Inclusion Properties for a Certain Class of Analytic Function Related to Linear Operator

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Abstract: In this paper, we introduce a new class of analytic functions defined by a new convolution operator $L^t_{\alpha,\beta}(\rho; h)$ in $U^* = \{ z : 0 < |z| < 1 \}$ is defined by means of a hypergeometric function with an integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination. The authors also introduces and investigates various properties of certain classes of meromorphically univalent functions.

Key–Words: Analytic function; Convex function; Starlike function; Prestarlike function; Meromorphic function; Hurwitz Zeta function; Linear operator; Hadamard product.

1 Introduction

A meromorphic function is a single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities it must go to infinity like a polynomial (i.e., those exceptional points must be poles and not essential singularities). A simpler definition states that a meromorphic function $f(z)$ is a function of the form

$$f(z) = \frac{g(z)}{h(z)},$$

where $g(z)$ and $h(z)$ are entire functions with $h(z) \neq 0$ (see [1], p. 64). A meromorphic function therefore may only have finite-order, isolated poles and zeros and no essential singularities in its domain. An equivalent definition of a meromorphic function is a complex analytic map to the Riemann sphere. For example the Gamma function is meromorphic in the whole complex plane $\mathbb{C}$ (see [2], [3] and [4]).

Let $A$ be the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in the open unit disk $U = U^* \cup \{0\}$ and for which

$$\Re \{ h(z) \} > 0 \quad (z \in U).$$

(1)

For functions $f$ and $g$ analytic in $U$, we state that $f$ is subordinate to $g$ and write

$$f \prec g \text{ in } U \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U)$$

if there exists an analytic function $w(z)$ in $U$ such that

$$|w(z)| \leq |z| \quad \text{and} \quad f(z) = g(w(z)), \quad (z \in U).$$

Furthermore, if the function $g$ is univalent in $U$, then

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(U) = g(U), \quad (z \in U).$$

In the present paper, we initiate the study of functions which are meromorphic in the punctured disk $U^* = \{ z : 0 < |z| < 1 \}$ with a Laurent expansion about the origin, see [5]. Also, we shall use the operator $L^t_{\alpha,\beta}(\alpha, \beta)$ to introduce some new classes of meromorphic functions. We also, introduce and investigate various inclusion relationships and convolution properties of a certain class of meromorphic functions, which are defined in this paper by means of a linear operator.

2 Preliminaries

Let $\Sigma$ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

(2)

which are analytic in the punctured unit disk

$$U^* = \{ z : z \in C \quad \text{and} \quad 0 < |z| < 1 \} \ U \setminus \{0\},$$
C being (as usual) the set of complex numbers. We denote by $\Sigma S^*(\beta)$ and $\Sigma K(\beta)$ ($\beta \geq 0$) the subclasses of $\Sigma$ consisting of all meromorphic functions which are, respectively, starlike of order $\beta$ and convex of order $\beta$ in $U^*$ (see also the recent works [6] and [7]).

For functions $f_j(z) \ (j = 1, 2)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n \quad (j = 1, 2),$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 \ast f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n.$$

Let us consider the function $\tilde{\phi}(\alpha; \beta; z)$ defined by

$$\tilde{\phi}(\alpha; \beta; z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{1}{(\beta)_{n+1}} a_n z^n$$

where

$$\beta \in C \setminus Z^{-}; \ \alpha \in C$$

Here, and in the remainder of this paper, $(\lambda)_{\kappa}$ denotes the general Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_{\kappa} := \frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)}$$

where

$$\kappa = \begin{cases} 1 & (\kappa = \gamma \in C \setminus \{0\}) \\ \lambda (\lambda + 1) (\lambda + n - 1) & (\kappa = n \in N; \lambda \in C) \end{cases}$$

it being understood conventionally that $(0)_0 := 1$ and assumed tacitly that the $\Gamma$-quotient exists (see, for details, [8, p. 21 et seq.]), $N$ being the set of positive integers.

We recall here a general Hurwitz-Lerch-Zeta function, which is defined in [9, 10] by the following series:

$$\Phi\left(z, t, a\right) = \frac{1}{a^t} + \sum_{n=1}^{\infty} \frac{z^n}{(n+a)^t}$$

where $a \in C \setminus Z^{-}, Z^{-} = \{0, -1, -2, \ldots\}; t \in C$ when $z \in U = U^* \cup \{0\}; \ \Re(t) > 1$ when $z \in \partial U$.

Important special cases of the function $\Phi\left(z, t, a\right)$ include, for example, the Reimann zeta function $\zeta\left(t\right) = \Phi\left(1, t, 1\right)$, the Hurwitz zeta function $\zeta\left(t, a\right) = \Phi\left(1, t, a\right)$, the Lerch zeta function $L_t\left(\zeta\right) = \Phi\left(\exp^{2\pi i \xi}, t, 1\right), (\xi \in \mathbb{R}, \Re(t) > 1)$, the polylogarithm $L_t^\gamma\left(z\right) = z \Phi\left(z, t, a\right)$ and so on. Recent results on $\Phi\left(z, t, a\right)$, can be found in the expositions [[11], [12]]. By making use of the following normalized function we define:

$$G_{t,a}(z) = (1 + a)^t \left[\Phi(z, t, a) - a^t + \frac{1}{z (1 + a)^t}\right]$$

where

$$\begin{align*}
G_{t,a}(z) &= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1 + a}{n + a}\right)^t a_n z^n, \\
(z & \in U^*).
\end{align*}$$

Using the functions $G_{t,a}(z)$ with the Hadamard product for $f(z) \in \Sigma$, a new linear operator $L_{t,a}(\alpha, \beta)$ on $\Sigma$ will be define by the following series::

$$L_{t,a}^{(t)}(\alpha, \beta) f(z) = \phi(\alpha, \beta; z) + G_{t,a}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \left(\frac{1 + a}{n + a}\right)^t a_n z^n.$$  \hspace{1cm} (5)

$(z \in U^*).$

Many papers considered the above operator along with the meromorphic functions and generalized hypergeometric functions, see for example [[6], [13], [14], [15],[16], and [17]].

It follows from (5) that

$$z \left(L_{t,a}^{(t)}(\alpha, \beta) f(z)\right)' = \alpha \left(L_{t,a}^{(t)}(\alpha + 1, \beta) f(z)\right) - (\alpha + 1) L_{t,a}^{(t)}(\alpha, \beta) f(z).$$  \hspace{1cm} (6)

Let $\Omega$ represent the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in the open unit disk $U = U^* \cup \{0\}$.

**Definition I** A function $f \in \Sigma$ is said to be in the class $\Sigma_{\alpha, \beta}(\rho; h)$, if it satisfies the subordination condition

$$(1 + \rho) z \left(L_{t,a}^{(t)}(\alpha, \beta) f(z)\right) + \rho z^2 \left(L_{t,a}^{(t)}(\alpha, \beta) f(z)\right)' < h(z)$$

where $\rho$ is a complex number and $h(z) \in \Omega$.  \hspace{1cm} (7)
Let $A$ be class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (8)$$

which are analytic in $U$. A function $h(z) \in A$ is said to be in the class $S^*(\gamma)$, if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma \quad (z \in U).$$

For some $\gamma (\gamma < 1)$. When $0 < \gamma < 1$, $S^*(\gamma)$ is the class of starlike functions of order $\gamma$ in $U$. A function $h(z) \in A$ is said to be prestarlike of order $\gamma$ in $U$, if

$$\frac{z}{(1-z)^{2(1-\gamma)}} f(z) \in S^*(\gamma) \quad (\gamma < 1)$$

where the symbol $\ast$ is used to refer to the familiar Hadamard product (or convolution) of two analytic functions in $U$. We denote this class by $R(\gamma)$. A function $f(z) \in A$ is in the class $R(0)$, if and only if $f(z)$ is convex univalent in $U$ and

$$R\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right)$$

3 Main results

In order to establish our main results, the following lemmas will be required:

**Lemma 2** (See [18]) Let $g(z)$ be analytic in $U$, and $h(z)$ be analytic and convex univalent in $U$ with $h(0) = g(0)$. If,

$$g(z) + \frac{1}{\mu} z g'(z) < h(z) \quad (9)$$

where $\Re \mu \geq 0$ and $\mu \neq 0$, then

$$g(z) < h(z) = \mu z^{-\mu} \int_{0}^{z} t^{\mu-1} h(t) \, dt < h(z)$$

and $\tilde{h}(z)$ is the best dominant of (9).

**Lemma 3** [19] Let $\Re \alpha \geq 0$ and $\alpha \neq 0$. Then,

$$S_{\alpha,\beta}^{a,t}(\rho; h) = S_{\alpha,\beta}^{a,t}(\rho; \tilde{h}),$$

where

$$\tilde{h}(z) = az^{-\alpha} \int_{0}^{z} t^{\alpha-1} h(t) \, dt < h(z).$$

**Lemma 4** [19] Let $f(z) \in S_{\alpha,\beta}^{a,t}(\rho; h)$, $g(z) \in \Sigma$ and

$$\Re (zg(z)) > \frac{1}{2} \quad (z \in U).$$

Then,

$$(f * g)(z) \in S_{\alpha,\beta}^{a,t}(\rho; h)$$

**Lemma 5** (See [19]) Let $a < 1$, $f(z) \in S^*(a)$ and $g(z) \in R(a)$. For any analytic function $F(z)$ in $U$, then

$$\frac{g * (F)}{g * f}(z) \in \mathfrak{C}(F(U)),$$

where $\mathfrak{C}(F(U))$ denotes the convex hull of $F(U)$.

**Theorem 6** Let $f(z) \in S_{\alpha,\beta}^{a,t}(\rho; h)$. Then the function $F(z)$ defined by

$$F(z) = \frac{\mu - 1}{z^{\mu}} \int_{0}^{z} t^{\mu-1} f(t) \, dt \quad (\Re \mu > 1) \quad (10)$$

is in the class $S_{\alpha,\beta}^{a,t}(\rho; \tilde{h})$, where

$$\tilde{h}(z) = (\mu - 1) z^{-\mu} \int_{0}^{z} t^{\mu-2} h(t) \, dt < h(z)$$

**Proof:** For $f(z) \in \Sigma$ and $\Re \mu > 1$, we find from (10) that $F(z) \in \Sigma$ and

$$(\mu - 1) f(z) = \mu F(z) + z F'(z) \quad (11)$$

$F(z) \in \Sigma$.

Define $G(z)$ by

$$zG(z) = (1 + \rho) z \left( L_{\alpha}^{1} (\alpha, \beta) F(z) \right) + \rho z \left( L_{\alpha}^{1} (\alpha, \beta) F(z) \right)' \quad (12)$$

By differentiating both sides of (12) with respect to $z$, we get:

$$zG'(z) - G(z) = (1 + \rho) z \left( L_{\alpha}^{1} (\alpha, \beta) (zF'(z)) \right) + \rho z \left( L_{\alpha}^{1} (\alpha, \beta) (zF'(z)) \right)' \quad (13)$$

Furthermore, it follows from (11), (12) and (13) that:

$$(1 + \rho) z \left( L_{\alpha}^{1} (\alpha, \beta) f(z) \right) + \rho z \left( L_{\alpha}^{1} (\alpha, \beta) f(z) \right)'$$
Theorem 6. If
\( \left( 1 + \rho \right) z \left( L^t_a(\alpha, \beta) \left( \frac{\mu F(z) + zF'(z)}{\mu - 1} \right) \right) \)
\[ + \rho z^2 \left( L^t_a(\alpha, \beta) \left( \frac{\mu F(z) + zF'(z)}{\mu - 1} \right) \right) \]
\[ = \frac{\mu}{\mu - 1} G(z) + \frac{1}{\mu - 1} (zG'(z) - G(z)) \]
\[ = G(z) + \frac{zG'(z)}{\mu - 1}. \]  
(14)

Let \( f(z) \in \Sigma_{\alpha, \beta}^a(\rho; h) \). Then, by (14)
\[ G(z) + \frac{zG'(z)}{\mu - 1} \prec h(z) \quad (\Re \mu > 1), \]
aby using Lemma 2, we get
\[ G(z) \prec \tilde{h}(z) = (\mu - 1) z^{1-\mu} \int_0^z t^{\mu - 2} h(t) \, dt \]
\[ \prec h(z). \]
Hence, by Lemma 3, we arrive at:
\[ F(z) \in \Sigma_{\alpha, \beta}^{a,t}(\rho; \tilde{h}) \subset \Sigma_{\alpha, \beta}^a(\rho; h). \]

Theorem 7. Let \( f(z) \in \Sigma \) and \( F(z) \) be defined as in Theorem 6. If
\[(1 + \gamma) z \left( L^t_a(\alpha, \beta) F(z) \right) + \gamma z \left( L^t_a(\alpha, \beta) f(z) \right) \]
\[ \prec h(z) \quad (\gamma > 0), \]  
(15)
then \( F(z) \in \Sigma_{\alpha, \beta}^{a,t}(0, \tilde{h}) \), where \( \Re \mu > 1 \) and
\[ \tilde{h}(z) = \frac{(\mu - 1)}{\gamma} z^{1-\mu} \int_0^z t^{\mu - 1} h(t) \, dt \prec h(z). \]

Proof: Let us define,
\[ G(z) = z \left( L^t_a(\alpha, \beta) F(z) \right) \]  
(16)
Then the analytic function \( G(z) \) in the unit disk \( U \), with \( G(0) = 1 \), and
\[ zG'(z) = G(z) + z^2 \left( L^t_a(\alpha, \beta) F(z) \right)'. \]  
(17)
Making use of (11), (15), (16) and (17), we deduce that:
\[(1 + \gamma) z \left( L^t_a(\alpha, \beta) F(z) \right) + \gamma z \left( L^t_a(\alpha, \beta) f(z) \right) \]
\[ = (1 + \gamma) z \left( L^t_a(\alpha, \beta) F(z) \right) + \gamma \frac{1}{\mu - 1} \left( \mu z L^t_a(\alpha, \beta) F(z) \right) + z^2 \left( L^t_a(\alpha, \beta) F(z) \right)'. \]
\[ = G(z) + \frac{1}{\mu - 1} zG'(z) \prec h(z) \]
for \( \Re \mu > 1 \) and \( \gamma > 0 \).

Thus, an application of Lemma 2 evidently completes the proof of Theorem 7.

Theorem 8. Let \( f(z) \in \Sigma_{\alpha, \beta}^a(\rho; h) \). If the function \( f(z) \) is defined by
\[ F(z) = \frac{\mu - 1}{z^\mu} \int_0^z t^{\mu - 1} f(t) \, dt \quad (\mu > 1) \]  
(18)
then,
\[ \sigma f(\sigma z) \in \Sigma_{\alpha, \beta}^{a,t}(\rho; h) \]
where
\[ \sigma = \sigma(\mu) = \sqrt{\frac{\mu^2 - 2(\mu - 1)}{\mu - 1}} \in (0, 1). \]  
(19)
The bound \( \sigma \) is sharp when
\[ h(z) = \delta + (1 - \delta) \frac{1 + z}{1 - z} \quad (\delta \neq 1). \]  
(20)

Proof: For \( F(z) \in \Sigma_{\alpha, \beta}^a(\rho; h) \), we can verify that:
\[ F(z) = F(z) * \frac{z}{1-z} \quad \text{and} \quad zF'(z) = \]
\[ \frac{F(z) * \left( \frac{1}{z(1-z)^2} - \frac{2}{z(1-z)} \right)}{\mu - 1}. \]
Hence, by (18), we have:
\[ f(z) = \frac{\mu F(z) + zF'(z)}{\mu - 1} = (F * g)(z) \]  
(21)
\((z \in U^*, \mu > 1)\), where \( g(z) = \)
\[ \frac{1}{\mu - 1} \left( \frac{1}{(1-z)^2} + (\mu - 2) \frac{1}{z(1-z)} \right) \in \Sigma. \]  
(22)
We then show that:
\[ \Re \{ zg(z) \} > \frac{1}{2} \quad (|z| < \sigma), \]  
(23)
where \( \sigma = \sigma(\mu) \) is given by (19). Setting
\[ \frac{1}{1-z} = \Re e^{i\theta} \quad (R > 0, \ |z| = r < 1) \]
For $\mu > 1$ it follows from (29) and (32) that:

$$
2 \Re \{ z g (z) \} = \frac{2}{\mu - 1} \left[ (\mu - 2) R \cos \theta + R^2 \left( 2 \cos^2 \theta - 1 \right) \right]
$$

$$
= \frac{1}{\mu - 1} \left[ (\mu - 2) \left( 1 + R^2 \left( 1 - r^2 \right) \right) + (1 + R^2 \left( 1 - r^2 \right)^2 - 2 R^2 \right] = \frac{R^2}{\mu - 1} \left[ (1 - r^2)^2 + \frac{1}{\mu - 1} \right] + 1 = \frac{R^2}{\mu - 1} \left[ (1 - \mu) r^2 + \mu - 2 r - 1 \right] + 1.
$$

This evidently gives (31), which is equivalent to

$$
\Re \{ z \sigma g (\sigma z) \} > \frac{1}{2} \quad (z \in U). \quad (25)
$$

Let $F(z) \in \Sigma_{a, \beta}^{\Sigma, t} (\rho; h)$. Then, by using (28) and (33), an application of Lemma 4 yields:

$$
\sigma f (\sigma z) = F (z) + \sigma g (\sigma z) \in \Sigma_{a, \beta}^{\Sigma, t} (\rho; h).
$$

For $h (z)$ given by (27), we consider the function $F (z) \in \Sigma$ defined by:

$$
(1 + \lambda) z \left( L_a^t (\alpha, \beta) F(z) \right) + \lambda z^2 \left( L_a^t (\alpha, \beta) F(z) \right)' = \delta + (1 - \delta) \frac{1 + z}{1 - z}. \quad (26)
$$

$(\delta \neq 1)$. Then, by (34), (12) and (14) (used in the proof of Theorem thm1), we find that:

$$
(1 + \rho) z \left( L_a^t (\alpha, \beta) f(z) \right) + \rho z^2 \left( L_a^t (\alpha, \beta) f(z) \right)' = \delta + (1 - \delta) \frac{1 + z}{1 - z} + \frac{z}{\mu - 1} \left( \delta + (1 - \delta) \frac{1 + z}{1 - z} \right)' = \delta + (1 - \delta) \frac{1 + z}{1 - z} = \delta + \frac{(1 - \delta)(\mu + 2 z - 1 + (1 - \mu) z^2)}{(\mu - 1)(1 - z)^2} = \delta
$$

$(\sigma = -z)$.

Therefore, we conclude that the bound $\sigma = \sigma (\mu)$ cannot be increased for each $\mu (\mu > 1)$.

### 4 Inclusion relations

**Theorem 9** Let $0 \leq \rho_1 < \rho_2$. Then

$$
\Sigma_{\alpha, \beta}^{\alpha, t} (\rho_2; h) \subset \Sigma_{\alpha, \beta}^{\alpha, t} (\rho_1; h)
$$

**Proof:** Let $0 \leq \rho_1 < \rho_2$ and suppose that:

$$
g (z) < (1 + \rho_2) z \left( L_a^t (\alpha, \beta) f(z) \right) + \rho_2 z^2 \left( L_a^t (\alpha, \beta) f(z) \right)' = g(z) + \rho_2 z g' (z) < h (z).
$$

Hence an application of Lemma 2 with $m = \frac{1}{\rho_2} > 0$ yields:

$$
g (z) < h (z). \quad (29)
$$

Noting that $0 \leq \frac{\rho_1}{\rho_2} < 1$ and that $h (z)$ is convex univalent in $U$, it follows from (27), (28) and (29) that:

$$
(1 + \rho_1) z \left( L_a^t (\alpha, \beta) f(z) \right) + \rho_1 z^2 \left( L_a^t (\alpha, \beta) f(z) \right)' = \frac{\rho_1}{\rho_2} \left[ (1 + \rho_2) z \left( L_a^t (\alpha, \beta) f(z) \right) + \rho_2 z g' (z) \right] + \left( 1 - \frac{\rho_1}{\rho_2} \right) g(z) < h (z).
$$

Thus, $f(z) \in \Sigma_{\alpha, \beta}^{\alpha, t} (\rho_1; h)$ and the proof of Theorem 9 is complete.

**Theorem 10** Let,

$$
\Re \{ z \phi (\alpha_1, \alpha_2; z) \} > \frac{1}{2} \quad (30)
$$

$(z \in U; \alpha_2 \notin \{0, -1, -2, \ldots \})$,

where $\phi (\alpha_1, \alpha_2; z)$ is defined as in (??). Then,

$$
\Sigma_{\alpha_2, \beta}^{\alpha, t} (\rho; h) \subset \Sigma_{\alpha_1, \beta}^{\alpha, t} (\rho_1; h)
$$
Proof:
For \( f(z) \in \Sigma \), we can verify that:

\[
z \left( \int_a L^t_a (\alpha_1, \beta) f(z) \right)
\]

\[
= (z \bar{\alpha}(\alpha_1, \alpha_2; z) \ast \left( \int_a L^t_a (\alpha_2, \beta) f(z) \right))
\]

and

\[
z^2 \left( \int_a L^t_a (\alpha_1, \beta) f(z) \right)'
\]

\[
= \left( z \bar{\alpha}(\alpha_1, \alpha_2; z) \ast z^2 \left( \int_a L^t_a (\alpha_2, \beta) f(z) \right) \right)'.
\]

Let \( f(z) \in \Sigma^{a,t}_{\alpha_2, \beta}(\rho; h) \). Then from (31) and (32), we deduce that:

\[
(1 + \rho) z \left( \int_a L^t_a (\alpha_1, \beta) f(z) \right) + \rho z^2 \left( \int_a L^t_a (\alpha_1, \beta) f(z) \right)'
\]

\[
= \left( z \bar{\alpha}(\alpha_1, \alpha_2; z) \ast \Psi(z) \right)
\]

and

\[
\Psi(z) = (1 + \rho) z \left( \int_a L^t_a (\alpha_2, \beta) f(z) \right)
\]

\[
+ \rho z^2 \left( \int_a L^t_a (\alpha_2, \beta) f(z) \right)'
\]

In view of (30), the function \( z \bar{\alpha}(\alpha_1, \alpha_2; z) \) has the Herglotz representation:

\[
z \bar{\alpha}(\alpha_1, \alpha_2; z) = \int_{[x]} \frac{dm(x)}{1 - xz} \quad (z \in U),
\]

where \( m(x) \) is a probability measure defined on the unit circle \([x] = 1\) and

\[
\int_{[x]} dm(x) = 1.
\]

Since \( h(z) \) is convex univalent in \( U \), it follows from (33), (34) and (35) that:

\[
(1 + \rho) z \left( \int_a L^t_a (\alpha_1, \beta) f(z) \right) + \rho z^2 \left( \int_a L^t_a (\alpha_1, \beta) f(z) \right)'
\]

\[
= \int_{|x|=1} \Psi(xz) dm(x) \prec h(z)
\]

This shows that \( f(z) \in \Sigma^{a,t}_{\alpha_1, \beta}(\rho; h) \) and the theorem is proved.

**Theorem 11** Let \( 0 < \alpha_1 < \alpha_2 \). Then

\[
\Sigma^{a,t}_{\alpha_2, \beta}(\rho; h) \subset \Sigma^{a,t}_{\alpha_1, \beta}(\rho; h).
\]

**Proof:** Define,

\[
g(z) = z + \sum_{n=1}^{\infty} \left| \frac{\alpha_1}{\alpha_2} \right|^{n+1} z^{n+1}
\]

\[
(z \in \mathbb{U}; \ 0 < \alpha_1 < \alpha_2).
\]

Then,

\[
z^2 \bar{\alpha}(\alpha_1, \alpha_2; z) = g(z) \in A
\]

where \( \bar{\alpha}(\alpha_1, \alpha_2; z) \) is defined as in (??), and

\[
\frac{z}{(1 - z)^{\alpha_2}} \ast g(z) = \frac{z}{(1 - z)^{\alpha_1}}.
\]

By (37), we see that:

\[
\frac{z}{(1 - z)^{\alpha_2}} \ast g(z) \in S^*(1 - \frac{\alpha_1}{2}) \subset S^*(1 - \frac{\alpha_2}{2})
\]

for \( 0 < \alpha_1 < \alpha_2 \), which implies that:

\[
g(z) \in R \left( 1 - \frac{\alpha_2}{2} \right)
\]

Let \( f(z) \in \Sigma^{a,t}_{\alpha_2, \beta}(\rho; h) \). Then we deduce from (33), (34) and (36) that:

\[
(1 + \rho) z \left( \int_a L^t_a (\alpha_1, \beta) f(z) \right) + \rho z^2 \left( \int_a L^t_a (\alpha_1, \beta) f(z) \right)'
\]

\[
= \frac{g(z)}{z} \ast \Psi(z) = \frac{g(z) \ast (z \Psi(z))}{g(z) \ast z},
\]

\[
\Psi(z) = (1 + \rho) z \left( \int_a L^t_a (\alpha_2, \beta) f(z) \right)
\]

\[
+ \rho z^2 \left( \int_a L^t_a (\alpha_2, \beta) f(z) \right)'
\]

Since \( z \) belongs to \( S^*(1 - \frac{\alpha_2}{2}) \) and \( h(z) \) is convex univalent in \( U \), it follows from (38), (39), (40) and Lemma 5 that:

\[
(1 + \rho) z \left( \int_a L^t_a (\alpha_1, \beta) f(z) \right)
\]

\[
+ \rho z^2 \left( \int_a L^t_a (\alpha_1, \beta) f(z) \right)'
\]

Thus \( f(z) \in \Sigma^{a,t}_{\alpha_1, \beta}(\rho; h) \) and the proof is completed.
References:


