Component connectivity of crossed cubes

LITAO GUO
School of Applied Mathematics
Xiamen University of Technology
Xiamen Fujian 361024
P.R.CHINA
ltguo2012@126.com

Abstract: Let $G = (V, E)$ be a connected graph. A r-component cut of G is a set of vertices whose deletion results in a graph with at least r components. r-component connectivity $c\kappa_r(G)$ of G is the size of the smallest r-component cut. The r-component edge connectivity $c\lambda_r(G)$ can be defined similarly. In this paper, we determine the r-component (edge) connectivity of crossed cubes CQ_n for small r. And we also prove other properties of CQ_n.

Key–Words: Interconnection networks; Fault tolerance; r-component edge connectivity

1 Introduction

A network is often modeled by a graph $G = (V, E)$ with the vertices representing nodes such as processors or stations, and the edges representing links between the nodes. One fundamental consideration in the design of networks is reliability [2, 16]. Let $G = (V, E)$ be a connected graph, $N_G(v)$ the neighbors of a vertex v in G (simply $N(v)$), $E(v)$ the edges incident to v. Moreover, for $S \subset V$, $G[S]$ is the subgraph induced by S, $N_G(S) = \cup_{v\in S}N(v) - S, N_G[S] = N_G(S) \cup S$, and $G - S$ denotes the subgraph of G induced by the vertex set of $V \setminus S$. If $u, v \in V$, $d(u, v)$ denotes the length of a shortest (u, v)-path. For $X, Y \subset V$, denote by $[X, Y]$ the set of edges of G with one end in X and the other in Y. A connected graph G is called supper-κ (resp. super-λ) if every minimum vertex cut (edge cut) of G is the set of neighbors of some vertex in G, that is, every minimum vertex cut (edge cut) isolates a vertex. If G is super-κ or super-λ, then $\kappa(G) = \delta(G)$ or $\lambda(G) = \delta(G)$. For graph-theoretical terminology and notation not defined here we follow [1]. All graphs considered in this paper are simple, finite and undirected.

A r-component cut of G is a set of vertices whose deletion results in a graph with at least r components. r-component connectivity $c\kappa_r(G)$ of G is the size of the smallest r-component cut. The r-component edge connectivity $c\lambda_r(G)$ can be defined correspondingly. We can see that $c\kappa_{r+1}(G) \geq c\kappa_r(G)$ for each positive integer r. The connectivity $\kappa(G)$ is the 2-component connectivity $c\kappa_2(G)$. The r-component (edge) connectivity was introduced in [3] and [19] independently. Fábrega and Fiol introduced extraconnectivity in [7]. Let $F \subseteq V$ be a vertex set, F is called extra-cut, if $G - F$ is not connected and each component of $G - F$ has more than k vertices. The extraconnectivity $\kappa_k(G)$ is the cardinality of the minimum extra-cuts.

Two binary strings $x = x_1x_0$ and $y = y_1y_0$ are pair-related, denoted $x \sim y$, if and only if $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$; if x and y are not pair-related, we write $x \nsim y$.

The crossed cube CQ_n with 2^n vertices was introduced by Efe [5]. It can be defined inductively as follows: CQ_1 is K_2, the complete graph with labels 0 and 1. For $n > 1$, CQ_n contains CQ^0_{n-1} and CQ^1_{n-1} joined according to the following rule: the vertex $u = 0u_{n-2}\cdots u_0$ from CQ^0_{n-1} and the vertex $v = 1v_{n-2}\cdots v_0$ from CQ^1_{n-1} are adjacent if and only if

(1) $u_{n-2} = v_{n-2}$ if n is even, and

(2) for $0 \leq i < \lfloor (n-1)/2 \rfloor$, $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$.

From the definition, we can see that each vertex of CQ_n with a leading 0 bit has exactly one neighbor with a leading 1 and vice versa. It is a n-regular graph. In fact, some pairs of parallel edges are changed to some pairs of cross edges. Furthermore, CQ_n can be obtained by adding a perfect matching M between CQ^0_{n-1} and CQ^1_{n-1}. Hence CQ_n can be viewed as $G(CQ^0_{n-1}, CQ^1_{n-1}, M)$ or $CQ^0_{n-1} \odot CQ^1_{n-1}$ briefly. For any vertex $u \in V(CQ_n)$, $e_M(u)$ is the edge incident to u in M.

The crossed cube is an attractive alternative to hypercubes Q_n. The diameter of CQ_n is approximately half that of Q_n. For more references, we can see

[4, 6, 8, 9, 10, 18].

The fault tolerance analysis of other cubes has recently attracted the attention of many researchers [11, 14, 17, 20, 21, 22, 23, 24, 26, 27]. In [15], Hsu et al. determined the r-component connectivity of the hypercube Q_n for $r = 2, 3, \cdots, n+1$. In [25], Zhao et al. determined the r-component connectivity of the hypercube Q_n for $r = n+2, n+3, \cdots, 2n-4$. In [12], Guo et al. determined the r-component (edge) connectivity of the hypercube Q_n and the folded hypercube FQ_n for small r. In this paper, we obtain that:

(1) $c\kappa_2(CQ_n) = \kappa(CQ_n) = n (n \geq 2)$.
(2) $c\kappa_3(CQ_n) = 2n - 2 (n \geq 3)$.
(3) $c\lambda_2(CQ_n) = \lambda(CQ_n) = n$ for $n \geq 2$.
(4) $c\lambda_3(CQ_n) = 2n - 1$ for $n \geq 2$.
(5) $c\lambda_4(CQ_n) = 3n - 2$ for $n \geq 2$.

2 Component connectivity of crossed cubes

The hypercube Q_n has an important property as follows.

Lemma 2.1. *[24] Any two vertices of Q_n have exactly two common neighbors for $n \geq 3$ if they have any.*

The crossed cubes is obtained by interchanging a pair of edges of the hypercube. Then it appears two vertices which have only one common neighbor. So we have the following result similar to Lemma 2.1.

Lemma 2.2. *Any two vertices of CQ_n have at most two common neighbors for $n \geq 3$ if they have.*

Corollary 2.3. *For any two vertices $x, y \in V(CQ_n) (n \geq 3)$,*

(1) if $d(x, y) = 2$, then they have at most two common neighbors;

(2) if $d(x, y) \neq 2$, then they do not have common neighbors.

Lemma 2.4. *[12] Let x and y be any two vertices of $V(Q_n) (n \geq 3)$ such that have two common neighbors.*

(1) If $x \in V(Q^0_{n-1}), y \in V(Q^1_{n-1})$, then the one common neighbor is in Q^0_{n-1}, and the other one is in Q^1_{n-1}.

(2) If $x, y \in V(Q^0_{n-1})$ or $V(Q^1_{n-1})$, then the two common neighbors are in Q^0_{n-1} or Q^1_{n-1}.

According to the definition of CQ_n, if any two vertices of $V(CQ_n)$ have only one common neighbor, then it is obtained by interchanging a pair of edges of the hypercube. Hence similar to Lemma 2.4, we have

Lemma 2.5. *Let x and y be any two vertices of $V(CQ_n) (n \geq 3)$ such that have only two common neighbors.*

(1) If $x \in V(CQ^0_{n-1}), y \in V(CQ^1_{n-1})$, then the one common neighbor is in CQ^0_{n-1}, and the other one is in CQ^1_{n-1}.

(2) If $x, y \in V(CQ^0_{n-1})$ or $V(CQ^1_{n-1})$, then the two common neighbors are in CQ^0_{n-1} or CQ^1_{n-1}.

By the definition of CQ_n and above results, we have

Lemma 2.6. *If any two vertices of $V(CQ_n)$ have only one common neighbor, then the two vertices and their common neighbor are in some CQ_3.*

Lemma 2.7. *[13] $\kappa(CQ_n) = \lambda(CQ_n) = n (n \geq 2)$.*

Theorem 2.8. *CQ_n is super-λ for $n \geq 3$.*

Proof. By induction. It is true for $n \leq 4$. Let $n \geq 5$. Assume that it holds for $n < k$. We will show that it is true for $n = k$.

Let $F \subseteq E(CQ_n)$, $|F| = n$ and $CQ_n - F$ be not connected. Furthermore, $CQ_n - F$ has only two connected components. Without loss of generality, suppose $|F \cap E(CQ^0_{n-1})| \leq \lfloor n/2 \rfloor$. Then $CQ^0_{n-1} - F$ is connected.

Note that $|[CQ^0_{n-1}, CQ^1_{n-1}]| = 2^{n-1} > n (n \geq 5)$. If $CQ^1_{n-1} - F$ is connected, then $CQ_n - F$ is connected, a contradiction.

Assume that $CQ^1_{n-1} - F$ is not connected. We have $|F \cap E(CQ^1_{n-1})| \geq n-1$. If $|F \cap E(CQ^1_{n-1})| = n$, then $F \cap E(CQ^0_{n-1}) = \emptyset$ and $[CQ^0_{n-1}, CQ^1_{n-1}] \cap F = \emptyset$. And each vertex of $CQ^1_{n-1} - F$ has a neighbor in $CQ^0_{n-1} - F$, that is, $CQ_n - F$ is connected, a contradiction.

Hence $|F \cap E(CQ^1_{n-1})| = n - 1$. According to the inductive hypothesis, $CQ^1_{n-1} - F$ is super-λ. Suppose the isolated vertex x and G_1 are the only two components of $CQ^1_{n-1} - F$. And G_1 is connected to $CQ^0_{n-1} - F$. If $e_M(x) \notin F$, then $CQ_n - F$ is connected, a contradiction. So $e_M(x) \in F$. We have $F = e(x)$ and $CQ_n - F$ has only two components, one component is x. Hence CQ_n is super-λ. $\square$

Theorem 2.9. *CQ_n is super-κ for $n \geq 2$.*

The proof is similar to Theorem 2.9.

Theorem 2.10. *$c\kappa_2(CQ_n) = \kappa(CQ_n) = n (n \geq 2)$.*

Theorem 2.11. *$c\kappa_3(CQ_n) = 2n - 2 (n \geq 3)$.*

Proof. We choose two nonadjacent vertices x, y in a cycle C_4 which has two common neighbors. Then

$CQ_n - N(\{x,y\})$ has at least three connected components and $|N(\{x,y\})| = 2n-2$. That is $c\kappa_3(CQ_n) \leq 2n-2$.

We will show $c\kappa_3(CQ_n) \geq 2n-2$ by induction. It is easy to check that it is true for $n = 3, 4$. So we suppose $n \geq 5$. Suppose it is true for $n < k$. Let $n = k$.

Let $F \subseteq V(CQ_n)$ with $|F| \leq 2n-3$. And $CQ_n - F$ has at least three connected components, say, G_1, G_2 and G_3. We have $|F \cap V(CQ^0_{n-1})| \leq n-2$ or $|F \cap V(CQ^1_{n-1})| \leq n-2$. Without loss of generality, we set $|F \cap V(CQ^0_{n-1})| \leq n-2$. Hence $CQ^0_{n-1} - F$ is connected.

If $CQ^1_{n-1} - F$ has at least three components, from the inductive hypothesis, then $|F \cap V(CQ^1_{n-1})| \geq 2n-4$ and $|F \cap V(CQ^0_{n-1})| \leq 1$. Because each vertex of CQ^1_{n-1} has one neighbor in CQ^0_{n-1}, at most one vertex of $CQ^1_{n-1} - F$ has no neighbors in $CQ^0_{n-1} - F$. So $CQ_n - F$ has at most two connected components, a contradiction.

Hence $CQ^1_{n-1} - F$ has at most two components. At most one component of $CQ^1_{n-1} - F$ is not connected to $CQ^0_{n-1} - F$. And $CQ_n - F$ has at most two connected components, a contradiction. □

Theorem 2.12. $c\lambda_2(CQ_n) = \lambda(CQ_n) = n$ *for* $n \geq 2$.

Theorem 2.13. $c\lambda_3(Q_n) = 2n-1$ *for* $n \geq 2$.

Proof. Take an edge $e = uv$, then $|E(u) \cup E(v)| = 2n-1$. And $CQ_n - E(u) - E(v)$ has at least three connected components. That is $c\lambda_3(CQ_n) \leq 2n-1$.

Next we will show that $c\lambda_3(CQ_n) \geq 2n-1$ by induction. It is easy to check it is true for $n = 2, 3, 4$. So we suppose $n \geq 5$ and assume it is true for all $n < k$. We will prove that is true for $n = k$.

Let $F \subseteq E(CQ_n)$ with $|F| \leq 2n-2$, and $CQ_n - F$ has at least three components. Now since $CQ_n = CQ^0_{n-1} \odot CQ^1_{n-1}$, we have $|E(CQ^0_{n-1}) \cap F| \leq n-1$ or $|E(CQ^1_{n-1}) \cap F| \leq n-1$, say, $|E(CQ^0_{n-1}) \cap F| \leq n-1$. Since $\lambda(CQ_{n-1}) = n-1$, we have two cases.

Case 1. $CQ^0_{n-1} - F$ is not connected.

Then $|E(CQ^0_{n-1}) \cap F| = n-1$ and $CQ^0_{n-1} - F$ has only two components.

If $CQ^1_{n-1} - F$ is not connected, then $|E(CQ^1_{n-1}) \cap F| = n-1$. That is $[CQ^0_{n-1}, CQ^1_{n-1}] \cap F = \emptyset$. But each vertex of $CQ^1_{n-1} - F$ is connected to one component of $CQ^0_{n-1} - F$. Hence $CQ_n - F$ has only two components, a contradiction.

Note that $|[CQ^0_{n-1}, CQ^1_{n-1}]| = 2^{n-1} > n-1 (n \geq 5)$. If $CQ^1_{n-1} - F$ is connected, then $CQ^1_{n-1} - F$ is connected to one component of $CQ^0_{n-1} - F$. Hence $CQ_n - F$ has only two components, a contradiction.

Case 2. $CQ^0_{n-1} - F$ is connected.

If $CQ^1_{n-1} - F$ is connected, then we are done. We assume that $CQ^1_{n-1} - F$ is not connected. And $CQ^1_{n-1} - F$ has at most one isolated vertex since $|F| \leq 2n-2$.

If $CQ^1_{n-1} - F$ has at least three components, from the inductive hypothesis, then $|E(CQ^1_{n-1}) \cap F| \geq 2n-3$. Hence at most one of components of $CQ^1_{n-1} - F$ is not connected to $CQ^0_{n-1} - F$, $CQ_n - F$ has at most two components, a contradiction.

Therefore we assume that $CQ^1_{n-1} - F$ has only two components. But $2^{n-1} - (2n-2) > 0 (n \geq 5)$, $CQ_n - F$ has at most two components, a contradiction. □

Theorem 2.14. $c\lambda_4(CQ_n) = 3n-2$ *for* $n \geq 2$.

Proof. Take a path $P_3 = uvw$. Then $|E(u) \cup E(v) \cup E(w)| = 3n-2$. And $CQ_n - E(u) - E(v) - E(w)$ has at least four connected components. That is $c\lambda_4(CQ_n) \leq 3n-2$.

Next we will show that $c\lambda_4(CQ_n) \geq 3n-2$ by induction. It is easy to check it is true for $n = 2, 3, 4$. So we suppose $n \geq 5$ and assume this is true for all $n < k$. We will prove that is true for $n = k$.

Let $F \subseteq E(CQ_n)$ with $|F| \leq 3n-3$, and $CQ_n - F$ has at least four components. Now since $CQ_n = CQ^0_{n-1} \odot CQ^1_{n-1}$, we have $|E(CQ^0_{n-1}) \cap F| \leq [3n/2] - 2$ or $|E(CQ^1_{n-1}) \cap F| \leq [3n/2] - 2$, say, $|E(CQ^0_{n-1}) \cap F| \leq [3n/2] - 2$. Since $c\lambda_3(CQ_{n-1}) = 2n-3 > [3n/2] - 2 (n \geq 5)$, $CQ^0_{n-1} - F$ has at most two components.

Case 1. $CQ^0_{n-1} - F$ is connected.

If $CQ^1_{n-1} - F$ has at least four components, then $c\lambda_4(CQ_{n-1}) \geq 3n-5$ by the inductive hypothesis. We need delete at most two edges again. Since each vertex of CQ^1_{n-1} has a neighbor in CQ^0_{n-1} and $|[CQ^0_{n-1}, CQ^1_{n-1}]| = 2^{n-1} > 2 (n \geq 5)$, $CQ_n - F$ has at most three components, a contradiction.

Suppose $CQ^1_{n-1} - F$ has at most three components. Because of $|[CQ^0_{n-1}, CQ^1_{n-1}]| = 2^{n-1} > (3n-3)(n \geq 5)$, $CQ_n - F$ has at most three components, a contradiction.

Case 2. $CQ^0_{n-1} - F$ has only two connected components.

Then $|E(CQ^0_{n-1}) \cap F| \geq \lambda(CQ_{n-1}) = n-1$ and $|E(CQ^1_{n-1}) \cap F| \leq 2n-2$. Note that $c\lambda_3(CQ_{n-1}) = 2n-3$.

If $CQ^1_{n-1} - F$ has at least three components, then $|E(CQ^1_{n-1}) \cap F| \geq 2n-3$ and $|E(CQ^0_{n-1}) \cap F| \leq n$. But $|[CQ^0_{n-1}, CQ^1_{n-1}] \cap F| \leq 1$ and $2^{n-1} > 1 (n \geq$

5), $CQ_n - F$ has at most three components, a contradiction.

Hence $CQ^1_{n-1} - F$ has at most two components. We have $|[CQ^0_{n-1}, CQ^1_{n-1}]| > 3n - 3(n \geq 5)$, and $CQ_n - F$ has at most three components, a contradiction. □

References:

[1] J. Bondy, U. Murty, *Graph theory and its application*, Academic Press, 1976.

[2] E. Cheng, L. Lesniak, M. Lipman, L. Lipt*á*k, Conditional matching preclusion sets, *Information Sciences* 179, 2009, pp. 1092- 1101.

[3] G. Chartrand, S. Kapoor, L. Lesniak, D. Lick, Generalized connectivity in graphs, *Bull. Bombay Math. Colloq.* 2, 1984, pp. 1-6.

[4] C. Chang, C. Wu, Conditional fault diameter of crossed cubes, *J. Parallel Distrib. Comput.* 69, 2009, pp. 91-99.

[5] E. Efe, The crossed cube architecture for parallel computing, *IEEE Trans. Parallel Distrib. Systems* 3 (5), 1992, pp. 513-524.

[6] K. Efe, P.K. Blackwell, W. Slough, T. Shiau, Topological properties of the crossed cube architecture,*Parallel Computing* 20, 1994, pp. 1763-1775.

[7] J. Fábrega, M. Fiol, On the extraconnectivity of graphs, *Discr. Math.* 155, 1996, pp. 49 - 57.

[8] J. Fan, Diagnosability of crossed cubes under the comparison diagnosis model, *IEEE Transactions on Parallel and Distributed Systems* 13 (10), 2002, pp. 1099-1104.

[9] J. Fan, X. Lin, X. Jia, Optimal path embedding in crossed cubes, *IEEE Transactions on Parallel and Distributed Systems* 16 (2), 2004, pp. 1190-1200.

[10] J. Fan, X. Lin, X. Jia, Node-pancyclicity and edge-pancyclicity in crossed cubes, *IEEE Transactions on Parallel and Distributed Systems* 16 (2), 2004, pp. 1190-1200.

[11] L. Guo, X. Guo, Fault tolerance of hypercubes and folded hypercubes, *J Supercomput.* 68, 2014, pp. 1235-1240.

[12] L. Guo, Fault tolerance of hypercubes and folded hypercubes in terms of component connectivity , submitted.

[13] P. D. Kulasinghe, Connectivity of the crossed cube hypercubes, *Inform. Processing Let.* 61, 1997, pp. 221-226.

[14] S. Hsieh, Extra edge connectivity of hypercube-like networks, *Int. J. Parallel Emergent Distrib. Syst.* 28, 2013, pp. 123-133.

[15] L. Hsu, E. Cheng, L. Lipták, J. Tan, C. Lin, T. Ho, Component connectivity of the hypercubes, *Int. J. Comput. Math.* 89, 2012, pp. 137-145.

[16] M. Lin, M. Chang, D. Chen, Efficient algorithms for reliability analysis of distributed computing systems, *Inform. Sci.* 117, 1999, pp. 89 - 106.

[17] L. Lin, L. Xu, S. Zhou, Relating the extra connectivity and the conditional diagnosability of regular graphs under the comparison model, *Theoretical Comput. Sci.* 618, 2016, pp. 21-29.

[18] M. Ma, J. Xu, Edge-pancyclicity of crossed cubes, *J. China Univ. Sci. Tech.* 35 (3), 2005, pp. 329-333.

[19] E. Sampathkumar, Connectivity of a graphła generalization, *J. Comb. Inf. Syst. Sci.* 9, 1984, pp. 71-78.

[20] J. Xu, Q. Zhu, X. Hou, T. Zhou, On restricted connectivity and extra connectivity of hypercubes and folded hypercubes, *J. Shanghai Jiaotong Univ., Sci.* 10(2), 2005, pp. 203-207.

[21] W. Yang, H. Li, On reliability of the folded hypercubes in terms of the extra edge-connectivity, *Inform. Sci.* 272, 2014, pp. 238-243.

[22] W. Yang, S. Zhao, S. Zhang, Strong Menger connectivity with conditional faults of folded hypercubes, *Inform. Processing Let.* 125, 2017, pp. 30-34.

[23] Q. Zhu, J. Xu, X. Hou, M. Xu, On reliability of the folded hypercubes, *Inform. Sci.* 177, 2007, pp. 1782 - 1788.

[24] Q. Zhu, J. Xu, On restricted edge connectivity and extra edge connectivity of hypercubes and foled hypercubes, *J. University of Science and Technology of China* 36(3), 2006, pp. 246 -253.

[25] S. Zhao, W. Yang, S. Zhang, Component connectivity of hypercubes, *Theoretical Comput. Sci.* 640, 2016, pp. 115-118.

[26] M. Zhang, J. Zhou, On g-extra connectivity of folded hypercubes, *Theoretical Comput. Sci.* 593, 2015, pp. 146-153.

[27] M. Zhang, L. Zhang, X. Feng, Reliability measures in relation to the h-extra edge-connectivity of folded hypercubes, *Theoretical Comput. Sci.* 615, 2016, pp. 71-77.