On the modes of some distributions of order k

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Abstract: This paper introduces three generalized geometric distributions: the binomial, negative binomial and Poisson distribution of the same order k. The generating functions and probability distributions of them are investigated, and then the corresponding modes of the distributions are discussed. By the Fibonacci sequence, the modes of the negative binomial distribution of order k are derived as $m_{X_{(2,2)}} = 6, 7, 8$ and $m_{X_{(2,3)}} = 13$. For the mode of the binomial distribution of order k, only a conjecture is proposed as an open question for the parameters $k = 2, n = 2\tilde{n}$ and $p = 0.5$. Finally, the modes of the Poisson distribution of order k are discussed in some cases.

Key–Words: mode; success run; probability generating function; negative binomial distribution of order k; Poisson distribution of order k; binomial distribution of order k

1 Introduction

The mode is an important statistic of probability distribution. Denote by m_X the mode of $P_n = P(X = n), n = 0, 1, \cdots$, i.e., the value of n for which P_n attains its maximum. The mode of geometric distribution of order k and some mode vectors of multivariate distributions were found by Shao *et al* in [17, 18]. the mode of the Poisson distribution of order k was solved partially by Georghiou and Philippou in [6]. To the best of our knowledge, many modes of other distributions presented in statistical literature of recent decades are still awaiting discovery. As a continuation of Shao's work in [17], the present paper discusses the mode of the negative binomial distribution of order k by the Fibonacci sequence, and investigates the properties of the binomial distribution of order k and the Poisson distribution of the same order including their modes.

2 On the mode of the negative binomial distribution of order k

Let $X_{(k,r)}$ be the number of trials until the rth occurrence of the success run with length k in Bernoulli trials with success probability p. Then we say that $X_{(k,r)}$ is distributed as the negative binomial distribution of order k with parameter vector (r, p), denoted by $NB_k(r, p)$ [9, 10]. Especially for $r = 1$, $X_{(k)} = X_{(k,1)}$ is distributed as the geometric distribution of order k with parameter p, denoted by $G_k(p)$ [1, 2, 10]. Note that $NB_k(r, p)$ is defined by success runs, the readers are referred to [4, 7, 8, 12] for more detail about the runs. The present section considers the generating function, probability distribution and modes of the distribution $NB_k(r, p)$.

Lemma 1 *[14, 16] The probability generating function of $X_{(k)}$ distributed as $G_k(p)$ is given by*

$$G_{X_{(k)}}(x) = \frac{p^k x^k - p^{k+1} x^{k+1}}{1 - x + qp^k x^{k+1}}.$$

Lemma 2 *[15] The probability generating function of $X_{(k,r)}$ distributed as $NB_k(r, p)$ is presented as*

$$G_{X_{(k,r)}}(x) = \left(\frac{p^k x^k - p^{k+1} x^{k+1}}{1 - x + qp^k x^{k+1}} \right)^r.$$

Theorem 3 *The modes of the random variable $X_{(2,2)}$ distributed as $NB_2(2, 0.5)$ are $m_{X_{(2,2)}} = 6, 7, 8$.*

Proof. Firstly, by Lemma 1, for $k = 2$ and $p = 0.5$, we consider the probability generating function

$$G_{X_{(2)}}(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x^2/4}{1 - x/2 - x^2/4},$$

where $P_n = P(X_{(2)} = n)$. Then

$$(P_0 + P_1 x + P_2 x^2 + \cdots) \cdot (1 - x/2 - x^2/4) = x^2/4.$$

Comparing the terms in both sides of the above equation, we find

$$\begin{cases} P_0 = P_1 = 0, P_2 = \dfrac{1}{4}, P_3 = \dfrac{1}{8}, \\ P_n = \dfrac{1}{2}P_{n-1} + \dfrac{1}{4}P_{n-2}, n \geq 4. \end{cases} \tag{1}$$

Let

$$P_n = F_{n-2}/2^n, n \geq 2. \tag{2}$$

Combining (1) with (2), we come to

$$\begin{cases} F_0 = F_1 = 1, \\ F_n = F_{n-1} + F_{n-2}, n \geq 2. \end{cases}$$

It implies that $\{F_n, n = 0, 1, \cdots\}$ is a Fibonacci sequence. That's why another name for the geometric distribution of order 2 with parameter $p = 0.5$ is the Fibonacci probability distribution [13].

Secondly, by Lemma 2, we present the probability generating function of $X_{(2,2)}$ as follows

$$G_{X_{(2,2)}}(x) = G^2_{X_{(2)}}(x) = (\sum_{n=2}^{\infty} P_n x^n)^2$$
$$= (\frac{F_0}{2^2}x^2 + \frac{F_1}{2^3}x^3 + \cdots)^2 = \frac{x^4}{2^4}\sum_{n=0}^{\infty} a_n x^n,$$

where

$$a_n = \sum_{i=0}^{n} \frac{F_i}{2^i} \cdot \frac{F_{n-i}}{2^{n-i}} = \frac{1}{2^n}\sum_{i=0}^{n} F_i F_{n-i} = \frac{B_n}{2^n}, n \geq 0. \tag{3}$$

Note that

$$B_n = F_0F_n + F_1F_{n-1} + \cdots + F_nF_0, n \geq 0,$$

which satisfying

$$B_n = B_{n-1} + B_{n-2} + F_n, n \geq 2. \tag{4}$$

Table 1: Numbers of F_n and B_n for $0 \leq n \leq 13$

F_0	F_1	F_2	F_3	F_4	F_5	F_6
1	1	2	3	5	8	13
B_0	B_1	B_2	B_3	B_4	B_5	B_6
1	2	5	10	20	38	71
F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}
21	34	55	89	144	233	377
B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}
130	235	420	744	1308	2285	3970

Combining Table 1 with formula (3), we get

$$a_0 = a_1 < a_2 = a_3 = a_4. \tag{5}$$

When $n \geq 4$, by (3) and (4), we have

$$\begin{aligned} & a_{n+1} - a_n = B_{n+1}/2^{n+1} - B_n/2^n \\ & = (B_{n+1} - 2B_n)/2^{n+1} \\ & = (F_{n+1} + B_{n-1} - B_n)/2^{n+1} \\ & = (F_{n+1} - F_n - B_{n-2})/2^{n+1} \\ & = (F_{n-1} - B_{n-2})/2^{n+1} \\ & = (F_{n-2} + F_{n-3} - B_{n-2})/2^{n+1} < 0. \end{aligned}$$

Then we arrive at

$$a_4 > a_5 > a_6 > \cdots . \tag{6}$$

By (5) and (6), we obtain

$$a_2 = a_3 = a_4 = \max\{a_n; n = 0, 1, 2, \cdots\}. \tag{7}$$

On the other hand,

$$G_{X_{(2,2)}}(x) = \frac{x^4}{2^4}\sum_{n=0}^{\infty} a_n x^n$$
$$= \sum_{n=0}^{\infty} \frac{a_n}{2^4} x^{n+4} = \sum_{n=4}^{\infty} \frac{a_{n-4}}{2^4} x^n.$$

Then

$$P(X_{(2,2)} = n) = \frac{a_{n-4}}{2^4}, n \geq 4.$$

By equation (7) we conclude that $P(X_{(2,2)} = n)$ attains its maximum at $X_{(2,2)} = 6, 7, 8$. This completes the proof of the theorem. □

Theorem 4 *The unique mode of the random variable $X_{(2,3)}$ distributed as $NB_2(3, 0.5)$ is $m_{X_{(2,3)}} = 13$.*

Proof. Similar to the proof of Theorem 3, let $G_{X_{(2)}}(x)$ be the probability generating function of $X_{(2)}$ distributed as $G_2(0.5)$. Then

$$G_{X_{(2)}}(x) = \sum_{n=2}^{\infty} P_n x^n = \sum_{n=2}^{\infty} \frac{F_{n-2}}{2^n} x^n,$$

where P_n and F_n are the same ones as that in the proof of Theorem 3. By Lemma 2, we obtain the probability generating function of $X_{(2,3)}$ as follows

$$\begin{aligned} & G_{X_{(2,3)}}(x) = (\sum_{n=2}^{\infty} P_n x^n)^3 \\ & = \frac{x^6}{2^6}(F_0 + \frac{F_1}{2}x + \frac{F_2}{2^2}x^2 + \cdots)^3 \\ & = \frac{x^6}{2^6}\sum_{n=0}^{\infty} c_n x^n, \end{aligned}$$

where

$$c_n = \frac{1}{2^n}\sum_{i=0}^{n} B_i F_{n-i}.$$

From Table 1, we calculate the terms

$$c_0 = 1,\ c_1 = 3/2,\ c_2 = 9/4,$$

which yielding

$$c_0 < c_1 < c_2. \tag{8}$$

When $n \geq 2$,

$$\begin{aligned}
&c_{n+1} - c_n \\
&= (\sum_{i=0}^{n+1} B_i F_{n+1-i} - 2\sum_{i=0}^{n} B_i F_{n-i})/2^{n+1} \\
&= (B_{n+1}F_0 + \sum_{i=0}^{n} B_i F_{n+1-i} - 2\sum_{i=0}^{n} B_i F_{n-i})/2^{n+1} \\
&= \{B_{n+1} + \sum_{i=0}^{n} B_i (F_{n+1-i} - F_{n-i}) \\
&- \sum_{i=0}^{n} B_i F_{n-i}\}/2^{n+1} \\
&= (B_{n+1} + \sum_{i=0}^{n-1} B_i F_{n-1-i} - \sum_{i=0}^{n} B_i F_{n-i})/2^{n+1} \\
&= \{B_{n+1} - B_n + \sum_{i=0}^{n-1} B_i (F_{n-1-i} - F_{n-i})\}/2^{n+1} \\
&= (B_{n+1} - B_n - \sum_{i=0}^{n-2} B_i F_{n-2-i})/2^{n+1} \\
&= (B_{n-1} + F_{n+1} - \sum_{i=0}^{n-2} B_i F_{n-2-i})/2^{n+1}.
\end{aligned}$$

For $n = 2, 3, 4, 5, 6$,

$$B_{n-1} + F_{n+1} - \sum_{i=0}^{n-2} B_i F_{n-2-i} > 0.$$

Then we have

$$c_2 < c_3 < c_4 < c_5 < c_6 < c_7. \tag{9}$$

When $n \geq 7$, if we define

$$\Delta_n = B_{n-1} + F_{n+1} - \sum_{i=0}^{n-2} B_i F_{n-2-i},$$

by Table 1, we may get $\Delta_7 < 0, \Delta_8 < 0$. Assume that $\Delta_n < 0, \Delta_{n+1} < 0$. Then

$$\begin{aligned}
&\Delta_{n+2} = B_{n+1} + F_{n+3} - \sum_{i=0}^{n} B_i F_{n-i} \\
&= (B_n + B_{n-1} + F_{n+1}) + (F_{n+2} + F_{n+1}) \\
&- \sum_{i=0}^{n-2} B_i (F_{n-1-i} + F_{n-2-i}) - B_{n-1} - B_n \\
&= 2F_{n+1} + F_{n+2} - \sum_{i=0}^{n-2} B_i (F_{n-1-i} + F_{n-2-i}) \\
&= 2F_{n+1} + F_{n+2} - \sum_{i=0}^{n-2} B_i F_{n-1-i} \\
&- \sum_{i=0}^{n-2} B_i F_{n-2-i} \\
&= (B_{n-1} + F_{n+1} - \sum_{i=0}^{n-2} B_i F_{n-2-i}) \\
&- \sum_{i=0}^{n-2} B_i F_{n-1-i} - B_{n-1} + F_{n+1} + F_{n+2} \\
&= \Delta_n - \sum_{i=0}^{n-1} B_i F_{n-1-i} + F_{n+1} + F_{n+2} \\
&= \Delta_n + (B_n + F_{n+2} - \sum_{i=0}^{n-1} B_i F_{n-1-i}) \\
&+ F_{n+1} - B_n \\
&= \Delta_n + \Delta_{n+1} + (F_n + F_{n-1} - B_n) < 0.
\end{aligned}$$

The inequality implies that for any a fixed $n \geq 7$, $\Delta_n < 0$, from which it follows that

$$c_7 > c_8 > c_9 > c_{10} > \cdots. \tag{10}$$

Combining (8), (9) with (10), we conclude that

$$c_7 = \max\{c_n; n = 0, 1, 2, \cdots\}.$$

Note that

$$P(X_{(2,3)} = n) = \frac{c_{n-6}}{2^6}, n \geq 6,$$

then the unique mode of $X_{(2,3)}$ is $m_{X_{(2,3)}} = 13$. □

3 On the modes of the binomial distribution of order k

Let $N_n^{(k)}$ be the number of success runs of length k in n Bernoulli trials with success probability p. The probability distribution of $N_n^{(k)}$ denoted by $B_k(n,p)$ is called the binomial distribution of order k with parameter vector (n,p) [5, 11]. Note that when $k=1$, $B_1(n,p)$ is the usual binomial distribution $B(n,p)$. Based on the distribution $NB_k(r,p)$, this section considers the probability distribution, the mean and the mode of $B_k(n,p)$.

Theorem 5 *If the random variable $X_{(k,r)}$ is distributed as $NB_k(r,p)$, then its probability distribution is given by*

$$P(X_{(k,r)}=n)=\sum_{\substack{n_1,n_2,\cdots,n_k\ni\\ n_1+2n_2+\cdots kn_k=n-kr}} \binom{n_1+n_2+\cdots+n_k+r-1}{n_1,n_2,\cdots,n_k,r-1}\left(\frac{q}{p}\right)^{\sum\limits_{j=1}^{k}n_j} p^n,$$

where $\quad n=kr,kr+1,kr+2,\cdots$.

Proof. Following Lemma 2, we get

$$\begin{aligned}
&G_{X_{(k,r)}}(x)\\
&=\left(\frac{p^kx^k}{1-qx(1+px+\cdots+p^{k-1}x^{k-1})}\right)^r\\
&=p^{kr}x^{kr}\left(\frac{1}{1-\frac{q}{p}(px+\cdots+p^kx^k)}\right)^r\\
&=p^{kr}x^{kr}\left(1-\frac{q}{p}(px+\cdots+p^kx^k)\right)^{-r}\\
&=p^{kr}x^{kr}\sum_{n=0}^{\infty}\binom{r+n-1}{r-1}\left(\frac{q}{p}\right)^n\left(\sum_{j=1}^{k}(px)^j\right)^n\\
&=p^{kr}x^{kr}\sum_{n=0}^{\infty}\binom{r+n-1}{r-1}\left(\frac{q}{p}\right)^n\times\\
&\sum_{\substack{n_1,\cdots,n_k\ni\\ n_1+\cdots+n_k=n}}\binom{n}{n_1,\cdots,n_k}(px)^{n_1+2n_2+\cdots+kn_k}\\
&=\sum_{n=0}^{\infty}\sum_{\substack{n_1,\cdots,n_k\ni\\ n_1+\cdots+n_k=n}}\binom{r+n-1}{r-1}\binom{n}{n_1,\cdots,n_k}\\
&\times\left(\frac{q}{p}\right)^n(px)^{n_1+2n_2+\cdots+kn_k+kr}\\
&=\sum_{n_1,n_2,\cdots,n_k}\binom{n_1+n_2+\cdots+n_k+r-1}{n_1,n_2,\cdots,n_k,r-1}\\
&\times\left(\frac{q}{p}\right)^{n_1+n_2+\cdots+n_k}(px)^{n_1+2n_2+\cdots+kn_k+kr}\\
&=\sum_{n=kr}^{\infty}\sum_{\substack{n_1,\cdots,n_k\ni\\ \sum_{i=1}^k in_i=n-kr}}\binom{n_1+\cdots+n_k+r-1}{n_1,\cdots,n_k,r-1}\\
&\times\left(\frac{q}{p}\right)^{n_1+n_2+\cdots+n_k}p^nx^n.
\end{aligned}$$

So, from the above we can come to the probability distribution of $X_{(k,r)}$. $\square$

Theorem 6 *The probability distribution of $B_k(n,p)$ is given by*

$$P(N_n^{(k)}=r)=\sum_{s=0}^{k-1}\sum_{\substack{m_1,m_2,\cdots,m_k\ni\\ m_1+2m_2+\cdots+km_k=n-s-kr}}\binom{m_1+\cdots+m_k+r}{m_1,m_2,\cdots,m_k,r}\left(\frac{q}{p}\right)^{m_1+\cdots+m_k}p^n,$$

where $r=0,1,\cdots,[n/k]$, and $[x]$ denotes the greatest integer not exceeding $x\in\mathbb{R}$.

Proof. Suppose $X_{(k,r)}$ is a variable distributed as $NB_k(r,p)$, the occurrence of $\{X_{(k,r+1)}=m\}$ means that either the $(m-k)$th trial is a failure and the subsequent k trials are successes, or the last $2k$ trials are successes in the m trials. Let $\odot$, $\oplus$ and $\ominus$ be a trial, a success and a failure respectively. For $m=n+k,n+k-1,\cdots,n+1$, we have the figure of the events $\{X_{(k,r+1)}=n+k-s\}$, $s=0,1,\cdots,k-1$ as follows, where $k=4$ for simplicity.

$$\begin{array}{ll}
\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot\odot\ominus}^{n\text{ trials}}\oplus\oplus\oplus\oplus & s=0\\
\odot\cdots\odot\odot\odot\odot\oplus\oplus\oplus\oplus\oplus\oplus\oplus\oplus & s=0\\
\odot\cdots\odot\odot\odot\odot\odot\odot\ominus\oplus\oplus\oplus\oplus & s=1\\
\odot\cdots\odot\odot\odot\oplus\oplus\oplus\oplus\oplus\oplus\oplus\oplus & s=1\\
\odot\cdots\odot\odot\odot\odot\odot\ominus\oplus\oplus\oplus\oplus & s=2\\
\odot\cdots\odot\odot\oplus\oplus\oplus\oplus\oplus\oplus\oplus\oplus & s=2\\
\odot\cdots\odot\odot\odot\odot\ominus\oplus\oplus\oplus\oplus & s=3\\
\odot\cdots\odot\oplus\oplus\oplus\oplus\oplus\oplus\oplus\oplus & s=3
\end{array}$$

Figure 1: The decomposition of $\{\xi_{(k,r+1)}=n+k-s\}$

Let $\{X_{(k,r+1)}=n+k-s|(k-s)\oplus\}$ denote the event that $(r+1)$ success runs of length k occur in $(n+k-s)$ trials and the last $(k-s)$ successes are deleted, where $s=0,1,\cdots,k-1$. By Figure 1, we shall find that $\bigcup_{s=0}^{k-1}\{X_{(k,r+1)}=n+k-s|(k-s)\oplus\}$ is equivalent to all the possible occurrence ways of the r success runs in n trials. Hence we have

$$\{N_n^{(k)}=r\}=\bigcup_{s=0}^{k-1}\{X_{(k,r+1)}=n+k-s|(k-s)\oplus\}.$$

Therefore

$$\begin{aligned}
&P(N_n^{(k)}=r)\\
&=\sum_{s=0}^{k-1}P\left(\{X_{(k,r+1)}=n+k-s|(k-s)\oplus\}\right)\\
&=\sum_{s=0}^{k-1}p^{-(k-s)}\cdot P\left(X_{(k,r+1)}=n+k-s\right)\\
&=\sum_{s=0}^{k-1}\sum_{\substack{m_1,m_2,\cdots,m_k\ni\\ m_1+2m_2+\cdots+km_k=n-s-kr}}\\
&\binom{m_1+m_2+\cdots+m_k+r}{m_1,m_2,\cdots,m_k,r}\left(\frac{q}{p}\right)^{\sum\limits_{j=1}^{k}m_j}p^n.
\end{aligned}$$

Theorem 6 has been proven. □

Theorem 7 *Let* $P_k^*(n,r)=P(N_n^{(k)}=r)$*. Then we have the recurrence*

$$\begin{aligned}
&P_k^*(n,r)=P_k^*(n-1,r)-qp^kP_k^*(n-k-1,r)\\
&+p^kP_k^*(n-k,r-1)-p^{k+1}P_k^*(n-k-1,r-1),
\end{aligned}$$

where $n>k$ *and* $0\le r\le[n/k]$*.*

Proof. Similar to Figure 1, we still assume that $k=4$ for simplicity. Consider the following two events

$$B_1=\{\underbrace{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot\odot}^{r\text{ runs in }(n-1)\text{ trials}}\odot}_{r\text{ runs in }n\text{ trials}}\},$$

$$B_2=\{\underbrace{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot\odot}^{(r-1)\text{ runs in }(n-1)\text{ trials}}\odot}_{r\text{ runs in }n\text{ trials}}\},$$

obviously, we have

$$\{N_n^{(k)}=r\}=B_1\cup B_2. \tag{11}$$

For the event B_1, regardless of success or failure, the nth trial doesn't change the number of success runs, hence it can be decomposed as

$$B_1=B_{11}\cup B_{12},$$

where

$$B_{11}=\{\underbrace{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot\odot}^{r\text{ runs in }(n-1)\text{ trials}}\ominus}_{r\text{ runs in }n\text{ trials}}\}$$

$$=\{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot\odot}^{r\text{ runs in }(n-1)\text{ trials}}\ominus\},$$

$$B_{12}=\{\underbrace{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot\odot}^{r\text{ runs in }(n-1)\text{ trials}}\oplus}_{r\text{ runs in }n\text{ trials}}\}$$

$$=\{\overbrace{\odot\cdots\odot\odot\odot\odot\odot\odot\odot}^{r\text{ runs in }(n-1)\text{ trials}}\oplus\}\backslash\{\overbrace{\underbrace{\odot\cdots\odot\odot}_{n-k-1\text{ trials}}\ominus\oplus\oplus\oplus\oplus}^{r\text{ runs in }(n-1)\text{ trials}}\oplus\}.$$

Hence we have

$$\begin{aligned}
P(B_1)&=P(B_{11})+P(B_{12})\\
&=qP_k^*(n-1,r)+pP_k^*(n-1,r)\\
&\quad-qp^kP_k^*(n-k-1,r)\\
&=P_k^*(n-1,r)-qp^kP_k^*(n-k-1,r). \quad (12)
\end{aligned}$$

On the other hand, for the event B_2, all the last k trials must be success, otherwise, the nth trial, i.e. the last one can't lead to a success run. So

$$B_2=\{\underbrace{\overbrace{\odot\cdots\odot\odot\odot\oplus\oplus\oplus}^{(r-1)\text{ runs in }(n-1)\text{ trials}}\oplus}_{r\text{ runs in }n\text{ trials}}\}$$

$$=\{\overbrace{\underbrace{\odot\cdots\odot\odot\odot}_{(r-1)\text{ runs}}}^{(n-k)\text{ trials}}\oplus\oplus\oplus\oplus\}\setminus\{\overbrace{\underbrace{\odot\cdots\odot\odot}_{(r-1)\text{ runs}}}^{n-k-1\text{ trials}}\oplus\oplus\oplus\oplus\oplus\}.$$

Then we get

$$P(B_2)=p^kP_k^*(n-k,r-1)-p^{k+1}P_k^*(n-k-1,r-1). \tag{13}$$

Combining (11), (12) with (13), we can derive the recurrence relation in Theorem 7. □

Theorem 8 *The mean of the random variable* $N_n^{(k)}$ *distributed as* $B_k(n,p)$ *is given by*

$$EN_n^{(k)}=\sum_{m=1}^{[n/k]}\{1+(n-mk)q\}p^{mk}.$$

Proof. Let $E_n=EN_n^{(k)}$, we find that

$$E_0=E_1=\cdots=E_{k-1}=0,E_k=p^k.$$

When $n\ge k+1$, following the equation in Theorem 7, we have

$$\begin{aligned}
&\sum_{r=1}^{[n/k]}r\cdot P_k^*(n,r)=\sum_{r=1}^{[n/k]}r\cdot P_k^*(n-1,r)\\
&-qp^k\sum_{r=1}^{[n/k]}r\cdot P_k^*(n-k-1,r)\\
&+p^k\sum_{r=1}^{[n/k]}(r-1+1)\cdot P_k^*(n-k,r-1)\\
&-p^{k+1}\sum_{r=1}^{[n/k]}(r-1+1)\cdot P_k^*(n-k-1,r-1),
\end{aligned}$$

that is

$$E_n = E_{n-1} - qp^k E_{n-k-1} \\ +p^k E_{n-k} + p^k - p^{k+1} E_{n-k-1} - p^{k+1},$$

or

$$E_n - p^k E_{n-k} = E_{n-1} - p^k E_{n-k-1} + qp^k.$$

Let $H_n = E_n - p^k E_{n-k}$. Then we get

$$H_k = E_k - p^k E_0 = p^k$$

and

$$H_n = H_{n-1} + qp^k, n \geq k+1.$$

By the above recurrence, we come to

$$H_n = H_k + (n-k)qp^k = p^k + (n-k)qp^k,$$

where $n \geq k+1$. That is

$$E_n - p^k E_{n-k} = p^k + (n-k)qp^k, n \geq k+1.$$

So, when $n \geq k$,

$$\begin{aligned} E_n &= p^k E_{n-k} + p^k + (n-k)qp^k \\ &= p^{2k} E_{n-2k} + p^k + p^{2k} + (n-k)qp^k \\ &+(n-2k)qp^{2k} = \cdots \\ &= p^{[\frac{n}{k}]k} E_{n-[\frac{n}{k}]k} + (p^k + p^{2k} + \cdots + p^{[\frac{n}{k}]k}) + \\ &(n-k)qp^k + (n-2k)qp^{2k} \cdots + (n-[\frac{n}{k}]k)qp^{[\frac{n}{k}]k} \\ &= 0 + \sum_{m=1}^{[n/k]} p^{mk} + \sum_{m=1}^{[n/k]} (n-mk)qp^{mk} \\ &= \sum_{m=1}^{[n/k]} \{1 + (n-mk)q\} p^{mk}. \end{aligned}$$

The proof is complete. □

When $p = 0.5$, $k = 2$, $n = 2\tilde{n}$, $\tilde{n} \in \mathbb{N}$, the mean of the variable $N_{2\tilde{n}}^{(2)}$ distributed as $B_2(2\tilde{n}, 0.5)$ is given by

$$EN_{2\tilde{n}}^{(2)} = \sum_{m=1}^{\tilde{n}} \frac{\tilde{n}+1-m}{2^{2m}} = \frac{\tilde{n}}{3} - \frac{1}{9}\left(1 - \frac{1}{4^{\tilde{n}}}\right).$$

Let $[x]$ be the greatest integer not exceeding $x \in \mathbb{R}$, we can show that $[EN_{2\tilde{n}}^{(2)}] = [(\tilde{n}-1)/3]$.

Conjecture 9 *If the random variable $N_{2\tilde{n}}^{(2)}$ has a distribution $B_2(2\tilde{n}, 0.5)$, then its unique mode is*

$$m_{N_{2\tilde{n}}^{(2)}} = [(\tilde{n}-1)/3].$$

For example, let $n = 2\tilde{n} = 10$, by Theorem 6,

$$\begin{aligned} &P(N_{10}^{(2)} = 0) = \frac{1}{2^{10}}\binom{10}{10,0,0} + \frac{1}{2^{10}}\binom{9}{8,1,0} \\ &+ \cdots + \frac{1}{2^{10}}\binom{5}{0,5,0} + \frac{1}{2^{10}}\binom{9}{9,0,0} \\ &+\frac{1}{2^{10}}\binom{8}{7,1,0} + \cdots + \frac{1}{2^{10}}\binom{5}{1,4,0} \\ &= \frac{144}{1024}, \end{aligned}$$

$$\begin{aligned} &P(N_{10}^{(2)} = 1) = \frac{1}{2^{10}}\binom{9}{8,0,1} + \frac{1}{2^{10}}\binom{8}{6,1,1} \\ &+ \cdots + \frac{1}{2^{10}}\binom{5}{0,4,1} + \frac{1}{2^{10}}\binom{8}{7,0,1} \\ &+\frac{1}{2^{10}}\binom{7}{5,1,1} + \frac{1}{2^{10}}\binom{6}{3,2,1} + \frac{1}{2^{10}}\binom{5}{1,3,1} \\ &= \frac{365}{1024}, \end{aligned}$$

$$\begin{aligned} &P(N_{10}^{(2)} = 2) = \frac{1}{2^{10}}\binom{8}{6,0,2} + \frac{1}{2^{10}}\binom{7}{4,1,2} \\ &+\frac{1}{2^{10}}\binom{6}{2,2,2} + \frac{1}{2^{10}}\binom{5}{0,3,2} + \frac{1}{2^{10}}\binom{7}{5,0,2} \\ &+\frac{1}{2^{10}}\binom{6}{3,1,2} + \frac{1}{2^{10}}\binom{5}{1,2,2} \\ &= \frac{344}{1024}, \end{aligned}$$

$$\begin{aligned} &P(N_{10}^{(2)} = 3) = \frac{1}{2^{10}}\binom{7}{4,0,3} + \frac{1}{2^{10}}\binom{6}{2,1,3} \\ &+\frac{1}{2^{10}}\binom{5}{0,2,3} + \frac{1}{2^{10}}\binom{6}{3,0,3} + \frac{1}{2^{10}}\binom{5}{1,1,3} \\ &= \frac{145}{1024}, \end{aligned}$$

$$\begin{aligned} &P(N_{10}^{(2)} = 4) = \frac{1}{2^{10}}\binom{6}{2,0,4} + \frac{1}{2^{10}}\binom{5}{0,1,4} \\ &+\frac{1}{2^{10}}\binom{5}{1,0,4} = \frac{25}{1024}, \end{aligned}$$

$$P(N_{10}^{(2)} = 5) = \frac{1}{2^{10}}\binom{5}{0,0,5} = \frac{1}{1024},$$

So, we have

$$P(N_{10}^{(2)} = 1) = \max_{0 \leq r \leq 5} \{P(N_{10}^{(2)} = r)\},$$

which yielding $m_{N_{10}^{(2)}} = 1$. On the other hand, $[(\tilde{n} - 1)/3] = 1$, we have $m_{N_{2\tilde{n}}^{(2)}} = [(\tilde{n} - 1)/3]$.

In Figure 2, the horizontal axis denotes the numbers of success run in $B_2(n, 0.5)$ and the vertical axis denotes the corresponding values of the probability. We observe clearly that when $n = 2 \times 5, 2 \times 8$ and 2×10, the modes of them are $[(5-1)/3] = 1$, $[(8-1)/3] = 2$ and $[(10-1)/3] = 3$ respectively.

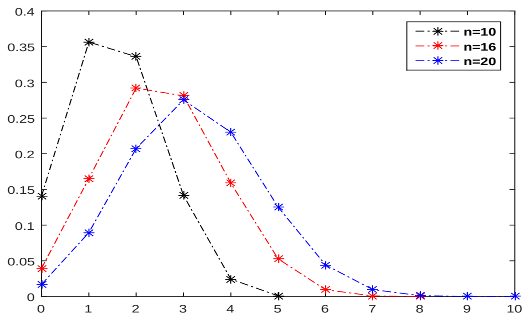

Figure 2: The The probability distributions of $B_2(n, 0.5)$

4 On the mode of the Poisson distribution of order k

This present section investigates the mode of the Poisson distribution of order k in some cases. Theorem 10 is inspired by Dai and Hou's work [3], where they derived the usual Poisson distribution $P(\lambda)$ from a negative binomial distribution.

Theorem 10 *In $NB_k(r,p)$, the variable $\eta_{(k,r)} = X_{(k,r)} - kr$ is the number of trials removing the r success runs. Let $q \to 0$ and $rq \to \lambda(\lambda > 0)$ as $r \to \infty$. Then the random variable $\eta_{(k)} = \eta_{(k,\infty)}$ is said to have the Poisson distribution of order k with parameter λ, to be denoted by $P_k(\lambda)$. The probability generating function of $\eta_{(k)}$ is given by*

$$G_{\eta_{(k)}}(x) = e^{\lambda(x+x^2+\cdots+x^k-k)}.$$

Proof. Combining the definition of probability generating function of $\eta_{(k,r)}$ with Lemma 2, we have

$$G_{\eta_{(k,r)}}(x) = \sum_{m=0}^{\infty} P(\eta_{(k,r)} = m)x^m$$
$$= \sum_{m=0}^{\infty} P(X_{(k,r)} = m + kr)x^m$$
$$= x^{-kr} \sum_{m=0}^{\infty} P(X_{(k,r)} = m + kr)x^{m+kr}$$
$$= x^{-kr} G_{X_{(k,r)}}(x) = \frac{p^{kr}(1-px)^r}{(1 - x + qp^k x^{k+1})^r}$$
$$= \left((1-q)^{\frac{-1}{q}}\right)^{-krq} \times$$
$$\left(\left(1 - qx\sum_{m=0}^{k-1}(px)^m\right)^{\frac{-1}{qx\sum_{m=0}^{k-1}(px)^m}}\right)^{rqx\sum_{m=0}^{k-1}(px)^m}.$$

Let $q \to 0$ and $rq \to \lambda(\lambda > 0)$ as $r \to \infty$. Then we get the probability generating function of $\eta_{(k)} = \eta_{(k,\infty)}$ as follows

$$G_{\eta_{(k)}}(x) = G_{\eta_{(k,\infty)}}(x) = \lim_{r\to\infty} G_{\eta_{(k,r)}}(x)$$
$$= e^{\lambda(x+x^2+\cdots+x^k-k)}.$$

Thus the proof is complete. □

Theorem 11 *Let $\eta_{(k)}$ be a random variable distributed as $P_k(\lambda)$. Then*

$$P(\eta_{(k)} = m) = \sum_{\substack{m_1,m_2,\cdots,m_k \ni \\ m_1+2m_2+\cdots+km_k=m}} \frac{\lambda^{m_1+m_2+\cdots+m_k}}{m_1!m_2!\cdots m_k!} e^{-\lambda k},$$

where $m = 0, 1, \cdots$.

Proof. Following Theorem 10, we get

$$G_{\eta_{(k)}}(x) = e^{\lambda(x_1+x_2+\cdots+x_k-k)}$$
$$= e^{-\lambda k} \sum_{m=0}^{\infty} \frac{\lambda^m (x + x^2 + \cdots + x^k)^m}{m!}$$
$$= e^{-\lambda k} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \sum_{\substack{m_1,m_2,\cdots,m_k \ni \\ m_1+2m_2+\cdots+km_k=m}} \binom{m}{m_1, m_2, \cdots, m_k} x^{m_1+2m_2+\cdots+km_k}$$
$$= \sum_{m=0}^{\infty} \sum_{\substack{m_1,m_2,\cdots,m_k \ni \\ m_1+m_2+\cdots+m_k=m}} \frac{\lambda^m e^{-\lambda k} x^{m_1+2m_2+\cdots+km_k}}{m_1!m_2!\cdots m_k!}$$
$$= \sum_{m=0}^{\infty} \sum_{\substack{m_1,m_2,\cdots,m_k \ni \\ m_1+2m_2+\cdots+km_k=m}} \frac{\lambda^{\sum_{j=1}^{k} m_j} e^{-\lambda k}}{m_1!m_2!\cdots m_k!} x^m.$$

Therefore,

$$P(\eta_{(k)} = m) = \sum_{\substack{m_1,m_2,\cdots,m_k \ni \\ m_1+2m_2+\cdots+km_k=m}} \frac{\lambda^{m_1+m_2+\cdots+m_k}}{m_1!m_2!\cdots m_k!} e^{-\lambda k},$$

where $m = 0, 1, \cdots$. □

Theorem 12 *Let $\eta_{(2)}$ be a random variable distributed as $P_2(\lambda)$. Then the mode of it is $m_{\eta_{(2)},\lambda} = 0$ if $0 < \lambda < \sqrt{3} - 1$, $m_{\eta_{(2)},\lambda} = 0$ or 2 if $\lambda = \sqrt{3} - 1$, $m_{\eta_{(2)},\lambda} = 2$ if $\sqrt{3} - 1 < \lambda \leq 1$.*

Proof. By Theorem 11, we have

$$P_m = P(\eta_{(2)} = m) = \sum_{\substack{m_1, m_2 \ni \\ m_1 + 2m_2 = m}} \frac{\lambda^{m_1+m_2} e^{-2\lambda}}{m_1! m_2!},$$

$$P_0 = e^{-2\lambda}, P_1 = e^{-2\lambda}\lambda, P_2 = e^{-2\lambda}(\frac{\lambda^2}{2} + \lambda). \quad (14)$$

Note that $0 < \lambda \leq 1$, when $n \geq 1$, we have

$$\begin{aligned} & P_{2n} - P_{2n+1} \\ &= \sum_{i=0}^{n} \frac{\lambda^{2n-i} e^{-2\lambda}}{(2n-2i)! i!} - \sum_{i=0}^{n} \frac{\lambda^{2n+1-i} e^{-2\lambda}}{(2n+1-2i)! i!} \\ &\geq \sum_{i=0}^{n} \frac{\lambda^{2n-i} e^{-2\lambda}}{(2n-2i)! i!} - \sum_{i=0}^{n} \frac{\lambda^{2n-i} e^{-2\lambda}}{(2n+1-2i)! i!} \\ &= \sum_{i=0}^{n} (\frac{\lambda^{2n-i} e^{-2\lambda}}{(2n-2i)! i!} - \frac{\lambda^{2n-i} e^{-2\lambda}}{(2n+1-2i)! i!}) > 0. \end{aligned} \quad (15)$$

$$\begin{aligned} & P_{2n+1} - P_{2n+2} = \\ &= \sum_{i=0}^{n} \frac{\lambda^{2n+1-i} e^{-2\lambda}}{(2n+1-2i)! i!} - \sum_{i=0}^{n+1} \frac{\lambda^{2n+2-i} e^{-2\lambda}}{(2n+2-2i)! i!} \\ &= \sum_{i=0}^{n-1} (\frac{\lambda^{2n+1-i} e^{-2\lambda}}{(2n+1-2i)! i!} - \frac{\lambda^{2n+2-i} e^{-2\lambda}}{(2n+2-2i)! i!}) \\ &+ e^{-2\lambda} \left(\frac{\lambda^{n+1}}{1! n!} - \frac{\lambda^{n+2}}{2! n!} - \frac{\lambda^{n+1}}{0!(n+1)!} \right) > 0. \end{aligned} \quad (16)$$

By (15) and (16), when $0 < \lambda \leq 1$, we get

$$P_2 > P_3 > P_4 > \cdots. \quad (17)$$

Combining (14) and (17), when $\lambda = \sqrt{3} - 1$, $P_0 = P_2 = \max\{P_m, m \geq 0\}$, this means $m_{\eta_{(2)},\lambda} = 0, 2$; when $0 < \lambda < \sqrt{3} - 1$, $P_0 = \max\{P_m, m \geq 0\}$, it yields $m_{\eta_{(2)},\lambda} = 0$; when $\sqrt{3} - 1 < \lambda \leq 1$, $P_2 = \max\{P_m, m \geq 0\}$, that is $m_{\eta_{(2)},\lambda} = 2$. □

Theorem 13 *Let $\eta_{(2)}$ be a random variable distributed as $P_2(\lambda)$. Then the mode of it is $m_{\eta_{(2)},\lambda} = (3\lambda - 1)$ if $\lambda > 1$ and $\lambda \in \mathbb{N}$.*

Proof. By Theorem 10, for $k = 2$, we have

$$G(x) = G_{\eta_{(2)}}(x) = e^{\lambda(x + x^2 - 2)},$$

differentiating $G(x)$, we get

$$G'(x) = \lambda(1 + 2x) G(x).$$

Differentiating $(n - 1)$ times both sides of the above equation with respect to x, then setting $x = 0$, we arrive at

$$G^{(n)}(0) = \lambda G^{(n-1)}(0) + 2\lambda(n-1) G^{(n-2)}(0),$$

We employ the fact $P_n = G^{(n)}(0)/n!$ and the above to obtain the recurrence

$$n P_n = \lambda(P_{n-1} + 2P_{n-2}). \quad (18)$$

Assume that $m_{\eta_{(2)},\lambda} = n^*$, so by (18),

$$n^* P_{n*} = \lambda(P_{n^*-1} + 2P_{n^*-2}) \leq 3\lambda P_{n^*},$$

which yielding

$$n^* \leq 3\lambda. \quad (19)$$

Let $\Delta_0 = P_0 > 0$, $\Delta_n = P_n - P_{n-1}, n \geq 1$. Then

$$\begin{cases} \Delta_1 = P_1 - P_0 = (\lambda - 1) e^{-2\lambda} > 0, \\ \Delta_2 = P_2 - P_1 = \dfrac{\lambda^2}{2} e^{-2\lambda} > 0. \end{cases} \quad (20)$$

By (18), we come to

$$\Delta_{n+2} = \frac{\lambda(\lambda + n)\Delta_n}{(n+1)(n+2)} + \frac{\lambda(3\lambda - n - 4)}{(n+1)(n+2)} P_{n-1}. \quad (21)$$

So, when $1 \leq n \leq 3\lambda - 4$, we have $\Delta_{n+2} > 0$, that is

$$\Delta_3 > 0, \Delta_4 > 0, \cdots, \Delta_{3\lambda-2} > 0. \quad (22)$$

Combining (20) with (22), we obtain

$$P_0 < P_1 < \cdots < P_{3\lambda-2},$$

which yielding

$$3\lambda - 2 \leq n^*. \quad (23)$$

Together with (19) and (23), we get

$$3\lambda - 2 \leq n^* \leq 3\lambda. \quad (24)$$

By (18), we arrive at

$$n\Delta_n = (\lambda - n)\Delta_{n-1} + (3\lambda - n) P_{n-2}. \quad (25)$$

Setting $n = 3\lambda$ in (25), we get

$$\Delta_{3\lambda} = -2\Delta_{3\lambda-1}. \quad (26)$$

Setting $n = 3\lambda - 3, 3\lambda - 5$ and $3\lambda - 7$ in (21) respectively, we get

$$\begin{cases} \Delta_{3\lambda-1} = \dfrac{\lambda(4\lambda-3)\Delta_{3\lambda-3} - \lambda P_{3\lambda-4}}{(3\lambda-2)(3\lambda-1)}, \\ \Delta_{3\lambda-3} = \dfrac{\lambda(4\lambda-5)\Delta_{3\lambda-5} + \lambda P_{3\lambda-6}}{(3\lambda-4)(3\lambda-3)}, \\ \Delta_{3\lambda-5} = \dfrac{\lambda(4\lambda-7)\Delta_{3\lambda-7} + 3\lambda P_{3\lambda-8}}{(3\lambda-6)(3\lambda-5)}. \end{cases} \tag{27}$$

By (18), we get

$$\begin{cases} P_{3\lambda-4} = \dfrac{\lambda(7\lambda-10)P_{3\lambda-6} + 2\lambda^2 P_{3\lambda-7}}{(3\lambda-4)(3\lambda-5)}, \\ P_{3\lambda-6} = \dfrac{\lambda P_{3\lambda-7} + 2\lambda P_{3\lambda-8}}{3\lambda-6}. \end{cases} \tag{28}$$

Together with (27) and (28), we have

$$\begin{aligned} &\frac{(3\lambda-1)\cdots(3\lambda-6)}{\lambda^3}\Delta_{3\lambda-1} \\ &= (64\lambda^3 - 267\lambda^2 + 360\lambda - 156)\Delta_{3\lambda-7} \\ &\quad + (3\lambda^2 + 24\lambda - 36)P_{3\lambda-8} > 0. \end{aligned}$$

Combining (26) with (29), we have $\Delta_{3\lambda-1} < 0$ and $\Delta_{3\lambda} > 0$, this means that $P_{3\lambda-1} > P_{3\lambda-2}$ and $P_{3\lambda-1} > P_{3\lambda}$, noting that the mode n^* satisfies (24), so we get the mode $m_{\eta_{(2)},\lambda} = n^* = 3\lambda - 1$. $\square$

At the end of this section, we give the probability distributions of $P_2(\lambda)$ when $\lambda = 0.5$ and $\lambda = 1, 2, 3$ in Figure 3, where the corresponding modes $m_{\eta_{(2)},0.5} = 0$ and $m_{\eta_{(2)},\lambda} = 3\lambda - 1 = 2, 5, 8$ are obvious.

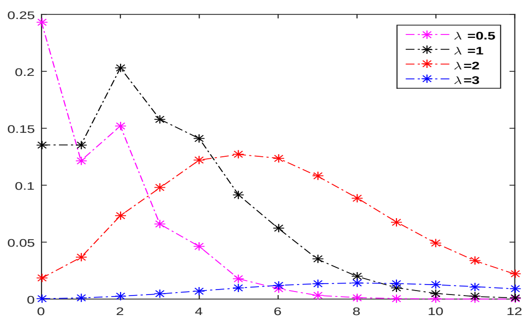

Figure 3: The probability distributions of $P_2(\lambda)$

5 Conclusion

In section 2, we discuss the mode of the random variable $X_{(k,r)}$ which has a distribution $NB_k(r, p)$. By the Fibonacci sequence, we obtain $m_{X_{(2,2)}} = 6, 7, 8$ and $m_{X_{(2,3)}} = 13$ if the parameter $p = 0.5$. In section 3, we investigate the probability distribution and mean of the variable $N_n^{(k)}$ distributed as $B_k(n, p)$ and propose the conjecture on $B_2(2\tilde{n}, 0.5)$, i.e. $m_{N_{2\tilde{n}}^{(2)}} = [(\tilde{n} - 1)/3]$. In section 4, based on $NB_k(r, p)$, we consider the probability generating function and probability distribution of $P_k(\lambda)$, furthermore, we obtain its modes in some cases: (1) $m_{\eta_{(2)},\lambda} = 0$ if $0 < \lambda < \sqrt{3} - 1$; $m_{\eta_{(2)},\lambda} = 0$ or 2 if $\lambda = \sqrt{3} - 1$; $m_{\eta_{(2)},\lambda} = 2$ if $\sqrt{3} - 1 < \lambda \le 1$. (2) $m_{\eta_{(2)},\lambda} = 3\lambda - 1$ if $\lambda \in \mathbb{N}$ and $\lambda > 1$. It's still an open question for all other cases.

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