Discrete Operators in Canonical Domains

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Abstract: In this paper, we consider a certain class of discrete pseudo-differential operators in a sharp convex cone and describe their invertibility conditions in L_2-spaces. For this purpose we introduce a concept of periodic wave factorization for elliptic symbol and show its applicability for the studying.

Key–Words: Discrete operator, Multidimensional periodic Riemann problem, Periodic wave factorization, Invertibility

1 Introduction

A classical pseudo-differential operator in Euclidean space $\mathbb{R}^m$ is defined by the formula [1, 2, 3, 4]

$$(Au)(x) = \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} \widetilde{A}(x,\xi)e^{i(\xi-y)}\tilde{u}(\xi)d\xi dy,$$

where the sign $\sim$ over a function denotes its discrete Fourier transform

$$\widetilde{u}(\xi) = \int\limits_{\mathbb{R}^m} u(x)e^{ix\cdot\xi}dx.$$

1.1 Multidimensional Fourier series and symbols

Given function u_d of a discrete variable $\tilde{x} \in \boldsymbol{Z}^m$ we define its discrete Fourier transform by the series

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x}\in\boldsymbol{Z}^m} e^{i\tilde{x}\cdot\xi}u(\tilde{x}),$$

$$\xi \in \boldsymbol{T}^m = [-\pi,\pi]^m,$$

where partial sums are taken over cubes

$$Q_N = \{\tilde{x} \in Z^m : \tilde{x} = (\tilde{x}_1,\cdots,\tilde{x}_m),$$

$$\max_{1\le k\le m}|\tilde{x}_k| \le N\}.$$

Let $D \subset \mathbb{R}^m$ be a sharp convex cone, $D_d \equiv D\cap\boldsymbol{Z}^m$, and $L_2(D_d)$ be a space of functions of discrete variable defined on D_d, and $A(\tilde{x})$ be a given function of a discrete variable $\tilde{x} \in \boldsymbol{Z}^m$. We consider the following types of operators

$$(A_d u_d)(\tilde{x}) = \int_{\boldsymbol{T}^m}\sum_{\tilde{y}\in D_d} e^{i(\tilde{y}-\tilde{x})\cdot\xi}\widetilde{A}(\xi)\tilde{u}_d(\xi)d\xi, \quad (1)$$

$$\tilde{x} \in D_d,$$

and introduce the function

$$\widetilde{A}_d(\xi) = \sum_{\tilde{x}\in\boldsymbol{Z}^m} e^{i\tilde{x}\cdot\xi}A(\tilde{x}), \quad \xi \in \boldsymbol{T}^m.$$

Definition 1 *The function $\widetilde{A}_d(\xi)$ is called a symbol of the operator A_d, and this symbol is called an elliptic symbol if $\widetilde{A}_d(\xi) \neq 0, \forall\xi \in \boldsymbol{T}^m$.*

Our main goal is describing a periodic variant of wave factorization for an elliptic symbol [9] and showing its usability for studying invertibility for the operator A_d.

1.2 Discrete projection operators

Let us denote P_{D_d} projection operator on D_d, $P_{D_d}: L_2(\boldsymbol{Z}^m) \to L_2(D_d)$ so that for arbitrary function $u_d \in L_2(\boldsymbol{Z}^m)$

$$(P_{D_d}u_d)(\tilde{x}) = \begin{cases} u_d(\tilde{x}), & \tilde{x} \in D_d \\ 0, & \tilde{x} \notin D_d. \end{cases}$$

1.2.1 Periodic Cauchy kernel

If we consider a half-space case, then the Fourier image of the operator P_{D_d} is evaluated [10, 11, 12] and we'll demonstrate it in the following

Example 2 *If $D = \mathbb{R}^m_+$ then*

$$(F_d P_{D_d} u_d)(\xi',\xi_m) =$$

$$\frac{1}{4\pi i}\lim_{\tau\to 0+}\int\limits_{-\pi}^{\pi} u_d(\xi',\eta_m)\cot\frac{\xi_m-\eta_m+i\tau}{2}d\eta_m.$$

1.2.2 Periodic Bochner kernel

If D is a sharp convex cone $C_+^a = \{\tilde{x} \in \boldsymbol{Z}^m : \tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_m), \tilde{x}_m > a|\tilde{x}'|, \tilde{x}' = (\tilde{x}_1, \cdots, \tilde{x}_{m-1}), a > 0\}$ then we introduce the function

$$B_d(z) = \sum_{\tilde{x} \in D_d} e^{i\tilde{x}\cdot z}, \quad z = \xi + i\tau, \quad \xi \in \boldsymbol{T}^m, \quad \tau \in C_+^a,$$

and define the operator

$$(B_d u)(\xi) = \lim_{\tau \to 0} \int_{\boldsymbol{T}^m} B_d(z - \eta) u_d(\eta) d\eta.$$

Lemma 3 *For arbitrary $u_d \in L_2(\boldsymbol{Z}^m)$ the following property*

$$F_d P_{D_d} u_d = B_d F_d u_d$$

holds.

Proof: Let $\chi_+(\tilde{x})$ be an indicator of the set D_d. Thus

$$(P_{D_d} u_d)(\tilde{x}) = \chi_+(\tilde{x}) \cdot u_d(\tilde{x}).$$

Further, since the function $\chi_+(\tilde{x})$ is not summable, we can't apply directly a convolution property of the Fourier transform. We choose the function $e^{i\tilde{x}\cdot\tau}$ so the product $\chi_+(\tilde{x}) e^{i\tilde{x}\cdot\tau}$ will be summable for some admissible τ. Taking into account a forthcoming passing to a limit under $\tau \to 0+$ we have

$$F_d(\chi_+(\tilde{x}) e^{i\tilde{x}\cdot\tau}) = B_d(z).$$

Thus we can use the Fourier transform obtaining convolution of functions $B_d(z)$ and $\widetilde{u}_d(\xi)$. It is left passing to a limit. $\square$

2 Multidimensional periodic Riemann boundary value problem

2.1 A half-space case and a periodic one-dimensional Riemann boundary value problem [10, 11, 12]

For $D = \mathbb{R}_+^m$ we will remind some author's constructions for discrete equations in a half-space. We have

$$B_d(z) = \cot\frac{z}{2}, \quad z = (\xi', \xi_m + i\tau),$$

$$\xi' = (\xi_1, \cdots, \xi_{m-1}), \tau > 0.$$

Thus (see example 1) we use a periodic one-dimensional Riemann problem with a parameter $\xi' \in \boldsymbol{T}^{m-1}$ which is the following. Finding a pair of functions $\Phi^\pm(\xi', \xi_m)$ which are boundary values of holomorphic in half-strips $\Pi_\pm = \{z \in \boldsymbol{C} : z = \xi_m \pm i\tau, \tau > 0\}$ such that these are satisfied a linear relation

$$\Phi^+(\xi)(\xi', \xi_m) = G(\xi', \xi_m)\Phi^-(\xi)(\xi', \xi_m) + g(\xi),$$

$$\xi \in \boldsymbol{T}^m,$$

for almost all $\xi' \in \boldsymbol{T}^{m-1}$, where $G(\xi), g(\xi)$ are given periodic functions.

2.2 Essential multidimensional case

Let $\overset{*}{D}$ be a conjugate cone for D i.e.

$$\overset{*}{D} = \{x \in \mathbb{R}^m : x \cdot y > 0, \ y \in D\},$$

and $T(\overset{*}{D}) \subset \boldsymbol{C}^m$ be a set of the type $\boldsymbol{T}^m + i \overset{*}{D}$. For $\boldsymbol{T}^m \equiv \mathbb{R}^m$ such a domain of multidimensional complex space is called a radial tube domain over the cone $\overset{*}{D}$ [7, 8, 9].

Let us define the subspace $A(\boldsymbol{T}^m) \subset L_2(\boldsymbol{T}^m)$ consisting of functions which admit a holomorphic continuation into $T(\overset{*}{D})$ and satisfy the following condition

$$\sup_{\tau \in \overset{*}{D}} \int_{\boldsymbol{T}^m} |\tilde{u}_d(\xi + i\tau)|^2 d\xi < +\infty. \qquad (2)$$

In other words, the space $A(\boldsymbol{T}^m) \subset L_2(\boldsymbol{T}^m)$ consists of boundary values of holomorphic in $T(\overset{*}{D})$ functions.

Let us denote

$$B(\boldsymbol{T}^m) = L_2(\boldsymbol{T}^m) \ominus A(\boldsymbol{T}^m),$$

so that $B(\boldsymbol{T}^m)$ is a direct complement of $A(\boldsymbol{T}^m)$ in $L_2(\boldsymbol{T}^m)$.

2.2.1 A jump problem

We formulate the problem by the following way: finding a pair of functions $\Phi^\pm, \Phi^+ \in A(\boldsymbol{T}^m), \Phi^- \in B(\boldsymbol{T}^m)$, such that

$$\Phi^+(\xi) - \Phi^-(\xi) = g(\xi), \quad \xi \in \boldsymbol{T}^m, \qquad (3)$$

where $g(\xi) \in L_2(\boldsymbol{T}^m)$ is given.

Lemma 4 *The operator $B_d : L_2(\boldsymbol{T}^m) \to A(\boldsymbol{T}^m)$ is a bounded projector. A function $u_d \in L_2(D_d)$ iff its Fourier transform $\tilde{u}_d \in A(\boldsymbol{T}^m)$.*

Proof: According to standard properties of the discrete Fourier transform F_d, we have

$$F_d(\chi_+(\tilde{x}) u_d(\tilde{x})) = \lim_{\tau \to 0} \int_{\boldsymbol{T}^m} B_d(z - \eta) \widetilde{u}_d(\eta) d\eta,$$

where $\chi_+(\tilde{x})$ is an indicator of the set D_d. It implies a boundedness of the operator B_d. The second assertion follows from holomorphic properties of the kernel $B_d(z)$. In other words for arbitrary function $v \in A(\boldsymbol{T}^m)$ we have

$$v(z) = \int_{\boldsymbol{T}^m} B_d(z-\eta)v(\eta)d\eta, \quad z \in T(\overset{*}{D}).$$

It is an analogue of the Cauchy integral formula. □

Theorem 5 *The jump problem has unique solution for arbitrary right-hand side from $L_2(\boldsymbol{T}^m)$.*

Proof: Indeed it is equivalent to one-to-one representation of the space $L_2(D_d)$ as a direct sum of two subspaces. If we'll denote $\chi_+(x), \chi_-(x)$ indicators of discrete sets $D_d, \boldsymbol{Z}^m \setminus D_d$ respectively then the following representation

$$u_d(\tilde{x}) = \chi_+(\tilde{x})u_d(\tilde{x}) + \chi_-(\tilde{x})u_d(\tilde{x})$$

is unique and holds for arbitrary function $u_d \in L_2(\boldsymbol{Z}^m)$. After applying the discrete Fourier transform we have

$$F_d u_d = F_d(\chi_+ u_d) + F_d(\chi_- u_d),$$

where $F_d(\chi_+ u_d) \in A(\boldsymbol{T}^m)$ according to lemma 2, and thus $F_d(\chi_- u_d) = F_d u_d - F_d(\chi_+ u_d) \in B(\boldsymbol{T}^m)$ because $F_d u_d \in L_2(\boldsymbol{T}^m)$. □

Example 6 *If $m = 2$ and C^2_+ is the first quadrant in a plane then a solution of the jump problem is given by formulas*

$$\Phi^+(\xi) = \frac{1}{(4\pi i)^2} \lim_{\tau \to 0} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cot\frac{\xi_1 + i\tau_1 - t_1}{2} \times$$

$$\cot\frac{\xi_2 + i\tau_2 - t_2}{2} g(t_1,t_2)dt_1dt_2$$

$$\Phi^-(\xi) = \Phi^+(\xi) - g(\xi), \quad \tau = (\tau_1,\tau_2) \in C^2_+.$$

2.2.2 A general statement

It looks as follows. Finding a pair of functions $\Phi^\pm, \Phi^+ \in A(\boldsymbol{T}^m), \Phi^- \in B(\boldsymbol{T}^m)$, such that

$$\Phi^+(\xi) = G(\xi)\Phi^-(\xi) + g(\xi), \quad \xi \in \boldsymbol{T}^m, \tag{4}$$

where $G(\xi), g(\xi)$ are given periodic functions. If $G(\xi) \equiv 1$ we have the jump problem 3).

Like classical studies [5, 6] we want to use a special representation for an elliptic symbol to solve the problem (4).

2.2.3 Associated singular integral equation

We can easily obtain so-called characteristic singular integral equation associated with multidimensional periodic Riemann boundary value problem (4).

Let us denote $Q_{D_d} = I - P_{D_d}$ and consider so-called paired operator composed by two operators $A^{(1)}_d, A^{(2)}_d$ of the following type

$$A^{(1)}_d P_{D_d} + A^{(2)}_d Q_{D_d} : L_2(\boldsymbol{Z}^m) \to L_2(\boldsymbol{Z}^m) \tag{5}$$

One can easily obtain the following

Property 7 *The invertibility of the operator (1) in the space $L_2(D_d)$ is equivalent to invertibility of the operator (5) in the space $L_2(\boldsymbol{Z}^m)$ with $A^{(1)}_d = A_d, A^{(2)}_d = I$.*

The Fourier image for the operator (5) is the following operator

$$\tilde{u}_d(\xi) \longmapsto ((A^{(1)}_d(\xi)B_d + A^{(2)}_d(\xi)(I - B_d)\tilde{u}_d)(\xi) \tag{6}$$

If $D = \mathbb{R}^m_+$ then the operator (6) is a one-dimensional singular integral operator with periodic Cauchy kernel and a parameter ξ' [10, 11, 12].

3 Periodic wave factorization

Definition 8 *Periodic wave factorization for elliptic symbol $\tilde{A}(\xi)$ is called its representation in the form*

$$\tilde{A}_d(\xi) = \tilde{A}_{\neq}(\xi)\tilde{A}_{=}(\xi)$$

where the factors $A^{\pm 1}_{\neq}(\xi), A^{\pm 1}_{=}(\xi)$ admit bounded holomorphic continuation into domains $T(\pm \overset{*}{D})$.

3.1 Sufficient conditions

We'll give here certain sufficient conditions for an existence of the periodic wave factorization for an elliptic symbol.

Theorem 9 *Let an elliptic symbol $\tilde{A}_d(\xi) \in C(\boldsymbol{T}^m)$ be a such that*

$$supp\, F^{-1}_d(\ln \tilde{A}_d(\xi)) \subset D_d \cup (-D_d), \tag{7}$$

$$\int_{-\pi}^{\pi} d \arg \tilde{A}_d(\cdots,\xi_k,\cdots) = 0,\ k = 1,\cdots,m. \tag{8}$$

Then the symbol $\tilde{A}_d(\xi)$ admits the wave factorization.

Proof: If we start from equality

$$\tilde{A}_d(\xi) = \tilde{A}_{\neq}(\xi)\tilde{A}_{=}(\xi)$$

then by logarithm we obtain

$$\ln \tilde{A}_d(\xi) = \ln \tilde{A}_{\neq}(\xi) + \ln \tilde{A}_{=}(\xi)$$

and we have a special kind of a jump problem.

Namely if we will denote by $A_1(\boldsymbol{T}^m)$ a subspace of the space $L_2(\boldsymbol{T}^m)$ consisting of functions which admit a holomorphic continuation into $T(-\overset{*}{D})$ and satisfy the condition (2) for $\tau \in -\overset{*}{D}$. So evidently we speak on a possibility of decomposition of the function $\ln \tilde{A}_d(\xi)$ into two summands one of which belongs to the space $A(\boldsymbol{T}^m)$ and the second one belongs to the space $A_1(\boldsymbol{T}^m)$. Let us denote

$$F^{-1}(\ln \tilde{A}_d(\xi)) \equiv v(x).$$

If $supp\ v \subset D_d \cup (-D_d)$ then we have the unique representation

$$v = \chi_+ v + \chi_- v$$

where $\chi_\pm$ is an indicator of the discrete set $\pm D_d$.

Further passing to the Fourier transform and potentiating we obtain the required factorization. □

Remark 10 *The condition (7) is not necessary but we have no an algorithm for constructing a periodic wave factorization. For $D = \mathbb{R}^m_+$ a such algorithm exists always (see [12]).*

3.2 Factorization and index

There is one point in previous considerations from proof of the Theorem 2 for which one needs an explanation. Indeed the function $\ln \tilde{A}(\xi)$ is defined correctly because the condition (8) provides an absence of bifurcation points. That's why one can call this factorization with vanishing index.

4 Invertibility of discrete operators

Lemma 11 *If $f \in B(\boldsymbol{T}^m), g \in A_1(\boldsymbol{T}^m)$ then $f \cdot g \in B(\boldsymbol{T}^m)$.*

Proof: According to properties of discrete Fourier transform F_d we have

$$(F_d^{-1}(f \cdot g))(\tilde{x}) - ((F_d^{-1}f) * (F_d^{-1}g))(\tilde{x}) \equiv$$

$$\sum_{\tilde{y} \in \boldsymbol{Z}^m} f_1(\tilde{x} - \tilde{y}) g_1(\tilde{y}) = \sum_{\tilde{y} \in -D_d} f_1(\tilde{x} - \tilde{y}) g_1(\tilde{y}),$$

where $f_1 = F_d^{-1}f, g_1 = F_d^{-1}g$ and according to lemma 2 $supp\ g_1 \subset -D_d$.

Further since we have $supp\ f_1 \subset \boldsymbol{Z}^m \setminus (-D_d)$ then for $\tilde{x} \in D_d, \tilde{y} \in -D_d$ we have $\tilde{x} - \tilde{y} \in D_d$ so that $f_1(\tilde{x} - \tilde{y}) = 0$ for such $\tilde{x}, \tilde{y}$. Thus $supp\ f_1 * g_1 \subset \boldsymbol{Z}^m \setminus D_d$. □

Theorem 12 *If the elliptic symbol $\tilde{A}_d(\xi) \in C(\boldsymbol{T}^m)$ admits periodic wave factorization then the operator A_d is invertible in the space $L_2(D_d)$.*

Proof: We will remind that according to the property 1 an invertibility of the operator A_d in the space $L_2(D_d)$ is equivalent to an invertibility of the operator $A_d P_{D_d} + IQ_{D_d}$ in the space $L_2(\boldsymbol{Z}^m)$. It is easily concluding the last invertibility is equivalent to solving the Riemann problem (4) for arbitrary right-hand side $g(\xi) \in L_2(\boldsymbol{Z}^m)$ with $G(\xi) \equiv \tilde{A}_d^{-1}(\xi)$. If we have the periodic wave factorization for the symbol $\tilde{A}_d(\xi)$ then

$$\tilde{A}_{\neq}(\xi)\Phi^+(\xi) = \tilde{A}_{=}^{-1}(\xi)\Phi^-(\xi) + \tilde{A}_{\neq}(\xi)g(\xi), \quad (9)$$

$$\xi \in \boldsymbol{T}^m,$$

and we have a jump problem.

The first summand $\tilde{A}_{\neq}(\xi)\Phi^+(\xi) \in A(\boldsymbol{T}^m)$ according to a holomorphic property, and the second one $\tilde{A}_{=}^{-1}(\xi)\Phi^-(\xi) \in B(\boldsymbol{T}^m)$ according to the lemma 3. Taking into account the theorem 2 we conclude that the Riemann problem (9) has a unique solution for arbitrary $g(\xi) \in L_2(\boldsymbol{T}^m)$. □

Conclusion

These "discrete" considerations can be transferred on more general situations and operators. It will be a subject of forthcoming papers of the author.

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