# Inclusion Properties for a Certain Class of Analytic Function Related to Linear Operator

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*Abstract:* In this paper, we introduce a new class of analytic functions define by a new convolution operator $L_a^t(\alpha,\beta)$. The new class of analytic functions $\Sigma_{\alpha,\beta}^{a,t}(\rho;\, h)$ in $U^* = \{z : 0 < |z| < 1\}$ is define by means of a hypergeometric function with an integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination. The authors also introduces and investigates various properties of certain classes of meromorphically univalent functions.

*Key–Words:* Analytic function; Convex function; Starlike function; Prestarlike function; Meromorphic function; Hurwitz Zeta function; Linear operator; Hadamard product.

## 1 Introduction

A meromorphic function is a single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities it must go to infinit like a polynomial (i.e., these exceptional points must be poles and not essential singularities). A simpler definitio states that a meromorphic function $f(z)$ is a function of the form

$$f(z) = \frac{g(z)}{h(z)},$$

where $g(z)$ and $h(z)$ are entire functions with $h(z) \neq 0$ (see [1], p. 64). A meromorphic function therefore may only have finite-orde , isolated poles and zeros and no essential singularities in its domain. An equivalent definitio of a meromorphic function is a complex analytic map to the Riemann sphere. For example the Gamma function is meromorphic in the whole complex plane C (see [2], [3] and [4]).

Let $A$ be the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in the open unit disk $U = U^* \cup \{0\}$ and for which

$$\Re\{h(z)\} > 0 \qquad (z \in U). \tag{1}$$

For functions $f$ and $g$ analytic in $U$, we state that $f$ is subordinate to $g$ and write

$$f \prec g \quad in \quad U \quad or \quad f(z) \prec g(z) \quad (z \in U)$$

if there exists an analytic function $w(z)$ in $U$ such that

$$|w(z)| \leq |z| \ \ and \ \ f(z) = g(w(z)), \ \ (z \in U).$$

Furthermore, if the function $g$ is univalent in $U$, then

$$f(z) \prec g(z) \Leftrightarrow \ f(0) = g(0) \ \ and \ f(U) = g(U),$$

$(z \in U)$.

In the present paper, we initiate the study of functions which are meromorphic in the punctured disk $U^* = \{z : 0 < |z| < 1\}$ with a Laurent expansion about the origin, see [5]. Also, we shall use the operator $L_a^t(\alpha,\beta) f(z)$ to introduce some new classes of meromorphic functions. We also, introduce and investigate various inclusion relationships and convolution properties of a certain class of meromorphic functions, which are define in this paper by means of a linear operator.

## 2 Preliminaries

Let $\Sigma$ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{2}$$

which are analytic in the punctured unit disk

$$U^* = \{z : z \in C \quad \text{and} \quad 0 < |z| < 1\}\ U \setminus \{0\},$$

$C$ being (as usual) the set of complex numbers. We denote by $\Sigma S^*(\beta)$ and $\Sigma K(\beta)$ $(\beta \geq 0)$ the subclasses of $\Sigma$ consisting of all meromorphic functions which are, respectively, starlike of order $\beta$ and convex of order $\beta$ in $U^*$ (see also the recent works [6] and [7]).

For functions $f_j(z)$ $(j = 1, 2)$ define by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n \qquad (j = 1, 2),$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n.$$

Let us consider the function $\widetilde{\phi}(\alpha, \beta; z)$ define by

$$\widetilde{\phi}(\alpha, \beta; z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} a_n z^n$$

$$\left(\beta \in C \setminus Z_0^-;\ \alpha \in C\right),$$

where

$$Z_0^- = \{0, -1, -2, \cdots\} = Z^- \cup \{0\}.$$

Here, and in the remainder of this paper, $(\lambda)_\kappa$ denotes the general Pochhammer symbol defined in terms of the Gamma function, by

$$(\lambda)_\kappa := \frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)}$$

$$= \begin{cases} 1 \qquad , & (\kappa =; \lambda \in C\backslash\{0\}) \\ \lambda(\lambda+1)(\lambda+n-1) & (\kappa = n \in N; \lambda \in C) \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the $\Gamma$-quotient exists (see, for details, [8, p. 21 *et seq.*]), $N$ being the set of positive integers.

We recall here a general Hurwitz-Lerch-Zeta function, which is define in [[9], [10]] by the following series:

$$\Phi(z, t, a) = \frac{1}{a^t} + \sum_{n=1}^{\infty} \frac{z^n}{(n+a)^t} \qquad (3)$$

$\Big(a \in C\backslash Z_0^-, Z_0^- = \{0, -1, -2, ...\}; t \in C \ when$
$z \in U = U^* \cup \{0\}; \Re(t) > 1 \, when \, z \in \partial U\Big)$

Important special cases of the function $\Phi(z, t, a)$ include, for example, the Reimann zeta function $\zeta(t) = \Phi(1, t, 1)$, the Hurwitz zeta function $\zeta(t, a) = \Phi(1, t, a)$, the Lerch zeta function $l_t(\zeta) = \Phi\left(\exp^{2\pi i \xi}, t, 1\right), (\xi \in \mathbf{R}, \Re(t) > 1)$, the polylogarithm $L_t^i(z) = z\Phi(z, t, a)$ and so on. Recent results on $\Phi(z, t, a)$, can be found in the expositions [[11], [12]]. By making use of the following normalized function we define

$$G_{t,a}(z) = (1+a)^t \left[\Phi(z, t, a) - a^t + \frac{1}{z(1+a)^t}\right]$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1+a}{n+a}\right)^t z^n, \qquad (4)$$

$(z \in U^*)$.

Using the functions $G_{t,a}(z)$ with the Hadamard product for $f(z) \in \Sigma$, a new linear operator $L_{t,a}(\alpha, \beta)$ on $\Sigma$ will be defin by the following series::

$$L_a^t(\alpha, \beta) f(z) = \phi(\alpha, \beta; z) * G_{t,a}(z) =$$

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \left(\frac{1+a}{n+a}\right)^t a_n z^n. \qquad (5)$$

$(z \in U^*)$.

Many papers considered the above operator along with the meromorphic functions and generalized hypergeometric functions, see for example [[6], [13], [14], [15],[16], and [17]].

It follows from (5) that

$$z\left(L_a^t(\alpha, \beta) f(z)\right)' =$$

$$\alpha\left(L_a^t(\alpha+1, \beta) f(z)\right) - (\alpha+1) L_a^t(\alpha, \beta) f(z). \quad (6)$$

Let $\Omega$ represent the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in the open unit disk $U = U^* \cup \{0\}$.

**Definition 1** *A function $f \in \Sigma$ is said to be in the class $\Sigma_{\alpha,\beta}^{a,t}(\rho;\ h)$, if it satisfies the subordination condition*

$$(1+\rho) z\left(L_a^t(\alpha, \beta) f(z)\right) + \rho z^2 \left(L_a^t(\alpha, \beta) f(z)\right)'$$

$$\prec h(z) \qquad (7)$$

*where $\rho$ is a complex number and $h(z) \in \Omega$.*

Let $A$ be class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{8}$$

which are analytic in $U$. A function $h(z) \in A$ is said to be in the class $S^*(\gamma)$, if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \qquad (z \in U).$$

For some $\gamma(\gamma < 1)$. When $0 < \gamma < 1$, $S^*(\gamma)$ is the class of starlike functions of order $\gamma$ in $U$. A function $h(z) \in A$ is said to be prestarlike of order $\gamma$ in $U$, if

$$\frac{z}{(1-z)^{2(1-\gamma)}} * f(z) \in S^*(\gamma) \qquad (\gamma < 1)$$

where the symbol $*$ is used to refer to the familiar Hadamard product (or convolution) of two analytic functions in $U$. We denote this class by $R(\gamma)$. A function $f(z) \in A$ is in the class $R(0)$, if and only if $f(z)$ is convex univalent in $U$ and

$$R\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right)$$

# 3 Main results

In order to establish our main results, the following lemmas will be required:

**Lemma 2** *(See [18]) Let $g(z)$ be analytic in $U$, and $h(z)$ be analytic and convex univalent in $U$ with $h(0) = g(0)$. If,*

$$g(z) + \frac{1}{\mu} z\, g'(z) \prec h(z) \tag{9}$$

*where $\Re\,\mu \geq 0$ and $\mu \neq 0$, then*

$$g(z) \prec \widetilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t)\, dt \prec h(z)$$

*and $\widetilde{h}(z)$ is the best dominant of (9).*

**Lemma 3** *[19] Let $\Re\,\alpha \geq 0$ and $\alpha \neq 0$. Then,*

$$\Sigma_{\alpha,\beta}^{a,t}(\rho;\, h) \subset \Sigma_{\alpha,\beta}^{a,t}\left(\rho;\, \widetilde{h}\right),$$

*where*

$$\widetilde{h}(z) = \alpha z^{-\alpha} \textstyle\int_0^z t^{\alpha-1} h(t) dt \prec h(z).$$

**Lemma 4** *[19] Let $f(z) \in \Sigma_{\alpha,\beta}^{a,t}(\rho;\, h)$, $g(z) \in \Sigma$ and*

$$\Re(zg(z)) > \frac{1}{2} \qquad (z \in U).$$

*Then,*

$$(f * g)(z) \in \Sigma_{\alpha,\beta}^{a,t}(\rho;\, h)$$

**Lemma 5** *(See [19]) Let $a < 1$, $f(z) \in S^*(a)$ and $g(z) \in R(a)$. For any analytic function $F(z)$ in $U$, then*

$$\frac{g * (fF)}{g * f}(U) \subset \overline{co}(F(U)),$$

*where $\overline{co}(F(U))$ denotes the convex hull of $F(U)$.*

**Theorem 6** *Let $f(z) \in \Sigma_{\alpha,\beta}^{a,t}(\rho;\, h)$. Then the function $F(z)$ defined by*

$$F(z) = \frac{\mu - 1}{z^{\mu}} \int_0^z t^{\mu-1} f(t)\, dt \qquad (\Re\mu > 1) \tag{10}$$

*is in the class $\Sigma_{\alpha,\beta}^{a,t}\left(\rho;\, \widetilde{h}\right)$, where*

$$\widetilde{h}(z) = (\mu - 1)\, z^{1-\mu} \int_0^z t^{\mu-2} h(t)\, dt \prec h(z)$$

**Proof:** For $f(z) \in \Sigma$ and $\Re\,\mu > 1$, we fin from (10) that $F(z) \in \Sigma$ and

$$(\mu - 1)\, f(z) = \mu F(z) + zF'(z) \tag{11}$$

$F(z) \in \Sigma$.
Defin $G(z)$ by

$$zG(z) = (1+\rho)\, z\left(L_a^t(\alpha,\, \beta)\, F(z)\right)$$

$$+\rho z\left(L_a^t(\alpha,\, \beta)\, F(z)\right)'. \tag{12}$$

By differentiating both sides of (12) with respect to $z$, we get:

$$zG'(z) - G(z) = (1+\rho)\, z\left(L_a^t(\alpha,\, \beta)\,(zF'(z))\right)$$

$$+\rho z^2\left(L_a^t(\alpha,\, \beta)\,(zF'(z))\right)'. \tag{13}$$

Furthermore, it follows from (11), (12) and (13) that:

$$(1+\rho)\, z\left(L_a^t(\alpha,\, \beta)\, f(z)\right) + \rho z^2\left(L_a^t(\alpha,\, \beta)\, f(z)\right)'$$

$$= (1+\rho)\, z \left( L_a^t\,(\alpha,\,\beta) \left( \frac{\mu F(z) + zF'(z)}{\mu - 1} \right) \right)$$

$$+\rho z^2 \left( L_a^t\,(\alpha,\,\beta) \left( \frac{\mu F(z) + zF'(z)}{\mu - 1} \right) \right)'$$

$$= \frac{\mu}{\mu-1} G\,(z) + \frac{1}{\mu-1}\,(zG'(z) - G(z))$$

$$= G(z) + \frac{zG'(z)}{\mu - 1}. \qquad (14)$$

Let $f(z) \in \Sigma_{\alpha,\beta}^{a,t}\,(\rho;\, h)$. Then, by (14)

$$G(z) + \frac{zG'(z)}{\mu-1} \prec h\,(z) \qquad (\Re\,\mu > 1)\,,$$

aby using Lemma 2, we get

$$G(z) \prec \widetilde{h}\,(z) = (\mu - 1)\, z^{1-\mu} \int_0^z t^{\mu-2} h\,(t)\, dt$$

$$\prec h\,(z)\,.$$

Hence, by Lemma 3, we arrive at:

$$F(z) \in \Sigma_{\alpha,\beta}^{a,t} \left(\rho;\, \widetilde{h}\right) \subset \Sigma_{\alpha,\beta}^{a,t}\,(\rho;\, h)\,.$$

**Theorem 7** *Let $f(z) \in \Sigma$ and $F(z)$ be defined as in Theorem 6. If*

$$(1+\gamma)\, z \left( L_a^t\,(\alpha,\,\beta)\, F(z) \right) + \gamma z \left( L_a^t\,(\alpha,\,\beta)\, f(z) \right)$$

$$\prec h\,(z) \qquad (\gamma > 0)\,, \qquad (15)$$

*then $F(z) \in \Sigma_{\alpha,\beta}^{a,t} \left(0,\, \widetilde{h}\right)$, where $\Re\,\mu > 1$ and*

$$\widetilde{h}\,(z) = \frac{(\mu-1)}{\gamma} z^{\frac{1-\mu}{\gamma}} \int_0^z t^{\frac{\mu-1}{\gamma}-1} h\,(t)\, dt \prec h\,(z)\,.$$

**Proof:** Let us define

$$G(z) = z \left( L_a^t\,(\alpha,\,\beta)\, F(z) \right) \qquad (16)$$

Then the analytic function $G(z)$ in the unit disk $U$, with $G(0) = 1$, and

$$zG'(z) = G(z) + z^2 \left( L_a^t\,(\alpha,\,\beta)\, F(z) \right)'. \qquad (17)$$

Making use of (11), (15), (16) and (17), we deduce that:

$$(1+\gamma)\, z \left( L_a^t\,(\alpha,\,\beta)\, F(z) \right) + \gamma z \left( L_a^t\,(\alpha,\,\beta)\, f(z) \right)$$

$$= (1+\gamma)\, z \left( L_a^t\,(\alpha,\,\beta)\, F(z) \right) +$$

$$\frac{\gamma}{\mu-1} \left( \mu z L_a^t\,(\alpha,\,\beta)\, F(z) \right) + z^2 \left( L_a^t\,(\alpha,\,\beta)\, F(z) \right)'$$

$$= G\,(z) + \frac{1}{\mu-1} zG'(z) \prec h\,(z)$$

for $\Re\,\mu > 1$ and $\gamma > 0$.

Thus, an application of Lemma 2 evidently completes the proof of Theorem 7.

**Theorem 8** *Let $F(z) \in \Sigma_{\alpha,\beta}^{a,t}\,(\rho;\, h)$. If the function $f(z)$ is defined by*

$$F\,(z) = \frac{\mu-1}{z^\mu} \int_0^z t^{\mu-1} f\,(t)\, dt \qquad (\mu > 1) \qquad (18)$$

*then,*

$$\sigma f\,(\sigma z) \in \Sigma_{\alpha,\beta}^{a,t}\,(\rho;\, h)$$

*where*

$$\sigma = \sigma\,(\mu) = \frac{\sqrt{\mu^2 - 2\,(\mu-1)} - 1}{(\mu-1)} \in (0, 1)\,. \qquad (19)$$

*The bound $\sigma$ is sharp when*

$$h(z) = \delta + (1-\delta)\, \frac{1+z}{1-z} \qquad (\delta \neq 1)\,. \qquad (20)$$

**Proof:** For $F(z) \in \Sigma_{\alpha,\beta}^{a,t}\,(\rho;\, h)$, wecan verify that:

$F(z) = F(z) * \frac{z}{1-z}$ and $zF'(z) =$

$$F(z) * \left( \tfrac{1}{z(1-z)^2} - \tfrac{2}{z(1-z)} \right).$$

Hence, by (18), we have:

$$f\,(z) = \frac{\mu F\,(z) + zF'(z)}{\mu-1} = (F * g)\,(z) \qquad (21)$$

$(z \in U^*, \mu > 1)$, where $g\,(z) =$

$$\frac{1}{\mu-1} \left( \frac{1}{(1-z)^2} + (\mu-2)\, \frac{1}{z\,(1-z)} \right) \in \Sigma. \qquad (22)$$

We then show that:

$$\Re\,\{zg\,(z)\} > \frac{1}{2} \qquad (|z| < \sigma)\,, \qquad (23)$$

where $\sigma = \sigma\,(\mu)$ is given by (19). Setting

$$\frac{1}{1-z} = \mathrm{Re}^{i\theta} \qquad (R > 0,\, |z| = r < 1)$$

we fin that:

$$\cos\theta = \frac{1+R^2\left(1-r^2\right)}{2R} \quad and \quad R \geq \frac{1}{1+r} \tag{24}$$

For $\mu > 1$ it follows from (29) and (32) that:

$$2\,\Re\left\{zg\left(z\right)\right\} =$$

$$\frac{2}{\mu-1}\left[\left(\mu-2\right)R\cos\theta + R^2\left(2\cos^2\theta - 1\right)\right]$$

$$= \frac{1}{\mu-1}\left[\left(\mu-2\right)\left(1+R^2\left(1-r^2\right)\right) +\right.$$

$$\left.\left(1+R^2\left(1-r^2\right)\right)^2 - 2R^2\right] =$$

$$\frac{R^2}{\mu-1}\left[R^2\left(1-r^2\right)^2 + \mu\left(1-r^2\right) - 2\right] + 1 \geq$$

$$\frac{R^2}{\mu-1}\left[\left(1-r^2\right)^2 + \mu\left(1-r^2\right) - 2\right] + 1 =$$

$$\frac{R^2}{\mu-1}\left[\left(1-\mu\right)r^2 + \mu - 2r - 1\right] + 1.$$

This evidently gives (31), which is equivalent to

$$\Re\left\{z\sigma g\left(\sigma z\right)\right\} > \frac{1}{2} \qquad \left(z \in U\right). \tag{25}$$

Let $F(z) \in \Sigma_{\alpha,\beta}^{a,t}\left(\rho;\, h\right)$. Then, by using (28) and (33), an application of Lemma 4 yields:

$$\sigma f\left(\sigma z\right) = F\left(z\right) * \sigma g\left(\sigma z\right) \in \Sigma_{\alpha,\beta}^{a,t}\left(\rho;\, h\right).$$

For $h\left(z\right)$ given by (27). we consider the function $F\left(z\right) \in \Sigma$ define by:

$$\left(1+\lambda\right)z\left(L_a^t\left(\alpha,\,\beta\right)F(z)\right) +$$

$$\lambda z^2\left(L_a^t\left(\alpha,\,\beta\right)F(z)\right)' = \delta + \left(1-\delta\right)\frac{1+z}{1-z}. \tag{26}$$

$\left(\delta \neq 1\right)$. Then, by (34), (12) and (14) (used in the proof of Theorem thm1), we fin that:

$$\left(1+\rho\right)z\left(L_a^t\left(\alpha,\beta\right)f(z)\right) + \rho z^2\left(L_a^t\left(\alpha,\beta\right)f(z)\right)' =$$

$$\delta + \left(1-\delta\right)\frac{1+z}{1-z} + \frac{z}{\mu-1}\left(\delta + \left(1-\delta\right)\frac{1+z}{1-z}\right)' =$$

$$\delta + \frac{\left(1-\delta\right)\left(\mu+2z-1+\left(1-\mu\right)z^2\right)}{\left(\mu-1\right)\left(1-z\right)^2} = \delta$$

$\left(\sigma = -z\right)$.
Therefore, we conclude that the bound $\sigma = \sigma\left(\mu\right)$ cannot be increased for each $\mu\left(\mu > 1\right)$.

# 4 Inclusion relations

**Theorem 9** *Let* $0 \leq \rho_1 < \rho_2$. *Then*

$$\Sigma_{\alpha,\,\beta}^{a,\,t}\left(\rho_2;\,h\right) \subset \Sigma_{\alpha,\,\beta}^{a,\,t}\left(\rho_1;\,h\right)$$

**Proof:** Let $0 \leq \rho_1 < \rho_2$ and suppose that:

$$g\left(z\right) = z\left(L_a^t\left(\alpha,\beta\right)f(z)\right) \tag{27}$$

for $f(z) \in \Sigma_{\alpha,\,\beta}^{a,\,t}\left(\rho_2;\,h\right)$. Then the function $g(z)$ is analytic in $U$ with $g(0) = 1$. Differentiating both sides of (27) with respect to $z$ and using (6), we have:

$$\left(1+\rho_2\right)z\left(L_a^t\left(\alpha,\beta\right)f(z)\right) + \rho_2 z^2\left(L_a^t\left(\alpha,\beta\right)f(z)\right)'$$

$$= g(z) + \rho_2 z g'\left(z\right) \prec h\left(z\right). \tag{28}$$

Hence an application of Lemma 2 with $m = \frac{1}{\rho_2} > 0$ yields:

$$g\left(z\right) \prec h(z). \tag{29}$$

Noting that $0 \leq \frac{\rho_1}{\rho_2} < 1$ and that $h(z)$ is convex univalent in $U$, it follows from (27), (28 and (29) that:

$$\left(1+\rho_1\right)z\left(L_a^t\left(\alpha,\beta\right)f(z)\right) + \rho_1 z^2\left(L_a^t\left(\alpha,\beta\right)f(z)\right)'$$

$$= \frac{\rho_1}{\rho_2}\left[\left(1+\rho_2\right)z\left(L_a^t\left(\alpha,\beta\right)f\left(z\right)\right)\right.$$

$$\left.+\rho_2 z^2\left(L_a^t\left(\alpha,\beta\right)f\left(z\right)\right)'\right] + \left(1-\frac{\rho_1}{\rho_2}\right)g(z)$$

$$\prec h\left(z\right).$$

Thus, $f(z) \in \Sigma_{\alpha,\,\beta}^{a,\,t}\left(\rho_1;\,h\right)$ and the proof of Theorem 9 is complete.

**Theorem 10** *Let,*

$$\Re\left\{z\widetilde{\phi}\left(\alpha_1,\,\alpha_2\,;z\right)\right\} > \frac{1}{2} \tag{30}$$

$$\left(z \in U;\ \alpha_2 \notin \left\{0,-1,-2,...\right\}\right),$$

*where* $\widetilde{\phi}\left(\alpha_1,\,\alpha_2\,;z\right)$ *is defined as in (**??**). Then,*

$$\Sigma_{\alpha_2,\,\beta}^{a,\,t}\left(\rho;\,h\right) \subset \Sigma_{\alpha_1,\,\beta}^{a,\,t}\left(\rho_1;\,h\right)$$

**Proof:**

For $f(z) \in \Sigma$, we can verify that:

$$z\left(L_a^t(\alpha_1,\beta) f(z)\right)$$

$$= \left(z\widetilde{\phi}(\alpha_1,\alpha_2;z) * \left(zL_a^t(\alpha_2,\beta) f(z)\right)\right) \quad (31)$$

and

$$z^2\left(L_a^t(\alpha_1,\beta) f(z)\right)'$$

$$= \left(z\widetilde{\phi}(\alpha_1,\alpha_2;z) * z^2\left(L_a^t(\alpha_2,\beta) f(z)\right)'\right). \quad (32)$$

Let $f(z) \in \Sigma_{\alpha_2,\beta}^{a,t}(\rho;h)$. Then from (31) and (32), we deduce that:

$$(1+\rho)z\left(L_a^t(\alpha_1,\beta) f(z)\right) + \rho z^2\left(L_a^t(\alpha_1,\beta) f(z)\right)'$$

$$= \left(z\widetilde{\phi}(\alpha_1,\alpha_2;z)\right) * \Psi(z) \quad (33)$$

and

$$\Psi(z) = (1+\rho)z\left(L_a^t(\alpha_2,\beta) f(z)\right)$$

$$+\rho z^2\left(L_a^t(\alpha_2,\beta) f(z)\right)' \prec h(z) \quad (34)$$

In view of (30), the function $z\widetilde{\phi}(\alpha_1,\alpha_2;z)$ has the Herglotz representation:

$$z\widetilde{\phi}(\alpha_1,\alpha_2;z) = \int_{|x|=1} \frac{dm(x)}{1-xz} \qquad (z \in U), \quad (35)$$

where $m(x)$ is a probability measure define on the unit circle $|x|=1$ and

$$\int_{|x|=1} dm(x) = 1.$$

Since $h(z)$ is convex univalent in $U$, it follows from (33), (34) and (35) that:

$$(1+\rho)z\left(L_a^t(\alpha_1,\beta) f(z)\right) + \rho z^2\left(L_a^t(\alpha_1,\beta) f(z)\right)'$$

$$= \int_{|x|=1} \Psi(xz)\, dm(x) \prec h(z)$$

This shows that $f(z) \in \Sigma_{\alpha_1,\beta}^{a,t}(\rho;h)$ and the theorem is proved.

**Theorem 11** *Let $0 < \alpha_1 < \alpha_2$. Then*

$$\Sigma_{\alpha_2,\beta}^{a,t}(\rho;h) \subset \Sigma_{\alpha_1,\beta}^{a,t}(\rho;h).$$

**Proof:** Define

$$g(z) = z + \sum_{n=1}^{\infty} \left|\frac{(\alpha_1)_{n+1}}{(\alpha_2)_{n+1}}\right| z^{n+1}$$

$$(z \in U;\ \ 0 < \alpha_1 < \alpha_2).$$

Then,

$$z^2\widetilde{\phi}(\alpha_1,\alpha_2;z) = g(z) \in A \quad (36)$$

where $\widetilde{\phi}(\alpha_1,\alpha_2;z)$ is define as in (**??**), and

$$\frac{z}{(1-z)^{\alpha_2}} * g(z) = \frac{z}{(1-z)^{\alpha_1}}. \quad (37)$$

By (37), we see that:

$$\frac{z}{(1-z)^{\alpha_2}} * g(z) \in S^*\left(1-\frac{\alpha_1}{2}\right) \subset S^*\left(1-\frac{\alpha_2}{2}\right)$$

for $0 < \alpha_1 < \alpha_2$, which implies that:

$$g(z) \in R\left(1-\frac{\alpha_2}{2}\right) \quad (38)$$

Let $f(z) \in \Sigma_{\alpha_2,\beta}^{a,t}(\rho;h)$. Then we deduce from (33), (34) and (36) that:

$$(1+\rho)z\left(L_a^t(\alpha_1,\beta) f(z)\right) + \rho z^2\left(L_a^t(\alpha_1,\beta) f(z)\right)'$$

$$= \frac{g(z)}{z} * \Psi(z) = \frac{g(z) * (z\Psi(z))}{g(z) * z}, \quad (39)$$

where

$$\Psi(z) = (1+\rho)z\left(L_a^t(\alpha_2,\beta) f(z)\right)$$

$$+\rho z^2\left(L_a^t(\alpha_2,\beta) f(z)\right)' \prec h(z). \quad (40)$$

Since $z$ belongs to $S^*\left(1-\frac{\alpha_2}{2}\right)$ and $h(z)$ is convex univalent in $U$, it follows from (38), (39), (40) and Lemma 5 that:

$$(1+\rho)z\left(L_a^t(\alpha_1,\beta) f(z)\right)$$

$$+\rho z^2\left(L_a^t(\alpha_1,\beta) f(z)\right)' \prec h(z)$$

Thus$f(z) \in \Sigma_{\alpha_1,\beta}^{a,t}(\rho;h)$ and the proof is completed.

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