On the construction of the forward-jumping method and its application to solving of the Volterra integral equations with symmetric boundaries

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Abstract: - As is known the theoretical and so practical interest are represent the numerical methods with the high order of accuracy, which have the extended stability area and have uses the minimal volume of computational work at the each step size. For the construction such methods, here has expound the new way to solving Volterra integral equation by using which, here are constructed the stable forward-jumping methods with the order of accuracy $p > 2[k/2]+2$. And also have determined the maximal value of the order of accuracy of the proposed methods. In the construction of the algorithms have considered to using of the forward-jumping methods with the new properties and application them to solving Volterra integral equation with the symmetric bounders. For the construction of the methods with the high order of accuracy, here are used the forward-jumping methods of hybrid type. The received results are illustrated by solving the model equations.

Key-Words: - Volterra integral equation, symmetric boundaries, model equation.

1 Introduction

One of the main questions in the study of numerical methods consist in finding reliable information about of the solution of the considering problem. As is known, all the numerical methods are used to find approximately solutions of the considering problems. Note that the estimates for these methods, are valid for sufficiently small values of h, which is commonly referred to as the step size of integration. Therefore, some scientists have proposed to construct methods, after application which to solve a practical problems, can be obtain a discrete solution having some properties of exact solution of the considering problems, as increases and decreases, as well as and some other properties to accurately solving these problems. Such approaches are relevant with the solution of the Volterra integral equations with the symmetric boundaries, which are connected with the fact that in solving such equations we must to found the value $y(x)$ and $y(-x)$ of the solution of original problem.

Note that to obtain reliable results in solving of some problems, one can use the two sided methods or by using the predictor-corrector method that can help to find the interval for the changing of step size h. However, here we want to construct the methods for solving Volterra integral equations with the symmetric boundaries by using the information about the solution of the considering problem in the previous and the next mesh points.

Consider to solving of the following Volterra integral equation with the symmetric boundaries:

$$y(x) = f(x) + \int_{-x}^{x} K(x, s, y(s))ds, \quad x_0 \le s \le x \le X. \quad (1)$$

Assume that the equation (1) has a unique continuous solution defined on the segment $[-X, X]$. To find the approximately values of the solution of equation (1) on some mesh points, let us divide the segment $[x_0, X]$ into N equal parts by the mesh points $x_i = x_0 + ih (i = 0,1,..., N)$. Here $0 < h$ - is a step size. Let us also denote by the y_i and y_{-i} approximately values, and by the $y(x_i)$ and $y(-x_i)$ through, the exact value of the solution of equation (1) at the mesh points $x_{\pm i}$ $(i = 0,1,..., N)$, respectively.

By solving equations similarly of equation (1) one can be study of earthquakes and natural periodic seismic processes, and also studying of the variation of the tension on the thickness rod the investigation to transmit of the signals and etc. (see [1]-[7]). It should be noted that the relationship between Volterra integral equations with symmetric boundary and the symmetric methods have been studied in the work [8], in which is given the way for determining the effective methods for solving of the Volterra integral equation with the symmetric boundaries. Here, in contrast to the title of the work is suggested a method for the construction of some algorithms for possessing any properties of the solutions of the equation (1).

Such methods are applied in solving of the initial value problem for ordinary differential equations which are studied by different authors (see eg. [9]-[12]). One can be fined the information about of the two-sided methods in the work [13].

As is known, depending from the accuracy of the considered method, which has applied to solving of the equation (1), the various conditions are imposed on the kernel $K(x,s,z)$. Here, we assume that the function $K(x,s,z)$, is continuous to the totality of variables and is defined in the set $G=\{x_0 \le s \le x \le X;\ |z| \le a\}$, and also it has the continuous partial derivatives up to some order p, inclusively. Sometimes it is necessary to investigation of the equation (1) in a ε-extension area G, which is defined as: $\overline{G}=\{x_0 \le s \le x+\varepsilon \le X+\varepsilon;\ |z| \le a\}$ (see eg. [14]). However, the use of such an extension domain of the function $K(x,s,z)$ is not essential in the study of the numerical solution of the equation (1). Therefore, we further believe that $\varepsilon = 0$.

Obviously, equation (1) can be written as follows:

$$y(x) = f(x) + \int_0^x \varphi(x,s,y(s))ds,\ \ x_0 \le s \le x \le X, \tag{2}$$

where the integral kernel $\varphi(x,s,z)$ is defined as:

$$\varphi(x,s,y(s)) = K(x,s,y(s)) + K(x,-s,y(-s)) \tag{3}$$

Thus, formally to solving of the equation (1) can be applied one of the known methods, which have used in solving of the Volterra integral equations (2) (see eg.[15]-[22]).

But in the last time some of specialists have constructed the efficient methods for solving of the Voltera integral equations of the type (2).Remark, that in above noted references does not investigated the maximal value of the order of accuracy of the proposed method. Therefore, let us consider to determined the maximal order of the accuracy of proposed methods which can be written as follows

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = \sum_{i=0}^{k}\alpha_i f_{n+i} + h\sum_{i=0}^{k}\sum_{j=0}^{k}\beta_i^{(j)}\varphi(x_{n+j},x_{n+i},y_{n+i}),\beta_i^{(j)}=0,(i>j) \tag{4}$$

But in the next, consider determining the necessary conditions for the convergence of the method (4) and some conditions imposed on its coefficients.

2 The conditions have imposed to the coefficients of the method of (4).

It is known, that the basic properties of the methods of the type (4) are define by the values of the coefficients of the method (4). Therefore, consider to the determined of some natural conditions imposed on the coefficients of the method (4), which can be formulated in the following form:

A. The coefficients $\alpha_i, \beta_i^{(j)} (i,j=0,1,2,...,k)$ are real numbers, and $\alpha_k \ne 0$.

B. The characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k}\alpha_i\lambda^i,\ \sigma(\lambda) \equiv \sum_{i=0}^{k}\beta_i(\lambda)\lambda^i\ \ (\beta_i(\lambda)=\sum_{j=i}^{k}\beta_i^{(j)}\lambda^j)$$

have no common factors differ from the a constant.

C. $\sigma(1) \ne 0$, and the order of the accuracy for the method (4) satisfies the condition $p \ge 1$. If we take into account that the required value of the quantity y_{n+k} is a real number, then satisfying of the condition A is obvious. Therefore we consider to satisfying of the condition B.

Let us rewrite the method of (4) in the following form:

$$\rho(E)(y_n - f_n) = h\sigma(E)\varphi(x_n,x_n,y_n). \tag{5}$$

Here, E is the shift operator, it is to say that $Ey(x) = y(x+h)$

Assume that the polynomials $\rho(\lambda)$ and $\sigma(\lambda)$ has the common factor. If suppose that $\psi(\lambda)$ is the common factor then by taking into account the assumptions in the finite-difference equation (5), we have:

$$\psi(E)\rho^*(E)(y_n - f_n) = h\psi(E)\sigma^*(E)\varphi(x_n,x_n,y_n).$$

If the use the condition $\psi(x) \ne const$, then from here ,one can be write the following:

$$\rho^*(E)(y_n - f_n) = h\sigma^*(E)\varphi(x_n,x_n,y_n). \tag{6}$$

It is easy to verify that the order of the resulting difference equation is less than the order of the difference equation of (5).Thus we obtain that the equation (6) have unique solution for given initial value conditions in an amount less than k-order of the difference equation (5).

From the theory of finite-difference equations it is known that if the number of initial conditions less than the order of the difference equation, then its solution is not unique. However, the finite-difference equations (5) and (6) are equivalent. Consequently, the difference equation (5) has a unique solution for given initial conditions less than order k. The obtained contradiction shows that our assumption does not hold. This implies the need for assumptions of B.

In the relation (5) we pass to the limit for $h \to 0$. Then, taking into account the continuity of the functions $f(x)$ and $\varphi(x,s,y)$, obtain that:

$$\rho(1)(y(x) - f(x)) = 0,$$

where $x = x_0 + nh$ is a fixed point. It follows from here, that :

$$\rho(1) = 0, \tag{7}$$

which is a necessary condition for the convergence of the method (4).

By taking in account of the condition of (7) into equality of (5), we have:

$$(E-I)\rho_1(E)(y_l - f_l) - h\sigma(E)\varphi(x_l,x_l,y_l) = 0.$$

From this we get that

$$\rho_1(E)(y_{l+1} - f_{l+1} - y_l + f_l) - h\sigma(E)\varphi(x_l,x_l,y_l) = 0. \tag{8}$$

Note that

$$\sigma(E)K(x_l,x_l,y_l)=\sum_{i=0}^{k}\sum_{j=0}^{k}\beta_i^{(j)}\varphi(x_{l+j},x_{l+i},y_{l+i})\cdot$$

Then, for a sufficiently small h, relations (8) can be rewritten in the following form:

$$\rho_1(E)(y_{l+1}-f_{l+1}-y_l+f_l)-h\sigma_1(E)\varphi(x,x_l,y_l)=0\cdot \qquad (9)$$

Here

$$\sigma_1(\lambda)\equiv\beta_k(1)\lambda^k+\beta_{k-1}(1)\lambda^{k-1}+...+\beta_1(1)\lambda+\beta_0(1),$$

and x is a fixed point. Remark, that one can be consider to the following relation, $\sum_{j=0}^{k}\beta_i^{(j)}K(x_{l+j},x_{l+i},y_{l+i})$ as the approximation for the following function:

$$\beta_i(1)\varphi(x,x_{l+i},y_{l+i})\,.$$

If in the equality (9), the integer variable l to appropriate the values from 0 to n then by the summing of the obtained equations, we have:

$$\rho_1(E)(y_{n+1}-f_{n+1})-\sigma_1(E)h\sum_{\nu=0}^{n}\varphi(x,x_\nu,y_\nu)=0\ .$$

For the fixed values of $x=x_0+nh$ and for $h\to 0$ we have:

$$\rho_1(1)(y(x)-f(x))=\sigma_1(1)\int_{x_0}^{x}\varphi(x,s,y(s))ds\ . \qquad (10)$$

It follows from here, that $\sigma_1(1)\neq 0$ and

$$\rho_1(1)=\rho'(1)=\sigma_1(1). \qquad (11)$$

We note that the coefficients $\beta_i^{(j)}(i,j=0,1,2,...,k)$ can defined as the solution of the following system of the linear-algebraic system:

$$\sum_{j=0}^{k}\beta_i^{(j)}=\beta_i(1),(i=0,1,2,..,k)\cdot \qquad (12)$$

Amount of the solutions of the system (12) always are more than one. Obviously, if $\rho'(1)=0$, then from the equality of (10) does not follows equation (2). Therefore, $\sigma_1(1)\neq 0$. From the following conditions:

$$\rho'(1)=0\ ;\ \rho'(1)=\sigma_1(1)$$

follows that

$$\sum_{i=0}^{k}\alpha_i\big(y(x+ih)-f(x+ih)\big)-$$
$$-h\sum_{i=0}^{k}\beta_i(1)\varphi(x,x+ih,y(x+ih))=O(h^s),s\geq 2.$$

Thus we have proved that the fulfillment of the condition C, is also necessary. We assume that there conditions A, B and C are holds everywhere.

Now, let us consider to determination of the coefficients for the method (4).

To this end, consider the following equation:

$$y(x)=f(x)+\int_{x_0}^{x}\sum_{j=1}^{m}\gamma_j\varphi(x-jh,s,y(s))ds\ , \qquad (13)$$

obtained from the equation (2) by using the Lagrange interpolation polynomial.

Then, by putting $x=x_{n+k}$ in the correlation of (13) one can receive the following:

$$y_{n+k}=f_{n+k}+\sum_{j=0}^{k}\bar{\gamma}\int_{x_0}^{x_{n+k}}\varphi(x_n-jh,s,y(s))ds\cdot \qquad (14)$$

Obviously, the value of y_{n+k} can be found by the formula (14) does not coincide with the exact value of $y(x_{n+k})$ of the solution of equation (2). Note that, by addition the new mesh points in the construction of the Lagrange polynomial the its accuracy can be increased. But the accuracy for the value y_{n+k} ,which is found by the formula (14)in this case does not increases .Thus, we obtain that the study of the determined the order of the accuracy for the method (4)can be replaced by an investigation of the following Volterra integral equation:

$$y(x)=f(x)+\int_{x_0}^{x}F(s,y(s))ds\cdot \qquad (15)$$

Remark, that the equation (15) can be taken as the model equation for the investigation of the equation (2). If the method (4) is applied to solving of equation (15), then we have:

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=\sum_{i=0}^{k}\alpha_i f(x_i)+h\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)}(y'_{n+i}-f'_{n+i})\cdot \qquad (16)$$

Then by using the notation $z(x)=y(x)-f(x)$ and the Taylor expansions:

$$z_{n+i}=z_n+ihz'_n+\frac{(ih)^2}{2}z''_n+...+\frac{(ih)^p}{p!}z_n^{(p)}+O(h^{p+1}),$$
$$z'_{n+i}=z'_n+ihz''_n+\frac{(ih)^2}{2}z'''_n+...+\frac{(ih)^{p-1}}{(p-1)!}z_n^{(p)}+O(h^p)$$

in the relation of (16), we have:

$$\sum_{i=0}^{k}\alpha_i(z_n+ihz'_n+\frac{(ih)^2}{2}z''_n+...+\frac{(ih)^p}{p!}z_n^{(p)}+O(h^{p+1}))=$$
$$=\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)}(hz'_n+ih^2z''_n+\frac{i^2h^3}{2!}z'''_n+...+\frac{i^{p-1}h^p}{(p-1)!}z_n^{(p)}+O(h^{p+1})).$$

From this it follows that in order to the method (16) has the order of accuracy of p, the necessary and sufficient condition is the satisfaction of the following algebraic equations for the coefficients $\alpha_i,\beta_i^{(j)}(i,j=0,1,..,k)$:

$$\sum_{i=0}^{k}\alpha_i=0,\quad \sum_{i=0}^{k}i\alpha_i=\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)},$$
$$\sum_{i=0}^{k}\frac{i^l}{l!}\alpha_i=\sum_{j=0}^{k}\sum_{i=0}^{k}\frac{i^{l-1}}{(l-1)!}\beta_i^{(j)},(l=2,3,..,p). \qquad (17)$$

The resulting relation is a homogeneous system of linear algebraic equations, the number of equations in which is equal to $p+1$.

If in the method (16) we use the substitution

$$\sum_{j=0}^{k}\beta_i^{(j)}=\beta_i,(i=0,1,2..,k)\ , \qquad (18)$$

then the method (16) can be rewritten in the following form:

$$\sum_{j=0}^{k}\alpha_i z_{n+i}=h\sum_{i=0}^{k}\beta_i z'_{n+i}\ . \qquad (19)$$

Let us the use the solution of the linear system of algebraic equations of (18) in the system of (17), then receive that the number of the unknowns in the receive system is equal to $2k+2$.In this case receive that for the values $p \le 2k$ the received system has the nontrivial solution and the method with the degree $p = 2k$ is not unique, but in the case $K(x,s,y) = F(s,y)$, the method of the type of (4) with the degree $p = 2k$ is single [see for example[11],[13,[17]].As is known so theoretical as practical interest to represent the stable methods with the high order of accuracy. But the maximal value for the order of the accuracy of stable methods of type (19) is equal $2[k/2]+2$ (see, for example [23]-[25]). Remark that there are some ways for the extension the value of the order of the accuracy of the method one can use the Richardson's extrapolation, linear combination of linear multistep methods or the multistep multiderivative methods (MMM) (see, for example [26]-[29]). As the known, that the stable forward-jumping methods are more accurate (see [30]-[32]) for the construction the stable method with the order of accuracy more than $2[k/2]+2$, here proposed the use forward-jumping methods.

3 On a way to construction an algorithm to solving of the equation(1)

Remark, that we applied the method (4) to solving the equation (2) don`t taking into account the dependence of the function $\varphi(x,s,y)$ from the values of $y(-x_m)(m>0)$. Because let us consider to construct methods by using the calculation of the approximately values of the quantity $y(-x_i)$ $(i>0)$. Therefore, using the methods application to the solving of the Volterra integral equations with the fixed boundary do not taking into account the properties of the integral kernel, is not always possible for obtain acceptable results. Therefore, here consider the methods, which are constructed by using some of the properties of the integral kernel. To this end, let us consider to the following method with the constant coefficients:

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} \ \ (m \ge 1,\ n = 0,1,\dots,N-k). \quad (20)$$

Note that the notion of the degree and stability for the method (20) can be determined in the following form:

Definition 1. The integer value p is called a degree of the method (20), if the following is holds:

$$\sum_{i=0}^{k-m} \alpha_i y(x+ih) - h\sum_{i=0}^{k} \beta_i y'(x+ih) = O(h^{p+1}),\ h \to 0.$$

Definition 2. The method (20) is stable if the roots of its characteristic polynomial:

$$\rho(\lambda) = \alpha_{k-m}\lambda^{k-m} + \alpha_{k-m+1}\lambda^{k-m+1} + \dots + \alpha_1\lambda + \alpha_0,$$

lies inside of the unit circle, on the boundary of which there is no multiple roots.

The stability of the methods has as theoretical and practical interest. By using the definition 2 we define the characteristic polynomial as the following:

$$\rho(\lambda) = \alpha_{k-m}\lambda^{1+j} + \alpha_{k-m-1-j}\ (j = 0,1).$$

Here $\alpha_{k-m} \times \alpha_{k-m-1-j} < 0$.

By using the above mentioned method (4) can be written as follows:

$$y_{n+k-m} = y_{n+k-m-1-j} + h\sum_{i=0}^{k} \beta_i y'_{n+i}\ . \quad (21)$$

Note that from the method of (20) by the selection of coefficients α_i $(i = 0,1,\dots,k-m)$ can be obtained various stable methods, the maximum degree for which coincides with the maximum accuracy of the method (21). It is clear, that the condition $k-m>0$ is holds.

If we take into account that the values of k and m are integers, then receive, that $k \ge 2$. In the case $k = 2$, from the method of (20) we have:

$$y_{n+1} = y_n + h(5y'_n + 8y'_{n+1} - y'_{n+2})/12. \quad (22)$$

If the method (22) by using the relation (3), is applied to solving of equation (2), then, we have:

$$\begin{aligned} y_{i+1} = y_i + f_{i+1} - f_i + h(3k(x_{i+1}, x_{i+1}, \bar{y}_{i+1}) + \\ + 3k(x_i, x_i, y_i) + 5k(x_{i+2}, x_{i+1}, \bar{y}_{i+1}) \\ + 2k(x_{i+1}, x_i, y_i) - k(x_{i+2}, x_{i+2}, \hat{y}_{i+2}))/12 + \\ h(3k(x_{i+1}, -x_{i+1}, \bar{y}_{-i-1}) + 5k(x_{i+2}, -x_{i+1}, \bar{y}_{-i-1}) + \\ + 3k(x_i, -x_i, y_{-i}) + \\ + 2k(x_{i+1}, -x_i, y_{-i}) - k(x_{i+2}, -x_{i+2}, \hat{y}_{-i-2}))/12. \end{aligned} \quad (23)$$

$$\begin{aligned} y_{-i-1} = y_{-i} + f_{-i-1} - f_{-i} - h(3k(-x_{i+1}, x_{i+1}, \bar{y}_{i+1}) \\ + 3k(-x_i, x_i, y_i) + 5k(-x_{i+2}, x_{i+1}, \bar{y}_{i+1}) + \\ + 2k(-x_{i+1}, x_i, y_i) - k(-x_{i+2}, x_{i+2}, \hat{y}_{i+2}))/12 - \\ h(3k(-x_{i+1}, -x_{i+1}, \bar{y}_{-i-1}) + 3k(-x_i, -x_i, y_{-i}) + \\ + 5k(-x_{i+2}, -x_{i+1}, \bar{y}_{-i-1}) + \\ 2k(-x_{i+1}, -x_i, y_{-i}) - k(-x_{i+2}, -x_{i+2}, \hat{y}_{-i-2}))/12. \end{aligned} \quad (24)$$

Note that to calculation of the values $\bar{y}_{n+1}$ and $\bar{y}_{-n-1}$ one can be use the predictor-corrector methods in which as the predictor and corrector methods may be proposed the Euler's method and the trapezoidal rule, respectively. But to calculate the values $\hat{y}_{n+2}$ and $\hat{y}_{-n-2}$ one can be use the following midpoint method:

$$\begin{aligned} \hat{y}_{n+2} = y_n + f_{n+2} - f_n + h(k(x_{i+1}, x_{i+1}, y_{i+1}) + \\ k(x_{i+2}, x_{i+1}, y_{i+1})) + \\ + h(k(x_{i+1}, -x_{i+1}, y_{-i-1}) + k(x_{i+2}, -x_{i+1}, y_{-i-1}). \end{aligned}$$

Method (22) is symmetrical (see.[8]), because for application that to the determination the values of the solution of equation (2) at the mesh point x_{n+1}, must be known the values of the solution of the original problem in the previous mesh point x_n and the next mesh point x_{n+2}. Therefore, its application to solving of equation (1) gives the best result. Indeed, to finding the numerical solution of the equation (1), we used Simpson's method and the method (22). The results obtained by the method (22) are more accurate. To illustration the effect by using the information about the solution of the considering problem in the next mesh point, consider the following forward-jumping method:

$$y_{n+2} = (8y_{n+1} + 11y_n)/19 + \\ + h(10y'_n + 57y'_{n+1} + 24y'_{n+2} - y'_{n+3})/57 . \quad (25)$$

The both methods (22) and (25) have the type of the forward-jumping methods. If formally put $m = 0$, then the method (20) is transformed into a multistep method. As is known stable methods received from the multistep method in the case $k = 2$ and $k = 3$ has the maximum degree $p = 4$. As follows from here the degree of the forward-jumping methods in the first case is less, and in the second case more than the degree of the corresponding multistep methods. But the best results are obtained when these methods are using to solving of practical problems.

Remark that in using forward-jumping methods we have meet some difficulties in the selection of the methods which are applied to calculating the approximately values of the solution of the considering problem in the next mesh points.

Note, that some authors are considered the hybrid method as successful (see.eg.[33]-[37]). However, in their application appeases difficulty for computing the values of the problem in the hybrid points, which can be solved by the predictor-corrector method (see. eg.[37]). In the work [33] is proved, that in the class of methods (20) there are stable methods with the degree $p \le k + m + 1$. It seems that, by the increases the value of m, the values of p also is increases. But it is not right. Remark, that there exist some relation between quantity p and m, which can be written as: $p \le 2k - m$ (see [28]).

From here, receive relation between quantity m and k in the form: $m \le [(k-1)/2] + 1$ ($[a]$ is whole part of a). But the increases values of quantity m are complicate to application of the forward-jumping methods. Therefore we are basically investigated method of (20) for the value $m = 1$ and $m = 2$. Note that there are works in which proved the advantage of the forward-jumping methods and hybrid methods, which are applied to solving of the integral equations of Volterra type (see. eg. [8], [21], [24], [12]) And so, here for solving of integral equations of type (1) and (2) proposed to use the methods from the following classes:

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} + h\sum_{i=0}^{k} \gamma_i y'_{n+i+\nu_i} . \quad (26)$$
$$(|\nu_i| < 1, i = 0,1,...,k)$$

Here $\alpha_i, \beta_i, \gamma_i (i = 0,1,...,k)$ - some real numbers.

4 On the illustration of the advantage of the forward-jumping methods

We assume that the coefficient of the method (26) satisfies the following conditions:

A: The coefficients $\alpha_i, \beta_i, \gamma_i, \nu_i$ $(i = 0,1,2,...,k)$ are some real numbers, moreover, $\alpha_k \ne 0$.

B: Characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i; \quad \delta(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i; \; \gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^{i+\nu_i} .$$

have no common multipliers different from the constant.

C: $\delta(1) + \gamma(1) \ne 0$ and $p \ge 1$.

And now to consider construction the stable methods of type (10) having a high degree. To this end, put $m = 1$ and $k = 2$. Then, to determine the values of variables $\alpha_i, \beta_i, \gamma_i, \nu_i$ $(i = 0,1,...,k)$, we obtain the following system nonlinear algebraic equations:

$$\beta_2 + \beta_1 + \beta_0 + \gamma_2 + \gamma_1 + \gamma_0 = 1,$$
$$2^j \beta_2 + \beta_1 + l_2^j \gamma_2 + l_1^j \gamma_1 + l_0^j \gamma_0 = 1/(j+1) \; (j = 1,...,7). (27)$$

By using the solution of the system (11) one can be constructed stable methods with the degree $p \le 8$. If in the system (27) using the values $l_2 = 1 + \alpha$, $l_1 = 1$, $l_0 = 1 - \alpha$ $(|\alpha| < 1)$ then receive the following solution:

$$\beta_2 = -1/180; \beta_1 = 6/90; \beta_0 = 29/180; \; l_0 = 1/2,$$
$$\gamma_2 = 1/45, \quad \gamma_1 = 6/90; \quad \gamma_0 = 31/45; \quad l_2 = 3/2.$$

In this case the stable method can be written in the following form:

$$y_{n+1} = y_n + h(29y'_n + 24y'_{n+1} - y'_{n+2})/180 + \\ + h(62y'_{n+1/2} + 2y'_{n+3/2})/90. \quad (29)$$

This method is stable and has the degree $p = 5$.

In [12, p. 277] is proved that if the method (26) for $m = 1$ is stable and has a maximum degree then the coefficients β_k and β_{k-1} satisfied the following conditions $\beta_{k-1} > 0$, $\beta_{k-1} \cdot \beta_k < 0$ and $|\beta_{k-1}| > |\beta_k|$. As is known, if the method (26) for sufficiently smooth solution of the original problem has the degree of p, then its local error can be written in the following form

$$C_1 h^{p+1} y^{p+1} + C_2 h^{p+2} y^{n+2} + ... + O(h^{p+2}), \ \ h \to 0.$$

Usually when increases the value of p, the value of the coefficient C_1 is decreases. These parameters for the method of (28), are defined as $p = 5$ and $C_1 = -1/720$. For the obtained coefficients $\alpha_i, \beta_i, \gamma_i$ $(i = 0,1,2,...,k)$ and variables ν_i $(i = 0,1,...,k)$ one can be use the same standard program. If p is sufficiently large (in the redistribution of p = 10), the values of the constant C_1 may be at a short distance from the machine zero. In this case the degree of the method (26) can be extended. Therefore, in such cases it is desirable compliance with caution.

Now consider the application of the method (28) to solving of the equation (2). Then we receive have:

$$\begin{aligned} y_{n+1} &= y_n + f_{n+1} - f_n + h(10\varphi(x_n, x_n, y_n) + \\ &+ 10\varphi(x_{n+1}, x_n, y_n) + 9\varphi(x_{n+2}, x_n, y_n) + \\ &+ 6\varphi(x_{n+1}, x_{n+1}, y_{n+1}) + 6\varphi(x_{n+2}, x_{n+1}, y_{n+1}) - \\ &- \varphi(x_{n+2}, x_{n+2}, y_{n+2})/180 + h(31\varphi(x_{n+1}, x_{n+1/2}, y_{n+1/2}) + \\ &+ 31\varphi(x_{n+1/2}, x_{n+1/2}, y_{n+1/2}) + 6\varphi(x_{n+1}, x_{n+1}, y_{n+1}) + \\ &+ \varphi(x_{n+2}, x_{n+3/2}, y_{n+3/2}) + \varphi(x_{n+3/2}, x_{n+3/2}, y_{n+3/2}))/90. \end{aligned}$$

In the work [8] have investigated the comparison of the method (22) with the following hybrid method by using the solutions of equation (1):

$$y_{n+1} = y_n + h(y'_{n+1/2-\alpha} + y'_{n+1/2+\alpha}) \, (\alpha = \sqrt{3}/6). \quad (29)$$

Here, we study the comparison of the method (6) with the method (25) and show that these methods are use information about the solution of the considering problem, as in the previous and so in the next mesh points, and therefore they give the best results. To this end, we apply the method of (25) to solving of the equation (1). Then we have:

$$\begin{aligned} y_{i+2} &= (8y_{i+1} + 11y_i)/19 + (19f_{i+2} - 8f_{i+1} - 11f_i)/19 \\ &- h(k(x_{i+3}, x_{i+3}, y_{i+3}) - 12k(x_{i+2}, x_{i+2}, y_{i+2}) - \\ &- 12k(x_{i+3}, x_{i+2}, y_{i+2}) - 37k(x_{i+1}, x_{i+1}, y_{i+1}) - \\ &- 20k(x_{i+2}, x_{i+1}, y_{i+1}) - 10k(x_{i+1}, x_i, y_i))/57 - \quad (30) \\ &- h(k(x_{i+3}, -x_{i+3}, y_{-i-3}) - 12k(x_{i+2}, -x_{i+2}, y_{-i-2}) - \\ &- 12k(x_{i+3}, -x_{i+2}, y_{-i-2}) - 37k(x_{i+1}, -x_{i+1}, y_{-i-1}) - \\ &- 20k(x_{i+2}, -x_{i+1}, y_{-i-1}) - 10k(x_{i+1}, -x_i, y_{-i}))/57, \end{aligned}$$

$$\begin{aligned} y_{-i-2} &= (8y_{-i-1} + 11y_{-i})/19 + (19f_{-i-2} - 8f_{-i-1} - 11f_{-i})/19 + \\ &+ h(k(-x_{i+3}, x_{i+3}, y_{i+3}) - 12k(-x_{i+2}, x_{i+2}, y_{i+2}) - \\ &- 12k(-x_{i+3}, x_{i+2}, y_{i+2}) - 37k(-x_{i+1}, x_{i+1}, y_{i+1}) - \\ &- 20k(-x_{i+2}, x_{i+1}, y_{i+1}) - 10k(-x_{i+1}, x_i, y_i))/57 + \quad (31) \\ &+ h(k(-x_{i+3}, -x_{i+3}, y_{-i-3}) - 24k(-x_{i+2}, -x_{i+2}, y_{-i-2}) - \\ &- 57k(-x_{i+1}, -x_{i+1}, y_{-i-1}) - 10k(-x_i, -x_i, y_{-i}))/57. \end{aligned}$$

For the selection of effective methods, we have considered application of some multistep methods to solving of the Volterra integral equations. To illustrate the way to selection of effective methods, here have used compares of the methods (22), (25), Simpson, trapezoidal and midpoints rules. In the construction of algorithms for the applications of the implicit methods to solving of the equations (1) and (2) (these equations are considered to be equivalent only in the case when the integral kernel has the form (3)) was used predictor and corrector method, in which in the capacity of the predictor methods proposed the same methods in all the algorithms. Therefore, the order of accuracy of these algorithms can be considers the same.

4 The illustration of the received results

Note that in all the cases the results received by the forward jumping methods and the hybrid methods were the best. Following are proposed some of these fragments. To this end, investigated the following examples:

Example1. $y(x) = 1 + \int_0^x y(s)ds, \ \ 0 \le x \le 1$

(exact solution $y(x) = \exp(x)$),

Example2. $y(x) = 1 + x^2/2 + \int_0^x y(s)ds, \ \ 0 \le x \le 1$

(exact solution $y(x) = \exp(x) - x - 1$),

Example3. $y(x) = \exp(-mx) + m\int_{-x}^x y(s)ds, 0 \le x \le 1$

(exact solution $y(x) = \exp(mx)$).

Here we have applied the method (22) and the hybrid method (29) to solving of the equation (1). The results can be considered identical. But to solving of the example 2 have applied the method of (29) and the Simpson's rules. The results are placed in the following table 1:

Table 1.

Variable x	Hybrid	Simpson
0.1	2.6E-11	1.6E-06
0.2	5.4E-11	3.6E-06
0.3	8.3E-11	6.0E-06

0.5	1.4E-11	1.2E-05
1.00	3.4E-11	4.0E-05

For the comparison methods of (22) and (29), here considered to solving of the example 3 by the methods (22) and (29). And the next to solving example 3 are applied the methods (25) and the method (28). Results for them are placed in the table 2 and 3.

Table 2.

Step size	Variable x	method (22)	method (29)
$h = 0{,}05$	0.05	7.8E-07	1.6E-10
	0.1	1.6E-06	2.2E-08
	1.0	4.05E-05	6.3E-06

Table 3.

Step size	Variable x	method (25)	method (28)
$h = 0{,}05$	0.1	1.4E-11	7.1E-9
	0.4	1.5E-8	4.8E-8
	0.7	5.9E-8	9.4E-8
	1.0	1.4E-7	1.4E-7
$h = 0{,}01$	0.1	1.5E-12	1.9E-11
	0.4	3.2E-11	8.6E-11
	0.7	1.09E-10	1.6E-10
	1.0	2.4E-10	2.5E-10

5 Conclusion

For solving some problems of the natural sciences apiaries necessity to use theoretical results to achieve the desired expectation. Therefore in the section 1considered to determined the maximal accuracy of the proposed method. As is known, each method has its advantages and disadvantages. Often are available to construct methods, that when applied to the solving of some problems gives irregular results, than the known methods (see. eg. [37, p. 410-411]). And it is also known, that by modifying some specific problem one can receive the define class of problems for which purpose specially constructed methods gives the best results, that the known. However, it does not follow from here that the proposed method is the best. For example, one can be shows that the result obtained by the following method:

$$y_{n+1} = y_n + 50hf_n / 49$$

applied to solving of the problem $y' = y,\ y(0) = 1$ is not worse than the result obtained by the explicit Euler's method. As is known the methods which are constructed at the junction of some methods usually are the best than the methods which are used for its construction. Consequently by the above mentioned, we are comparing the quality of some of the known numerical methods, such as stability and accuracy, the region of stability, etc. constructed, here the methods which have the best qualities than the using methods. Remark that the forward-jumping and hybrid methods are constructed on the above said method. Therefore, we have constructed here a new method at the junction of those methods, believing that the method of (26) will be one of the most promising directions in the theory of numerical methods.

6 Acknowledgment

The authors express their thanks to the academician Ali Abbasov for his suggestion to investigate the computational aspects of our problem. This work was supported by the Science Development Foundation underthe President of the Republic of Azerbaijan – Grant № EIF-KETPL-2-20151(25)-56/07/1.

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