Discrete Operators in Canonical Domains

VLADIMIR VASILYEV
Belgorod National Research University
Chair of Differential Equations
Studencheskaya 14/1, 308007 Belgorod
RUSSIA
vladimir.b.vasilyev@gmail.com

Abstract: In this paper, we consider a certain class of discrete pseudo-differential operators in a sharp convex cone and describe their invertibility conditions in $L_2$-spaces. For this purpose we introduce a concept of periodic wave factorization for elliptic symbol and show its applicability for the studying.

Key–Words: Discrete operator, Multidimensional periodic Riemann problem, Periodic wave factorization, Invertibility

1 Introduction

A classical pseudo-differential operator in Euclidean space $\mathbb{R}^m$ is defined by the formula [1, 2, 3, 4]

$$ (Au)(x) = \int \int_{\mathbb{R}^m} \hat{A}(x, \xi)e^{i(x-y)}\tilde{u}(\xi)d\xi dy, $$

where the sign $\sim$ over a function denotes its discrete Fourier transform

$$ \tilde{u}(\xi) = \int u(x)e^{ix\cdot\xi}dx. $$

1.1 Multidimensional Fourier series and symbols

Given function $u_d$ of a discrete variable $\tilde{x} \in \mathbb{Z}^m$ we define its discrete Fourier transform by the series

$$ (F_d u_d)(\xi) = \tilde{u}_d(\xi) = \sum_{\tilde{x} \in \mathbb{Z}^m} e^{i\tilde{x}\cdot\xi} u(\tilde{x}), $$

where partial sums are taken over cubes

$$ Q_N = \{\tilde{x} \in \mathbb{Z}^m : \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_m), \max_{1 \leq k \leq m} |\tilde{x}_k| \leq N\}. $$

Let $D \subset \mathbb{R}^m$ be a sharp convex cone, $D_d = D \cap \mathbb{Z}^m$, and $L_2(D_d)$ be a space of functions of discrete variable defined on $D_d$, and $A(\tilde{x})$ be a given function of a discrete variable $\tilde{x} \in \mathbb{Z}^m$. We consider the following types of operators

$$ (A_d u_d)(\tilde{x}) = \int_{T^m} \sum_{\eta \in D_d} e^{i(\eta - \tilde{x})\cdot\xi} \hat{A}(\xi)\tilde{u}_d(\xi)d\xi, \hspace{1cm} (1) $$

and introduce the function

$$ \hat{A}_d(\xi) = \sum_{\tilde{x} \in \mathbb{Z}^m} e^{i\tilde{x}\cdot\xi} A(\tilde{x}), \hspace{1cm} \xi \in T^m. $$

Definition 1 The function $\hat{A}_d(\xi)$ is called a symbol of the operator $A_d$, and this symbol is called an elliptic symbol if $\hat{A}_d(\xi) \neq 0, \forall \xi \in T^m$.

Our main goal is describing a periodic variant of wave factorization for an elliptic symbol [9] and showing its usability for studying invertibility for the operator $A_d$.

1.2 Discrete projection operators

Let $P_{D_d}$ projection operator on $D_d$, $P_{D_d} : L_2(\mathbb{Z}^m) \to L_2(D_d)$ so that for arbitrary function $u_d \in L_2(\mathbb{Z}^m)$

$$ (P_{D_d} u_d)(\tilde{x}) = \begin{cases} u_d(\tilde{x}), & \tilde{x} \in D_d \\ 0, & \tilde{x} \notin D_d. \end{cases} $$

1.2.1 Periodic Cauchy kernel

If we consider a half-space case, then the Fourier image of the operator $P_{D_d}$ is evaluated [10, 11, 12] and we’ll demonstrate it in the following

Example 2 If $D = \mathbb{R}^m_+$ then

$$ (F_d P_{D_d} u_d)(\xi', \xi_m) = \lim_{\tau \to 0^+} \frac{1}{4\pi \tau} \int_{-\pi}^\pi u_d(\xi', \eta_m) \cot \frac{\xi_m - \eta_m + i\tau}{2} d\eta_m. $$
1.2.2 Periodic Bochner kernel

If $D$ is a sharp convex cone $C_+ = \{ \tilde{x} \in \mathbb{Z}^m : \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_m), \tilde{x}_m > a|\tilde{x}|, \tilde{x}' = (\tilde{x}_1, \ldots, \tilde{x}_{m-1}), a > 0 \}$ then we introduce the function

$$B_d(z) = \sum_{\tilde{z} \in D_d} e^{i\tilde{z} \cdot z}, \quad z = \xi + i\tau, \quad \xi \in T^m, \quad \tau \in C^+_1,$$

and define the operator

$$(B_d u)(\xi) = \lim_{\tau \rightarrow 0^+} \int_{T^m} B_d(z - \eta)u_d(\eta)d\eta.$$ 

Lemma 3 For arbitrary $u_d \in L_2(Z^m)$ the following property

$$F_dP_d u_d = B_d F_d u_d$$

holds.

Proof: Let $\chi_+(\tilde{x})$ be an indicator of the set $D_d$. Thus

$$(P_d u_d)(\tilde{x}) = \chi_+(\tilde{x}) \cdot u_d(\tilde{x}).$$

Further, since the function $\chi_+(\tilde{x})$ is not summable, we can’t apply directly a convolution property of the Fourier transform. We choose the function $e^{i\tilde{z} \cdot z}$ so the product $\chi_+(\tilde{x})e^{i\tilde{z} \cdot z}$ will be summable for some admissible $\tau$. Taking into account a forthcoming passing to a limit under $\tau \rightarrow 0^+$ we have

$$F_d(\chi_+(\tilde{x})e^{i\tilde{z} \cdot \tau}) = B_d(z).$$

Thus we can use the Fourier transform obtaining convolution of functions $B_d(z)$ and $\tilde{u}_d(\xi)$. It is left passing to a limit. □

2 Multidimensional periodic Riemann boundary value problem

2.1 A half-space case and a periodic one-dimensional Riemann boundary value problem [10, 11, 12]

For $D = \mathbb{R}^m_+$ we will remind some author’s constructions for discrete equations in a half-space. We have

$$B_d(z) = \cot \frac{z}{2}, \quad z = (\xi', \xi_m + i\tau),$$

$$\xi' = (\xi_1, \ldots, \xi_{m-1}), \tau > 0.$$ 

Thus (see example 1) we use a periodic one-dimensional Riemann problem with a parameter $\xi' \in T^{m-1}$ which is the following. Finding a pair of functions $\Phi^\pm(\xi', \xi_m)$ which are boundary values of holomorphic in half-strips $\Pi_\pm = \{ z \in C : z = \xi_m \pm i\tau, \tau > 0 \}$ such that these are satisfied a linear relation

$$\Phi^+(\xi')(\xi_m) = G(\xi', \xi_m)\Phi^-(\xi')(\xi_m) + g(\xi'), \quad \xi' \in T^m,$$

for almost all $\xi' \in T^{m-1}$, where $G(\xi), g(\xi)$ are given periodic functions.

2.2 Essential multidimensional case

Let $\mathring{D}$ be a conjugate cone for $D$ i.e.

$$\mathring{D} = \{ x \in \mathbb{R}^m : x \cdot y > 0, y \in D \},$$

and $T(\mathring{D}) \subset C^m$ be a set of the type $T^m + i\mathring{D}$. For $T^{m} = \mathbb{R}^m$ such a domain of multidimensional complex space is called a radial tube domain over the cone $\mathring{D}$ [7, 8, 9].

Let us define the subspace $A(T^m) \subset L_2(T^m)$ consisting of functions which admit a holomorphic continuation into $T(\mathring{D})$ and satisfy the following condition

$$\sup_{\tau \in \mathring{D}} \int_{T^m} |\tilde{u}_d(\xi + i\tau)|^2 d\xi < +\infty. \quad (2)$$

In other words, the space $A(T^m) \subset L_2(T^m)$ consists of boundary values of holomorphic in $T(\mathring{D})$ functions.

Let us denote

$$B(T^m) = L_2(T^m) \ominus A(T^m),$$

so that $B(T^m)$ is a direct complement of $A(T^m)$ in $L_2(T^m)$.

2.2.1 A jump problem

We formulate the problem by the following way: finding a pair of functions $\Phi^\pm \in A(T^m), \Phi^- \in B(T^m)$, such that

$$\Phi^+(\xi') - \Phi^-(\xi') = g(\xi'), \quad \xi' \in T^m, \quad (3)$$

where $g(\xi) \in L_2(T^m)$ is given.

Lemma 4 The operator $B_d : L_2(T^m) \rightarrow A(T^m)$ is a bounded projector. A function $u_d \in L_2(D_d)$ iff its Fourier transform $\tilde{u}_d \in A(T^m)$.

Proof: According to standard properties of the discrete Fourier transform $F_d$, we have

$$F_d(\chi_+(\tilde{x})u_d(\tilde{x})) = \lim_{\tau \rightarrow 0^+} \int_{T^m} B_d(z - \eta)\tilde{u}_d(\eta)d\eta,$$
where $\chi_+(\tilde{x})$ is an indicator of the set $D_d$. It implies a boundedness of the operator $B_d$. The second assertion follows from holomorphic properties of the kernel $B_d(z)$. In other words for arbitrary function $v \in A(T^m)$ we have

$$v(z) = \int B_d(z - \eta)v(\eta)d\eta, \quad z \in T(D).$$

It is an analogue of the Cauchy integral formula.

\[ \square \]

**Theorem 5** The jump problem has unique solution for arbitrary right-hand side from $L^2(T^m)$.

**Proof:** Indeed it is equivalent to one-to-one representation of the space $L_2(D_d)$ as a direct sum of two subspaces. If we’ll denote $\chi_+(x), \chi_-(x)$ indicators of discrete sets $D_d, Z^m \setminus D_d$ respectively then the following representation

$$u_d(\tilde{x}) = \chi_+(\tilde{x})u_d(\tilde{x}) + \chi_-(\tilde{x})u_d(\tilde{x})$$

is unique and holds for arbitrary function $u_d \in L_2(Z^m)$. After applying the discrete Fourier transform we have

$$F_d u_d = F_d(\chi_+ u_d) + F_d(\chi_- u_d),$$

where $F_d(\chi_+ u_d) \in A(T^m)$ according to lemma 2, and thus $F_d(\chi_- u_d) = F_d u_d - F_d(\chi_+ u_d) \in B(T^m)$ because $F_d u_d \in L_2(T^m)$.

\[ \square \]

**Example 6** If $m = 2$ and $C^2_\Omega$ is the first quadrant in a plane then a solution of the jump problem is given by formulas

$$\Phi^+(\xi) = \frac{1}{(4\pi i)^2} \lim_{\tau \to 0} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cot \frac{\xi_1 + i\tau_1 - t_1}{2} \times$$

$$\cot \frac{\xi_2 + i\tau_2 - t_2}{2} g(t_1, t_2) dt_1 dt_2$$

$$\Phi^-(\xi) = \Phi^+(\xi) - g(\xi), \quad \tau = (\tau_1, \tau_2) \in C^2_\Omega.$$

**2.2.2 A general statement**

It looks as follows. Finding a pair of functions $\Phi^\pm, \Phi^+ \in A(T^m), \Phi^- \in B(T^m)$, such that

$$\Phi^+(\xi) = G(\xi)\Phi^- (\xi) + g(\xi), \quad \xi \in T^m, \quad (4)$$

where $G(\xi), g(\xi)$ are given periodic functions. If $G(\xi) \equiv 1$ we have the jump problem 3).

Like classical studies [5, 6] we want to use a special representation for an elliptic symbol to solve the problem (4).

**2.2.3 Associated singular integral equation**

We can easily obtain so-called characteristic singular integral equation associated with multidimensional periodic Riemann boundary value problem (4).

Let us denote $Q_{D_d} = I - P_{D_d}$ and consider so-called paired operator composed by two operators $A_d(1), A_d(2)$ of the following type

$$A_d(1)P_{D_d} + A_d(2)Q_{D_d} : L_2(Z^m) \to L_2(Z^m) \quad (5)$$

One can easily obtain the following

**Property 7** The invertibility of the operator (1) in the space $L_2(D_d)$ is equivalent to invertibility of the operator (5) in the space $L_2(Z^m)$ with $A_d(1) = A_d, A_d(2) = I$.

The Fourier image for the operator (5) is the following operator

$$\tilde{u}_d(\xi) \longmapsto ((A_d(1))(\xi)B_d + A_d(2))(\xi)(I - B_d)\tilde{u}_d)(\xi) \quad (6)$$

If $D = \mathbb{R}_+^m$ then the operator (6) is a one-dimensional singular integral operator with periodic Cauchy kernel and a parameter $\xi'_{[10, 11, 12]}$.

**3 Periodic wave factorization**

**Definition 8** Periodic wave factorization for elliptic symbol $\tilde{A}(\xi)$ is called its representation in the form

$$\tilde{A}_d(\xi) = \tilde{A}_{\neq}(\xi)\tilde{A}_{\equiv}(\xi)$$

where the factors $A_{\neq}(\xi), A_{\equiv}(\xi)$ admit bounded holomorphic continuation into domains $T(\pm \frac{\pi}{2})$.

**3.1 Sufficient conditions**

We’ll give here certain sufficient conditions for an existence of the periodic wave factorization for an elliptic symbol.

**Theorem 9** Let an elliptic symbol $\tilde{A}_d(\xi) \in C(T^m)$ be a such that

$$\text{supp } F_d^{-1}(\ln \tilde{A}_d(\xi)) \subset D_d \cup (-D_d), \quad (7)$$

$$\int_{-\pi}^{\pi} d\arg \tilde{A}_d(\xi) = 0, \quad k = 1, \ldots, m. \quad (8)$$

Then the symbol $\tilde{A}_d(\xi)$ admits the wave factorization.
Proof: If we start from equality
\[ \tilde{A}_d(\xi) = \tilde{A}_\neq(\xi) \tilde{A}_=(\xi) \]
then by logarithm we obtain
\[ \ln \tilde{A}_d(\xi) = \ln \tilde{A}_\neq(\xi) + \ln \tilde{A}_=(\xi) \]
and we have a special kind of a jump problem.

Namely if we will denote by \( A_1(T^m) \) a subspace of the space \( L_2(T^m) \) consisting of functions which admit a holomorphic continuation into \( T(-D) \) and satisfy the condition (2) for \( \tau \in -D \). So evidently we speak on a possibility of decomposition of the function \( \ln \tilde{A}_d(\xi) \) into two summands one of which belongs to the space \( A(T^m) \) and the second one belongs to the space \( A_1(T^m) \). Let us denote
\[ F^{-1}(\ln \tilde{A}_d(\xi)) \equiv v(x). \]
If \( \text{supp } v \subset D_d \cup (-D_d) \) then we have the unique representation
\[ v = \chi_+ v + \chi_- v \]
where \( \chi_{\pm} \) is an indicator of the discrete set \( \pm D_d \).

Further passing to the Fourier transform and potentiating we obtain the required factorization. \( \square \)

Remark 10 The condition (7) is not necessary but we have no algorithm for constructing a periodic wave factorization. For \( D = \mathbb{R}_+^m \) a such algorithm exists always (see [12]).

3.2 Factorization and index

There is one point in previous considerations from proof of the Theorem 2 for which one needs an explanation. Indeed the function \( \ln \tilde{A}(\xi) \) is defined correctly because the condition (8) provides an absence of bifurcation points. That’s why one can call this factorization with vanishing index.

4 Invertibility of discrete operators

Lemma 11 If \( f \in B(T^m) \), \( g \in A_1(T^m) \) then \( f \cdot g \in B(T^m) \).

Proof: According to properties of discrete Fourier transform \( F_d \) we have
\[ (F^{-1}_d(f \cdot g))(\tilde{x}) - ((F^{-1}_d f)(F^{-1}_d g))(\tilde{x}) = \]
\[ \sum_{\tilde{y} \in Z^m} f_1(\tilde{x} - \tilde{y})g_1(\tilde{y}) = \sum_{\tilde{y} \in -D_d} f_1(\tilde{x} - \tilde{y})g_1(\tilde{y}), \]
where \( f_1 = F^{-1}_d f, g_1 = F^{-1}_d g \) and according to lemma 2 \( \text{supp } g_1 \subset -D_d \).

Further since we have \( \text{supp } f_1 \subset Z^m \setminus (-D_d) \) then for \( \tilde{x} \in D_d, \tilde{y} \in -D_d \) we have \( \tilde{x} - \tilde{y} \in D_d \) so that \( f_1(\tilde{x} - \tilde{y}) = 0 \) for such \( \tilde{x}, \tilde{y} \). Thus \( \text{supp } f_1 \cdot g_1 \subset Z^m \setminus D_d \).

Theorem 12 If the elliptic symbol \( \tilde{A}_d(\xi) \in C(T^m) \) admits periodic wave factorization then the operator \( A_d \) is invertible in the space \( L_2(D_d) \).

Proof: We will remind that according to the property 1 an invertibility of the operator \( A_d \) in the space \( L_2(D_d) \) is equivalent to an invertibility of the operator \( A_dP_{D_d} + IQ_{D_d} \) in the space \( L_2(Z^m) \). It is easily concluding the last invertibility is equivalent to solving the Riemann problem (4) for arbitrary right-hand side \( g(\xi) \in L_2(Z^m) \) with \( G(\xi) \equiv \tilde{A}^{-1}_d(\xi) \). If we have the periodic wave factorization for the symbol \( A_d(\xi) \) then
\[ \tilde{A}_\neq(\xi)\Phi^+(\xi) = \tilde{A}^{-1}_d(\xi)\Phi^-(\xi) + \tilde{A}_\neq(\xi)g(\xi), \]
\[ \xi \in T^m, \]
and we have a jump problem.

The first summand \( \tilde{A}_d(\xi)\Phi^+(\xi) \in A(T^m) \) according to a holomorphic property, and the second one \( \tilde{A}^{-1}_d(\xi)\Phi^-(\xi) \in B(T^m) \) according to the lemma 3. Taking into account the theorem 2 we conclude that the Riemann problem (9) has a unique solution for arbitrary \( g(\xi) \in L_2(T^m) \). \( \square \)

Conclusion

These “discrete” considerations can be transferred on more general situations and operators. It will be a subject of forthcoming papers of the author.

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