \cdots, n+1$.In [19], Zhao et al. determined the r-component connectivity of the hypercube Q_n for $r = n+2, n+3, \cdots, 2n-4$.

In this paper, we obtain that:

(1) $c\kappa_2(FQ_n) = \kappa(FQ_n) = n+1 (n \ge 4)$.

(2) $c\kappa_3(FQ_n) = 2n (n \ge 4)$.

(3) $c\kappa_4(FQ_n) = 3n-2 (n \ge 4)$.

(4) $c\lambda_2(Q_n) = \lambda(Q_n) = n$ for $n \ge 2$.

(5) $c\lambda_3(Q_n) = 2n-1$ for $n \ge 2$.

(6) $c\lambda_4(Q_n) = 3n-2$ for $n \ge 2$.

(7) $c\lambda_2(FQ_n) = \lambda(FQ_n) = n+1$ for $n \ge 3$.

(8) $c\lambda_3(FQ_n) = 2n+1$ for $n \ge 3$.

(9) $c\lambda_4(FQ_n) = 3n+1$ for $n \ge 3$.

2 Main results

For the sake of convenience, we denote the vertex whose i th coordinate of the binary string representation different from v's by v_i. Similarly, v_{ij} is the vertex whose n-bit binary string which differs in the j th position with v_i. Clearly, $v_{ii} = v$.

Lemma 2.1 [18]
Any two vertices of Q_n have exactly two common neighbors for $n \ge 3$ if they have any.

Lemma 2.2 [17]
Any two vertices of FQ_n have exactly two common neighbors for $n \ge 4$ if they have.

Corollary 2.3
For any two vertices $x, y \in V(Q_n)(n \ge 3)$ or $V(FQ_n)(n \ge 4)$,
(1) if $d(x, y) = 2$, then they have exactly two common neighbors;
(2) if $d(x, y) \ne 2$, then they do not have common neighbors.

Lemma 2.4
Let x and y be any two vertices of $V(Q_n)(n \ge 3)$ such that have two common neighbors.
(1) If $x \in V(Q_{n-1}^0), y \in V(Q_{n-1}^1)$, then the one common neighbor is in Q_{n-1}^0, and the other one is in Q_{n-1}^1.
(2) If $x, y \in V(Q_{n-1}^0)$ or $V(Q_{n-1}^1)$, then the two common neighbors are in Q_{n-1}^0 or Q_{n-1}^1.

Proof
(1) Let $x = 0u$ and $y = 1u_i$. Then x,y have two common neighbors $1u, 0u_i$. According to Lemma 2.1, the result holds.
(2) Let $x = 0u$ and $y = 0v$. Since they have two common neighbors, we assume that they are $0u_i, 0u_j$. And $0u_{ij}$ has two neighbors $0u_i, 0u_j$. According to Lemma 2.1, $y = 0v = 0u_{ij}$.

Analogue to Lemma 2.4, we have

Lemma 2.5
For any two vertices $x, y \in V(FQ_n)(n \ge 4)$, $FQ_n = G(Q_{n-1}^0, Q_{n-1}^1, M_0 + \bar{M})$, and x and y have two common neighbors.
(1) If $x \in V(Q_{n-1}^0), y \in V(Q_{n-1}^1)$, then one of the common neighbors is in Q_{n-1}^0, and the other one is in Q_{n-1}^1.
(2) If $x, y \in V(Q_{n-1}^0)$ or $V(Q_{n-1}^1)$, then both of the common neighbors are in Q_{n-1}^0 or Q_{n-1}^1.
The following results are about the extraconnectivit-

y of FQ_n, and we will use them in the following proof.

Lemma 2.6
(1) $\kappa_0(FQ_n)=\kappa(FQ_n)=n+1(n\geq 2)$. [4]
(2) $\kappa_1(FQ_n)=2n(n\geq 4), \kappa_2(FQ_n)=3n-2(n\geq 8)$. [12,17]

Lemma 2.7 [8] $c\kappa_{k+1}(Q_n)=kn-k(k+1)/2+1$, for $n\geq 2, 1\leq k\leq n$.

Lemma 2.8
(1) Let $u\in Q_n(n\geq 3)$. $\kappa(Q_n-N[u])=n-2$. [16]
(2)Let $u,v\in Q_n(n\geq 3), uv\in E$. Then we have

$$\kappa(Q_n-N[u,v])=n-2 .[15]$$

Theorem 2.9
$c\kappa_2(FQ_n)=\kappa(FQ_n)=n+1(n\geq 4)$.

Theorem 2.10 $c\kappa_3(FQ_n)=2n(n\geq 4)$.

Proof We choose two nonadjacent vertices x,y in a cycle C_4. Then $FQ_n-N(\{x,y\})$ has at least 3 connected components and $|N(\{x,y\})|=2n$. That is $c\kappa_3(FQ_n)\leq 2n$.

We will show $c\kappa_3(FQ_n)\geq 2n$. It is easy to check that it is true for $n=4$. So we suppose $n\geq 5$.

By contradiction. Let $F\subseteq V(FQ_n)$, with $|F|\leq 2n-1$. And FQ_n-F has at least 3 connected components, say, G_1, G_2 and G_3.

If FQ_n-F has at least 2 isolated vertices, then $|F|\geq 2n$, a contradiction. Hence FQ_n-F has at most one isolated vertex.

If each component of FQ_n-F has at least 2 vertices, then it contradicts to $\kappa_1(FQ_n)=2n$. Therefore, FQ_n-F has only one isolated vertex x. Because $FQ_n=G(Q_{n-1}^0, Q_{n-1}^1, M_0+\bar{M})$, we have $|F\cap V(Q_{n-1}^0)|\leq n-1$ or $|F\cap V(Q_{n-1}^1)|\leq n-1$. Without loss of generality, we set

$$|F\cap V(Q_{n-1}^0)|\leq n-1.$$

Case 1. Q_{n-1}^0-F is not connected.

Firstly, we assume that $x\in V(Q_{n-1}^1)$. Because $|F\cap V(Q_{n-1}^0)|\leq n-1, |N_{Q_{n-1}^1}(x)|=\kappa(Q_{n-1})=n-1$, we have $|F\cap V(Q_{n-1}^0)|=n-1$. By Lemma 2.8, $Q_{n-1}^1-(F\cup\{x\})$ is connected. Since $|F|\leq 2n-1$, we need delete the last one vertex z in $Q_{n-1}^1-N_{Q_{n-1}^1}[x]$. For any $u\in V(Q_{n-1}^0)-F$, u has at least one neighbor in $Q_{n-1}^1-(F\cup\{x\})$ or is connected to $Q_{n-1}^1-(F\cup\{x\})$ via $N_{Q_{n-1}^0}(u)$ according to Lemma 2.5. Then FQ_n-F has only two components, a contradiction.

Hence $x\in V(Q_{n-1}^0)$. So Q_{n-1}^0-F has only two components. For any $u\in V(Q_{n-1}^1)-F$, u and x have at most one common neighbor in Q_{n-1}^0 by Lemma 2.5. B ut u has two neighbors in Q_{n-1}^0, furthermore u has at least one neighbor in $V(Q_{n-1}^0)-F$. Then FQ_n-F has only two components, a contradiction.

Case 2. Q_{n-1}^0-F is connected.

Then $x\in V(Q_{n-1}^1)-F$. If there is a neighbor in Q_{n-1}^0-F for any $y\in V(Q_{n-1}^1)-(F\cup\{x\})$, then FQ_n-F has only two components, a contradiction.

We assume that there is a vertex $y\in V(Q_{n-1}^1)-(F\cup\{x\})$ such that there exits no neighbor in Q_{n-1}^0-F. There must be a neighbor of y in Q_{n-1}^1-F because of $|F|\leq 2n-1$. Since $|N_{FQ_n}(x)|=n+1$ and $|N_{Q_{n-1}^0}(y)|=2$, we need delete at most $n-4$ vertices in

$$FQ_n-N_{FQ_n}(x)-N_{Q_{n-1}^0}(y).$$

Whether x and y have common neighbors in Q_{n-1}^1 or not, y has at least $n-4$ neighbors in $FQ_n-N_{FQ_n}(x)-N_{Q_{n-1}^0}(y)$. And these neighbors are in Q_{n-1}^1. Note that each vertex of Q_{n-1}^1 has two neighbors in Q_{n-1}^0. According to Pigeonhole principle, y is connected to Q_{n-1}^0-F. Hence FQ_n-F has only two components, a contradiction.

Lemma 2.11 [12]
$\kappa_1(Q_n)=2n-2(n\geq 3)$.

Lemma 2.12 [14]

Let FQ_n be a folded hypercube with $n \geq 8$, and $F \subseteq V(FQ_n)$ with $|F| \leq 3n-3$, then there is a connected component C in $FQ_n - F$ such that $|V(C)| \geq 2^n - |F| - 2$.

Theorem 2.13

$c\kappa_4(FQ_n) = 3n-2 (n \geq 4)$.

Proof

We choose a Q_3 and two 4-cycles, say C_1, C_2, of Q_3. Take two nonadjacent vertices x, y in C_1, and take a vertex z in C_2 such that $d(y,z)=2$ (see Fig.1).

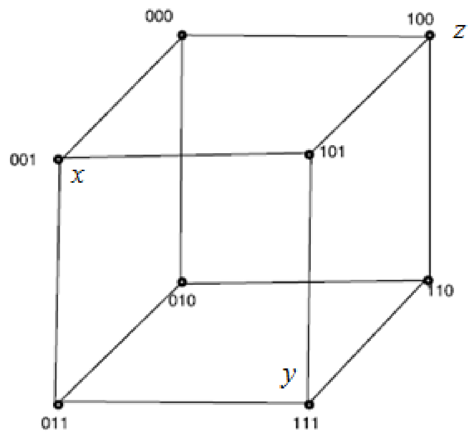

Fig.1

Then $|N(\{x,y,z\})| = 3n-2$ and $FQ_n - N(\{x, y,z\})$ has at least 4 components. Hence $c\kappa_4(FQ_n) \leq 3n-2$.

We will show $c\kappa_4(FQ_n) \geq 3n-2$. It is easy to check that holds for $n = 4,5$. So we suppose $n \geq 6$.

By contradiction. Let $F \subseteq V(FQ_n)$, with $|F| \leq 3n-3$. If $n \geq 8$, then by Lemma 2.12, $FQ_n - F$ has at most 3 connected components, a contradiction. We need show $c\kappa_4(FQ_n) \geq 3n-2$ for $n = 6,7$.

Suppose $n = 6$, we will show $c\kappa_4(FQ_6) \geq 16$. By contradiction. Let $F \subseteq V(FQ_n)$, with $|F| \leq 15$.

Because $FQ_6 = G(Q_5^0, Q_5^1, M_0 + \overline{M})$, we have $|F \cap V(Q_5^0)| \leq 7$ or $|F \cap V(Q_5^1)| \leq 7$. Without loss of generality, we set $|F \cap V(Q_5^0)| \leq 7$. And $FQ_6 - F$ has at most two isolated vertices.

Case 1. $FQ_6 - F$ has two isolated vertices x, y. Then at most one of x and y is in Q_5^0.

Subcase 1.1. $d(x,y) \neq 2$.

Hence $N(x) \cap N(y) = \varnothing$, $N(x) \cup N(y) \subseteq F$ and $|N(x)| + |N(y)| = 14$.

If x is in Q_5^0, and y is in Q_5^1, then $N_{Q_5^0}(x) \subseteq F, |N_{Q_5^0}(x)| = 5$. Note that

$$|N_{Q_5^0}(x)| + |N_{Q_5^0}(y)| = 7$$

and

$$|F \cap V(Q_5^0)| \leq 7.$$

Then $F \cap V(Q_5^0) = N_{Q_5^0}(x) \cup N_{Q_5^0}(y)$. According to Lemma 2.8, $\kappa(Q_5^0 - N_{Q_5^0}(x) - x) = 3$, hence $Q_5^0 - F - x$ is connected. Furthermore, for any $z \in Q_5^1 - (F \cup \{y\})$, z has at least one neighbor in $Q_5^0 - (F \cup \{x\})$ by Lemma 2.5. Therefore, $FQ_6 - F$ has at most three connected components, a contradiction.

If x and y are in Q_5^1, then

$$|V(Q_5^0) \cap F| \geq |N_{Q_5^0}(x)| + |N_{Q_5^0}(y)| = 4,$$

$$|V(Q_5^1) \cap F| \geq |N_{Q_5^1}(x)| + |N_{Q_5^1}(y)| = 10.$$

Since $c\kappa_3(Q_5) = 8 > 5$ by Lemma 2.7, $Q_5^0 - F$ has at most two components. For any $z \in Q_5^1 - F$, z has at least one neighbor in $Q_5^0 - F$ by Lemma 2.5. Then $FQ_6 - F$ has at most three connected components, a contradiction.

Subcase 1.2. $d(x,y) = 2$.

It is similar to that of Subcase 1.1, for any $z \in Q_5^1 - F$, z has at least one neighbor in $Q_5^0 - F$ or can be connected to $Q_5^0 - F$ by a path.

Case 2. $FQ_6 - F$ has only one isolated vertex x.

Subcase 2.1. $x \in V(Q_5^0)$.

Because $|N_{Q_5^0}(x)|=5, |V(Q_5^0)\cap F|\le 7$,according to Lemma 2.7, Q_5^0-F has only two components. At most one of vertex, say y, of Q_5^1-F does not have neighbors in Q_5^0-F. And y has a neighbor z in Q_5^1-F. There is at least one neighbor of z in Q_5^0-F by Lemma 2.5. Hence y is connected to Q_5^0-F. Then FQ_6-F has at most three connected components, a contradiction.

Subcase 2.2. $x \in V(Q_5^1)$.

Since $\kappa_1(Q_5)=8$ by Lemma 2.11, we can obtain that Q_5^0-F is connected or Q_5^0-F has an isolated vertex y and y has neighbors in FQ_6-F (that is, y is the isolated vertex of Q_5^0-F but not FQ_6-F).

We assume that Q_5^0-F is connected. We will show that for any $u\in Q_5^1-(F\cup\{x\})$, u is connected to Q_5^0-F. By contradiction. There is a vertex $u\in Q_5^1-(F\cup\{x\})$, u is not connected to Q_5^0-F. Then $N_{Q_5^0}(u)\subseteq F$. And u has a neighbor v in Q_5^1-F , v has no n eighbors in Q_5^0-F . Hence $N_{Q_5^0}(v)\subseteq F$.

If $FQ_6[\{u,v\}]$ is a connected component of FQ_6-F, then

$$N_{Q_5^1}(\{u,v\})\subseteq F, |N_{Q_5^1}(\{u,v\})|=8,$$
$$|N_{Q_5^0}(u)|=|N_{Q_5^0}(v)|=|N_{Q_5^0}(x)|=2$$

and $N_{Q_5^0}(u), N_{Q_5^0}(v), N_{Q_5^0}(x)$ are pairwise disjoint. Note that $|F|\le 15$. For any

$$w\in Q_5^1-F-N_{Q_5^1}(\{u,v\})-\{u,v,x\},$$

w has a neighbor in Q_5^0-F. Then FQ_6-F has at most three connected components, a contradiction.

Suppose that u has another neighbor, say w, different from v in Q_5^1-F . Because of $|F\cap V(Q_5^0)|\le 7$, w has a neighbor in Q_5^0-F. Then FQ_6-F has at most three connected components, a contradiction. If v has another neighbor, say w', different from u in Q_5^1-F, then it is similar to the front of the above case. We have a contradiction.

Now we assume that Q_5^0-F has an isolated vertex y and y has neighbors in Q_5^1-F. And $Q_5^0-(F\cup\{y\})$ is connected. Because

$$N_{Q_5^0}(x)\subseteq F, N_{Q_5^0}(y)\subseteq F,$$
$$|N_{Q_5^0}(y)|=5, |N_{Q_5^0}(x)|=2,$$
$$|F\cap V(Q_5^0)|\le 7.$$

For any $w\in Q_5^1-(F\cup\{x\})$, as the above discussion, w is connected to Q_5^0-F . Then FQ_6-F has at most three connected components, a contradiction.

Case 3. FQ_6-F has no isolated vertices.

Since $\kappa_1(Q_5)=8$ by Lemma 2.11 a nd $|F\cap V(Q_5^0)|\le 7$, we can obtain that Q_5^0-F is connected or Q_5^0-F has an isolated vertex, say y, such that y has neighbors in Q_5^1-F (that is, y is the isolated vertex of Q_5^0-F but not FQ_6-F).

Subcase 3.1. Q_5^0-F is connected.

We will show that for any $u\in Q_5^1-F$, u is connected to Q_5^0-F. By contradiction, we assume that there is a v ertex $u\in Q_5^1-F$, u is not connected to Q_5^0-F. Then $N_{Q_5^0}(u)\subseteq F$. And u has a neighbor v in Q_5^1-F, v has no neighbors in Q_5^0-F. Hence $N_{Q_5^0}(v)\subseteq F$.

If $FQ_6[\{u,v\}]$ is a connected component of FQ_6-F, then

$$N_{Q_5^1}(\{u,v\})\subseteq F,$$
$$|N_{Q_5^1}(\{u,v\})|=8,$$
$$|N_{Q_5^0}(u)|=|N_{Q_5^0}(v)|=2,$$

and $N_{Q_5^0}(u), N_{Q_5^0}(v)$ are disjoint.

Suppose that there is a vertex

$$w \in Q_5^1 - F - N_{Q_5^1}(\{u,v\}) - \{u,v\},$$

w is not connected to $Q_5^0 - F$. Then $N_{Q_5^0}(w) \subseteq F$. According to Lemma 2.5, u,v,w do not have common neighbors in Q_5^0. Because

$$|N_{Q_5^0}(u)| + |N_{Q_5^0}(v)| + |N_{Q_5^0}(w)| = 6,$$

$$|F \cap V(Q_5^0)| \le 7,$$

and w has a neighbor w_1 in

$$Q_5^1 - F - N_{Q_5^1}(\{u,v\}) - \{u,v\},$$

w_1 has a neighbor in $Q_5^0 - F$. Then $FQ_6 - F$ has at most three connected components, a contradiction.

Hence for any $w \in Q_5^1 - F - N_{Q_5^1}(\{u,v\}) - \{u,v\}$, w is connected to $Q_5^0 - F$. We obtain a contradiction.

Suppose that u has another neighbor w different from v in $Q_5^1 - F$. Then $N_{Q_5^0}(w) \subseteq F$. And

$$|N_{Q_5^0}(u)| + |N_{Q_5^0}(v)| + |N_{Q_5^0}(w)| = 6,$$

$$|F \cap V(Q_5^0)| \le 7.$$

For any $z \in Q_5^1 - F - N_{Q_5^1}(\{u,v,w\}) - \{u,v,w\}$, z is connected to $Q_5^0 - F$. We also obtain a contradiction. If v has another neighbor w' different from u in $Q_5^1 - F$, then it is similar to the front of the above case. We have a contradiction.

Subcase 3.2. $Q_5^0 - F$ has an isolated vertex y and y has neighbors in $Q_5^1 - F$ (that is, y is the isolated vertex of $Q_5^0 - F$ but not $FQ_6 - F$).

The proof is similar to that of Subcase 2.2, we get a contradiction.

For $n = 7$, we can show $c\kappa_4(FQ_7) = 19$ using the similar method.

Theorem 2.14
$c\lambda_2(Q_n) = \lambda(Q_n) = n$ for $n \ge 2$.

Theorem 2.15
$c\lambda_3(Q_n) = 2n-1$ for $n \ge 2$.
Proof

Take an edge $e = uv$, then $|E(u) \cup E(v)| = 2n-1$. And $Q_n - E(u) - E(v)$ has at least 3 connected components. That is $c\lambda_3(Q_n) \le 2n-1$.

Next we will show that $c\lambda_3(Q_n) \ge 2n-1$ by induction. It is easy to check it is true for $n = 2,3,4$. So we suppose $n \ge 5$ and assume it is true for all $k < n$. We will prove that is true for $k = n$.

Let $F \subseteq E(Q_n)$ with $|F| \le 2n-2$, and $Q_n - F$ has at least 3 components. Now since $Q_n = Q_{n-1}^0 \odot Q_{n-1}^1$, we have $|E(Q_{n-1}^0) \cap F| \le n-1$ or $|E(Q_{n-1}^1) \cap F| \le n-1$, say $|E(Q_{n-1}^0) \cap F| \le n-1$. Since $\lambda(Q_{n-1}) = n-1$, we have two cases.

Case 1. $Q_{n-1}^0 - F$ is not connected.

Then $|E(Q_{n-1}^0) \cap F| = n-1$ and $Q_{n-1}^0 - F$ has only two components.

If $Q_{n-1}^1 - F$ is not connected, then $|E(Q_{n-1}^1) \cap F| = n-1$. That is $[Q_{n-1}^0, Q_{n-1}^1] \cap F = \emptyset$. But each vertex of $Q_{n-1}^1 - F$ is connected to one component of $Q_{n-1}^0 - F$. Hence $Q_n - F$ has only two components, a contradiction.

Note that $|[Q_{n-1}^0, Q_{n-1}^1]| = 2^{n-1} > n-1 (n \ge 5)$. If $Q_{n-1}^1 - F$ is connected, then $Q_{n-1}^1 - F$ is connected to one component of $Q_{n-1}^0 - F$. Hence $Q_n - F$ has only two components, a contradiction.

Case 2. $Q_{n-1}^0 - F$ is connected.

If $Q_{n-1}^1 - F$ is connected, then we are done. We assume that $Q_{n-1}^1 - F$ is not connected. And $Q_{n-1}^1 - F$ has at most one isolated vertex since $|F| \le 2n-2$.

If $Q_{n-1}^1 - F$ has at least 3 components, from the inductive hypothesis, then $|E(Q_{n-1}^1) \cap F| \ge 2n-3$. Hence at most one of components of $Q_{n-1}^1 - F$ is not connected to $Q_{n-1}^0 - F$, $Q_n - F$ has at most two components, a contradiction.

Therefore we assume that $Q_{n-1}^1 - F$ has only two components. But $2^{n-1}-(2n-2)>0 (n\ge 5)$, $Q_n - F$ has at most two components, a contradiction.

Theorem 2.16
$c\lambda_4(Q_n)=3n-2$ for $n\ge 2$.

Proof
Take a path $P_3=uvw$. Then

$$|E(u)\cup E(v)\cup E(w)|=3n-2.$$

And $Q_n - E(u)-E(v)-E(w)$ has at least 4 connected components. That is $c\lambda_4(Q_n)\le 3n-2$.

Next we will show that $c\lambda_4(Q_n)\ge 3n-2$ by induction. It is easy to check it is true for $n=2,3,4$. So we suppose $n\ge 5$ and assume this is true for all $k<n$. We will prove that is true for $k=n$.

Let $F\subseteq E(Q_n)$ with $|F|\le 3n-3$, and $Q_n - F$ has at least 4 c omponents. Now since $Q_n = Q_{n-1}^0 \odot Q_{n-1}^1$, we have

$$|E(Q_{n-1}^0)\cap F|\le [3n/2]-2$$

or

$$|E(Q_{n-1}^1)\cap F|\le [3n/2]-2,$$

say, $|E(Q_{n-1}^0)\cap F|\le [3n/2]-2$.

Since $c\lambda_3(Q_{n-1})=2n-3>[3n/2]-2 (n\ge 5)$, $Q_{n-1}^0 - F$ has at most two components.

Case 1. $Q_{n-1}^0 - F$ is connected.

If $Q_{n-1}^1 - F$ has at least 4 components, then $c\lambda_4(Q_{n-1})\ge 3n-5$ by the inductive hypothesis. We need delete at most two edges again. Since each vertex of Q_{n-1}^1 has a neighbor in Q_{n-1}^0 and $|[Q_{n-1}^0, Q_{n-1}^1]|=2^{n-1}>2 (n\ge 5)$, $Q_n - F$ has at most 3 components, a contradiction.

Suppose $Q_{n-1}^1 - F$ has at most 3 components. Because $|[Q_{n-1}^0, Q_{n-1}^1]|=2^{n-1}-(3n-3)>0 (n\ge 5)$, $Q_n - F$ has at most 3 components, a contradiction.

Case 2. $Q_{n-1}^0 - F$ has only two connected components.

Then $|E(Q_{n-1}^0)\cap F|\ge \lambda(Q_{n-1})=n-1$ and $|E(Q_{n-1}^1)\cap F|\le 2n-2$. And $c\lambda_3(Q_{n-1})=2n-3$.

If $Q_{n-1}^1 - F$ has at least 3 components, then $|E(Q_{n-1}^1)\cap F|\ge 2n-3$ and $|E(Q_{n-1}^0)\cap F|\le n$. But $|[Q_{n-1}^0, Q_{n-1}^1]\cap F|\le 1$ and $2^{n-1}>1 (n\ge 5)$, $Q_n - F$ has at most two components, a contradiction.

Hence $Q_{n-1}^1 - F$ has at most two components. We have $|[Q_{n-1}^0, Q_{n-1}^1]|>3n-3 (n\ge 5)$, and $Q_n - F$ has at most 3 components, a contradiction.

And because the hypercube Q_n is the subgraph of the folded hypercube FQ_n , w e can apply the similar method to FQ_n . Hence we have the following theorem.

Theorem 2.17
(1) $c\lambda_2(FQ_n)=\lambda(FQ_n)=n+1$ for $n\ge 3$.

(2) $c\lambda_3(FQ_n)=2n+1$ for $n\ge 3$.

(3) $c\lambda_4(FQ_n)=3n+1$ for $n\ge 3$.

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References:

[1] J. Bondy, U. Murty, *Graph theory and its application*, Academic Press, 1976.
[2] E. Cheng, L. Lesniak, M. Lipman, L. Liptak, Conditional matching preclusion sets, *Information Sciences,* Vol. 179, 2009, pp. 1092- 1101.
[3] G. Chartrand, S. Kapoor, L. Lesniak, D. Lick, Generalized connectivity in graphs, *Bull. Bombay Math. Colloq.*, Vol. 2, 1984, pp.1-6.
[4] A. El-Amawy, S. Latifi, Properties and performance of folded hypercubes, *IEEE Trans. Parallel Distrib. Syst.* , Vol. 2, 1991, pp. 31 - 42.
[5] J. Fabrega, M. Fiol, On the extraconnectivity of graphs, *Discr. Math.* , Vol. 155, 1996, pp. 49 - 57.

[6] L. Guo, X. Guo, Fault tolerance of hypercubes and folded hypercubes, *J. Supercomput.* Vol. 68, 2014, pp. 1235-1240.
[7] S. Hsieh, Extra edge connectivity of hypercube-like networks, *Int. J. Parallel Emergent Distrib. Syst.*, Vol. 28, 2013, pp. 123-133.
[8] L. Hsu, E. Cheng, L. Liptak, J. Tan, C. Lin, T. Ho, Component connectivity of the hypercubes, *Int. J. Comput. Math.* Vol. 89, 2012, pp. 137-145.
[9] M. Lin, M. Chang, D. Chen, Efficient algorithms for reliability analysis of distributed computing systems, *Inform. Sci.*, Vol.117, 1999, pp. 89 - 106.
[10] L. Lin, L. Xu, S. Zhou, Relating the extra connectivity and the conditional diagnosability of regular graphs under the comparison model, *Theoretical Comput. Sci.,* Vol. 618, 2016, pp. 21-29.
[11] E. Sampathkumar, Connectivity of a graph—a generalization, *J. Comb.Inf. Syst. Sci.,* Vol. 9, 1984, pp.71-78.
[12] J. Xu, Q. Zhu, X. Hou, T. Zhou, On restricted connectivity and extra connectivity of hypercubes and folded hypercubes, *J. Shanghai Jiaotong Univ., Sci.* Vol. 10, 2005, pp. 203-207.
[13] W. Yang, H. Li, On reliability of the folded hypercubes in terms of the extra edge-connectivity, *Inform. Sci*., Vol. 272, 2014, pp.238-243.
[14] W. Yang, S. Zhao, S. Zhang, Strong Menger connectivity with conditional faults of folded hypercubes, *Inform. Processing Let.,*Vol. 125, 2017, pp.30-34.
[15] X. Yang, D. J. Evans, B. Chen, G. M. Megson, H. Lai, On the maximal connected component of hypercube with faulty vertices. *Int. J. Comp. Math.*, Vol. 81, 2004, pp. 515-525.
[16] X. Yang, F ault tolerance of hypercube with forbidden faulty sets. *Proc. 10th Chinese Conf. Fault-Tolerant Computing.* Peking, 2003, pp. 135-139.
[17] Q. Zhu, J. Xu, X. Hou, M. Xu, On reliability of the folded hypercubes, *Inform. Sci.,*Vol.177, 2007, pp. 1782 - 1788.
[18] Q. Zhu, J. Xu, On restricted edge connectivity and extra edge connectivity of hypercubes and foled hypercubes, *J. University of Science and Technology of China*, Vol. 36, 2006, pp. 246 - 253.
[19] S. Zhao, W. Yang, S. Zhang, Component connectivity of hypercubes, *Theoretical Comput. Sci.* Vol. 640, 2016,pp.115-118.
[20] M. Zhang, J. Zhou, On g-extra connectivity of folded hypercubes, *Theoretical Comput. Sci.* Vol. 593, 2015, pp.146-153.
[21] M. Zhang, L. Zhang, X. Feng, Reliability measures in relation to the h-extra edge-connectivity of folded hypercubes, *Theoretical Comput. Sci.* Vol. 615, 2016, pp.71-77.