Generalized Retarded Integral Inequalities of Gronwall Type and Applications

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Abstract: We present in this paper some retarded integral inequalities of Gronwall type. The obtained results can be used to discuss the behavior of integral equations.

Key–Words: Differential equations, inequalities of Gronwall type.

1 Introduction

The integral inequalities occupy privileged position in the theory of differential and integral equations. In the recent years nonlinear integral inequalities have received considerable attention because of their important applications to a variety of problems in diverse fields of nonlinear differential and integral equations.

In 1919, Gronwall [4] introduced the famous Gronwall inequality in the study of the solutions of differential equations. There exist many lemmas which carry the name of Gronwall's lemma. A main class may be identified is the integral inequality. The original lemma proved by Gronwall in 1919 [4], was the following

Lemma 1 (Gronwall) *Let $z : [a, a + h] \to \mathbb{R}$ be a continuous function that satisfies the inequality*

$$0 \leq z(x) \leq \int_a^x (A + Mz(s))ds,$$

for all $a \leq x \leq a+h$, where A, $M \geq 0$ are constants. Then

$$0 \leq z(x) \leq Ahe^{Mh},$$

for all $a \leq x \leq a + h$.

The above Lemma can be formulated by the following famous inequality, which is called the Gronwall inequality:

Let $u(.)$ be a continuous function defined on the interval $[t_0, t_1]$ and

$$u(t) \leq a + b \int_{t_0}^t u(s)ds,$$

where a and b are nonnegative constants. Then, for all $t \in [t_0, t_1]$, we have

$$u(t) \leq ae^{b(t-t_0)}.$$

After more than 20 years, Bellman [2] extended the last inequality, which reads in the following:

Let a be a positive constant, $u(.)$ and $b(.)$, $t \in [t_0, t_1]$ be real-valued continuous functions, $b(t) \geq 0$, satisfying

$$u(t) \leq a + \int_{t_0}^t b(s)u(s)ds, \ t \in [t_0, t_1].$$

Then, for all $t \in [t_0, t_1]$, we have

$$u(t) \leq ae^{\int_{t_0}^t b(s)ds}.$$

The somewhat more general extensions of the original Gronwall inequality can be found in [1], [3], [7]. Since that, a lot of contributions have been achieved by many researchers and is extensively applied in diverse areas including global existence, uniqueness and stability.

In this paper we are basically interested in retarded Gronwall like inequalities, we will give generalizations of those done in [6]. Some applications are also given to convey the importance of our results.

2 Retarded Integral Inequalities of Gronwall Type

Throughout this paper, we denote $R_0^+ = [0, \infty]$ and $R^+ = (0, \infty)$. We first recall some basic results.

Proposition 2 *([6]) Let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t) \leq t$ on R_0^+, f, g, $h \in C(R_0^+, R_0^+)$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$u(t) \leq a(t) + \int_0^t h(s)u(s)ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)u(\tau)d\tau ds, t \in R_0^+,$$

then we have:

$$u(t) \leq a(t)\exp(\int_0^t h(s)ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)d\tau ds), t \in R_0^+.$$

Proposition 3 *([6]) Let $f(t,s)$, $g(t,s)$, $h(t,s)$ be continuous on $(R_0^+ \times R_0^+, R_0^+)$ and nondecreasing in t for every s fixed. Moreover, let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t) \leq t$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$u(t) \leq a(t) + \int_0^t h(t,s)u(s)ds + \int_0^{\alpha(t)} f(t,s) \left[u(s) + \int_0^s g(s,\tau)u(\tau)d\tau\right] ds, t \in R_0^+,$$

then we have:

$$u(t) \leq a(t)\exp(\int_0^t h(t,s)ds + \int_0^{\alpha(t)} f(t,s) \left[1 + \int_0^s g(s,\tau)d\tau\right] ds), t \in R_0^+.$$

Proposition 4 *([6]) Let $a \in C(R_0^+, R^+)$ and $\alpha_i \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha_i(t) \leq t$ on R_0^+, f_i, g_i, $h \in C(R_0^+, R_0^+)$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$u(t) \leq a(t) + \int_0^t h(s)u(s)ds + \sum_{i=1}^n \int_0^{\alpha_i(t)} f_i(s) \int_0^s g_i(\tau)u(\tau)d\tau ds, t \in R_0^+,$$

then we have:

$$u(t) \leq a(t)\exp(\int_0^t h(s)ds + \sum_{i=1}^n \int_0^{\alpha_i(t)} f_i(s) \int_0^s g_i(\tau)d\tau ds), t \in R_0^+.$$

Proposition 5 *([6]) Let $f_i(t,s)$, $g_i(t,s)$, $h(t,s)$ be continuous on $(R_0^+ \times R_0^+, R_0^+)$ and nondecreasing in t for every s fixed. Moreover, let $a \in C(R_0^+, R^+)$ and $\alpha_i \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha_i(t) \leq t$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$u(t) \leq a(t) + \int_0^t h(t,s)u(s)ds + \sum_{i=1}^n \int_0^{\alpha_i(t)} f_i(t,s) \left[u(s) + \int_0^s g_i(s,\tau)u(\tau)d\tau\right] ds, t \in R_0^+,$$

then we have:

$$u(t) \leq a(t)\exp(\int_0^t h(t,s)ds + \sum_{i=1}^n \int_0^{\alpha_i(t)} f_i(t,s) \left[1 + \int_0^s g_i(s,\tau)d\tau\right] ds), t \in R_0^+.$$

3 Main Results

In this section, we present some generalizations of the previous results proved in ([6]).

Proposition 6 *Let $u(t)$ be a positive function satisfying the inequality:*

$$u(t) \leq K(t)\exp\int_0^t h(s)u(s)ds, t \in \mathbb{R}_+, \qquad (1)$$

where the functions $K(t)$ and $h(t)$ are nonnegative continuous, then:

$$u(t) \leq \frac{K(t)}{1 - \int_0^t h(s)K(s)ds}.$$

Under the assumption, for $t \in \mathbb{R}_+$,

$$1 - \int_0^t h(s)K(s)ds > 0.$$

Proof: Let $\varphi(t) = \int_0^t h(s)u(s)ds$, then it comes that $\varphi'(t) = h(t)u(t)$, using (3.1) it follows that

$$\varphi'(t) \leq h(t)K(t)e^{\varphi(t)},$$

or

$$\varphi'(t)e^{-\varphi(t)} \leq h(t)K(t).$$

By integration, we get

$$1 - e^{-\varphi(t)} \leq \int_0^t h(s)K(s)ds,$$

or

$$\varphi(t) \leq -\ln\left(1 - \int_0^t h(s)K(s)ds\right).$$

Finally, it follows that

$$u(t) \leq \frac{K(t)}{1 - \int_0^t h(s)K(s)ds}.$$

□

Next, we can prove the following result.

Proposition 7 *Let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t) \le t$ on R_0^+, f, g, $h \in C(R_0^+, R_0^+)$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$\begin{aligned} u(t) &\le a(t) + \int_0^t h(s)u^n(s)ds + \\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u(\tau)d\tau ds, t \in R_0^+, \end{aligned}$$

then we have:

$u(t) \le$

$$\frac{a(t)\exp\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds}{\left\{1-\int_0^t (n-1)h(s)a^{n-1}(s)\exp(n-1)\int_0^{\alpha(s)} f(r)\int_0^r g(\tau)d\tau drds\right\}^{\frac{1}{n-1}}}.$$

Under the assumption, for $t \in \mathbb{R}_+$,

$$1 - \int_0^t (n-1)h(s)a^{n-1}(s)\exp(n-1)\int_0^{\alpha(s)} f(r) \int_0^r g(\tau)d\tau drds > 0.$$

Proof: The function u satisfies

$$\begin{aligned} u(t) &\le a(t) + \int_0^t h(s)u^{n-1}(s)u(s)ds + \\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u(\tau)d\tau ds, t \in R_0^+, \end{aligned}$$

and using (2.1) we get

$$\begin{aligned} u(t) &\le a(t)\exp(\int_0^t h(s)u^{n-1}(s)ds + \\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds) \\ &\le a(t)\exp\left(\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right) \\ &\exp\int_0^t h(s)u^{n-1}(s)ds. \end{aligned}$$

Then,

$$\begin{aligned} u^{n-1}(t) &\le a^{n-1}(t)\exp(n-1)(\int_0^{\alpha(t)} f(s) \\ &\int_0^s g(\tau)d\tau ds)\exp\int_0^t (n-1)h(s)u^{n-1}(s)ds. \end{aligned}$$

Using (3.1), it follows that

$u^{n-1}(t) \le$

$$\frac{a^{n-1}(t)\exp(n-1)\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds}{1-\int_0^t (n-1)h(s)\left(a^{n-1}(s)\exp(n-1)\int_0^{\alpha(s)} f(r)\int_0^r g(\tau)d\tau dr\right)ds}.$$

Finally

$u(t) \le$

$$\frac{a(t)\exp\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds}{\left\{1-\int_0^t (n-1)h(s)a^{n-1}(s)\exp(n-1)\int_0^{\alpha(s)} f(r)\int_0^r g(\tau)d\tau drds\right\}^{\frac{1}{n-1}}}.$$

□

Proposition 8 *Assume that $a \ge 0$, $p \ge 1$, then for any $k > 0$ we have*

$$a^{\frac{1}{p}} \le \frac{1}{p}k^{\frac{1-p}{p}}a + \frac{p-1}{p}k^{\frac{1}{p}}. \qquad (2)$$

or equivalently $a^{\frac{1}{p}} \le m_1 a + m_2$ where $m_1 = \frac{1}{p}k^{\frac{1-p}{p}}$ and $m_2 = \frac{p-1}{p}k^{\frac{1}{p}}$.

Proof: Using that the function : $t \mapsto e^t$ is convex we can write $u^\alpha v^\beta \le \alpha u + \beta v$ for $\alpha + \beta = 1$. Taking $\alpha = \frac{1}{p}$, $\beta = \frac{p-1}{p}$, $u = k^{\frac{1-p}{p}}a$ and $v = k^{\frac{1}{p}}$ we get the desired result.

□

Thus, we have the following proposition.

Proposition 9 *Let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t) \le t$ on R_0^+, f, g, $h \in C(R_0^+, R_0^+)$, $p \ge 1$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$\begin{aligned} u^p(t) &\le a(t) + \int_0^t h(s)u^p(s)ds + \\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u(\tau)d\tau ds, t \in R_0^+, \end{aligned}$$

then we have:

$$\begin{aligned} u(t) &\le \left(a(t) + \int_0^{\alpha(t)} m_2 f(s)\int_0^s g(\tau)d\tau ds\right)^{\frac{1}{p}} \\ &\exp\frac{1}{p}\left(\int_0^t h(s)ds + \int_0^{\alpha(t)} m_1 f(s)\int_0^s g(\tau)d\tau ds\right). \end{aligned}$$

Proof: Let $z(t) = u^p(t)$, then z satisfies

$$z(t) \le a(t) + \int_0^t h(s)z(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)z^{\frac{1}{p}}(\tau)d\tau ds.$$

Using (3.2), we get

$$\begin{aligned} z(t) &\le a(t) + \int_0^t h(s)z(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau) \\ &[m_1 z(\tau) + m_2]\, d\tau ds \\ &\le a(t) + \int_0^t h(s)z(s)ds + \int_0^{\alpha(t)} f(s) \\ &\left[m_1\int_0^s g(\tau)z(\tau)d\tau + m_2\int_0^s g(\tau)d\tau\right] ds \\ &\le a(t) + \int_0^{\alpha(t)} m_2 f(s)\int_0^s g(\tau)d\tau ds + \\ &\int_0^t h(s)z(s)ds + \int_0^{\alpha(t)} m_1 f(s)\int_0^s g(\tau)z(\tau)d\tau ds. \end{aligned}$$

Using (2.1), we get

$$z(t) \le \left(a(t)+\int_0^{\alpha(t)} m_2 f(s)\int_0^s g(\tau)d\tau ds\right)\exp \left(\int_0^t h(s)ds+\int_0^{\alpha(t)} m_1 f(s)\int_0^s g(\tau)d\tau ds\right).$$

Finally, it comes that

$$u(t) \le \left(a(t)+\int_0^{\alpha(t)} m_2 f(s)\int_0^s g(\tau)d\tau ds\right)^{\frac{1}{p}}\exp \frac{1}{p} \left(\int_0^t h(s)ds+\int_0^{\alpha(t)} m_1 f(s)\int_0^s g(\tau)d\tau ds\right).$$

□

Remark 10 *If $p=1$, then we obtain Proposition 2.1.*

Proposition 11 *Let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t)\le t$ on R_0^+, f, g, $h \in C(R_0^+, R_0^+)$, $p \ge 1$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$u^p(t) \le a(t)+\int_0^t h(s)u(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u^p(\tau)d\tau ds, t\in R_0^+,$$

then we have:

$$u(t) \le \left(a(t)+m_2\int_0^t h(s)ds\right)^{\frac{1}{p}}\exp\frac{1}{p} \left(m_1\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right).$$

Proof: Let $z(t)=u^p(t)$, then z satisfies

$$z(t) \le a(t)+\int_0^t h(s)z^{\frac{1}{p}}(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)z(\tau)d\tau ds,$$

using (3.2), we get

$$\begin{aligned} z(t) &\le a(t)+\int_0^t h(s)[m_1 z(s)+m_2]ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)z(\tau)d\tau ds \\ &\le a(t)+m_2\int_0^t h(s)ds+m_1\int_0^t h(s)z(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)z(\tau)d\tau ds. \end{aligned}$$

Using (2.1), we get

$$z(t) \le \left(a(t)+m_2\int_0^t h(s)ds\right)\exp \left(m_1\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right).$$

It follows that,

$$u(t) \le \left(a(t)+m_2\int_0^t h(s)ds\right)^{\frac{1}{p}}\exp\frac{1}{p} \left(m_1\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right).$$

□

Remark 12 *If $p=1$, then we obtain Proposition 2.1.*

Proposition 13 *Let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t)\le t$ on R_0^+, f, g, $h \in C(R_0^+, R_0^+)$, $p\ge q \ge 1$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$u^p(t) \le 1+\int_0^t h(s)u^p(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u^q(\tau)d\tau ds, t\in R_0^+,$$

then we have:

$$u(t) \le \exp\frac{1}{p}\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right].$$

Proof: Define a function $z(t)$ by

$$z^p(t) = 1+\int_0^t h(s)u^p(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u^q(\tau)d\tau ds, \tag{3}$$

then $u^p(t)\le z^p(t)$ and $z^p(0)=1$. Differentiating (3.3) and using the fact that $u(t)\le z(t)$

and $z(t)$ is monotone nondecreasing for $t\in \mathbb{R}_+$, we obtain

$$\begin{aligned} pz'(t)z^{p-1}(t) &= h(t)u^p(t)+\alpha'(t)f(\alpha(t)) \int_0^{\alpha(t)} g(s)u^q(s)ds \\ &\le h(t)z^p(t)+\alpha'(t)f(\alpha(t)) \end{aligned}$$

$$\begin{aligned}&\int_0^{\alpha(t)} g(s)z^q(s)ds\\ \leq\ & h(t)z^p(t)+\alpha'(t)f(\alpha(t))z^q(t)\\ &\int_0^{\alpha(t)} g(s)ds\\ \leq\ & z^p(t)[h(t)+\alpha'(t)f(\alpha(t))\\ &\int_0^{\alpha(t)} g(s)ds].\end{aligned}$$

Then,

$$pz'(t)\leq z(t)\left[h(t)+\alpha'(t)f(\alpha(t))\int_0^{\alpha(t)} g(s)ds\right].$$

By integration, we get

$$z(t)\leq \exp\frac{1}{p}\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right].$$

Finally it comes that,

$$u(t)\leq \exp\frac{1}{p}\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right].$$

□

Remark 14 *If $p=1$, then we obtain proposition 2.1.*

Proposition 15 *Let $a\in C(R_0^+,R^+)$ and $\alpha\in C^1(R_0^+,R_0^+)$ be nondecreasing with $\alpha(t)\leq t$ on R_0^+, f, g, $h\in C(R_0^+,R_0^+)$, $p\geq q\geq 1$. If $u\in C(R_0^+,R_0^+)$ satisfies*

$$\begin{aligned}u^p(t)\ \leq\ & a(t)+\int_0^t h(s)u^p(s)ds+\\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u^q(\tau)d\tau ds, t\in R_0^+,\end{aligned}$$

then we have:

$$\begin{aligned}u(t)\ \leq\ & a^{\frac{1}{p}}(t)\exp\frac{1}{p}[\int_0^t h(s)ds+\\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds].\end{aligned}$$

Proof: We have

$$\begin{aligned}\frac{u^p(t)}{a(t)}\ \leq\ & 1+\int_0^t h(s)\frac{u^p(s)}{a(s)}ds+\\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)\frac{u^q(\tau)}{a(\tau)}d\tau ds\\ \leq\ & 1+\int_0^t h(s)\frac{u^p(s)}{a(s)}ds+\\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)\frac{u^q(\tau)}{a^{\frac{q}{p}}(\tau)}d\tau ds.\end{aligned}$$

Or equivalently

$$\begin{aligned}\left(\frac{u(t)}{a^{\frac{1}{p}}(t)}\right)^p\ \leq\ & 1+\int_0^t h(s)\left(\frac{u(s)}{a^{\frac{1}{p}}(s)}\right)^p ds+\\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)\left(\frac{u(\tau)}{a^{\frac{1}{p}}(\tau)}\right)^q d\tau ds.\end{aligned}$$

Using the last proposition, we get

$$\frac{u(t)}{a^{\frac{1}{p}}(t)}\leq \exp\frac{1}{p}\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right].$$

Finally

$$u(t)\leq a^{\frac{1}{p}}(t)\exp\frac{1}{p}\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right].$$

□

Remark 16 *If $p=1$, then we obtain Proposition 2.1.*

Proposition 17 *Let $a\in C(R_0^+,R^+)$ and $\alpha\in C^1(R_0^+,R_0^+)$ be nondecreasing with $\alpha(t)\leq t$ on R_0^+, f, g, $h\in C(R_0^+,R_0^+)$, $p\geq q\geq 1$. If $u\in C(R_0^+,R_0^+)$ satisfies*

$$u^p(t)\leq 1+\int_0^t h(s)u^q(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u(\tau)d\tau ds,$$

then we have:

$$u(t)\leq \exp\frac{1}{p}\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right].$$

Proof: Define a function $z(t)$ by

$$z^p(t)=1+\int_0^t h(s)u^q(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u(\tau)d\tau ds, \tag{4}$$

then $u^p(t) \leq z^p(t)$ and $z^p(0) = 1$. Differentiating (3.4) and using the fact that $u(t) \leq z(t)$

and $z(t)$ is monotone nondecreasing for $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned} pz'(t)z^{p-1}(t) &= h(t)u^q(t) + \alpha'(t)f(\alpha(t)) \\ &\quad \int_0^{\alpha(t)} g(s)u(s)ds \\ &\leq h(t)z^q(t) + \alpha'(t)f(\alpha(t)) \\ &\quad \int_0^{\alpha(t)} g(s)z(s)ds \\ &\leq h(t)z^p(t) + \alpha'(t)f(\alpha(t))z^p(t) \\ &\quad \int_0^{\alpha(t)} g(s)ds \\ &\leq z^p(t)[h(t) + \alpha'(t)f(\alpha(t)) \\ &\quad \int_0^{\alpha(t)} g(s)ds]. \end{aligned}$$

then

$$pz'(t) \leq z(t)\left[h(t) + \alpha'(t)f(\alpha(t))\int_0^{\alpha(t)} g(s)ds\right].$$

By integration we get

$$z(t) \leq \exp\frac{1}{p}\left[\int_0^t h(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right].$$

Finally, it follows that

$$u(t) \leq \exp\frac{1}{p}\left[\int_0^t h(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right].$$

□

Remark 18 *If $p = 1$, then we obtain Proposition 2.1.*

Proposition 19 *Let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t) \leq t$ on R_0^+, f, g, $h \in C(R_0^+, R_0^+)$, $p \geq q \geq 1$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$u(t) \leq 1+\int_0^t h(s)u^p(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u^q(\tau)d\tau ds,$$

then we have:
$$u(t) \leq \frac{1}{\left\{1-(p-1)\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right]\right\}^{\frac{1}{p-1}}},$$

under the assumption, for $t \in \mathbb{R}_+$

$$1-(p-1)\left[\int_0^t h(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right] > 0.$$

Proof: Define a function $z(t)$ by

$$z(t) = 1+\int_0^t h(s)u^p(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u^q(\tau)d\tau ds, \tag{5}$$

then $u(t) \leq z(t)$ and $z(0) = 1$. Differentiating (3.4) and using the fact that $z(t)$ is monotone

nondecreasing for $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned} z'(t) &= h(t)u^p(t) + \alpha'(t)f(\alpha(t))\int_0^{\alpha(t)} g(s)u^q(s)ds \\ &\leq h(t)z^p(t) + \alpha'(t)f(\alpha(t))\int_0^{\alpha(t)} g(s)z^q(s)ds \\ &\leq z^p(t)\left[h(t) + \alpha'(t)f(\alpha(t))\int_0^{\alpha(t)} g(s)ds\right]. \end{aligned}$$

By integration we get
$$z(t) \leq \frac{1}{\left\{1-(p-1)\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right]\right\}^{\frac{1}{p-1}}}.$$
Finally
$$u(t) \leq \frac{1}{\left\{1-(p-1)\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right]\right\}^{\frac{1}{p-1}}}.$$

□

Proposition 20 *Let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t) \leq t$ on R_0^+, f, g, $h \in C(R_0^+, R_0^+)$, $b \in C^1(R_0^+, R_0^+)$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$\begin{aligned} u(t) &\leq a(t) + b(t)\int_0^t h(s)u(s)ds + \\ &\quad \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u(\tau)d\tau ds, t \in R_0^+, \end{aligned}$$

then we have:

$$u(t) \leq a(t)\exp\left[b(t)\int_0^t h(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right].$$

Proof: Since $a(t)$ is positive and nondecreasing, from (3.6) we have

$$\begin{aligned}\frac{u(t)}{a(t)} &\leq 1+b(t)\int_0^t h(s)\frac{u(s)}{a(t)}ds+\\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)\frac{u(\tau)}{a(t)}d\tau ds\\ &\leq 1+b(t)\int_0^t h(s)\frac{u(s)}{a(s)}ds+\\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)\frac{u(\tau)}{a(\tau)}d\tau ds.\end{aligned}$$

We define a function $z(t)$ on R_0^+ by

$$\begin{aligned}z(t) &= 1+b(t)\int_0^t h(s)\frac{u(s)}{a(s)}ds+\\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)\frac{u(\tau)}{a(\tau)}d\tau ds,\end{aligned}$$

then $z(t)$ is positive and nondecreasing. We have $z(0)=1$, $\frac{u(t)}{a(t)} \leq z(t)$, $t \in R_0^+$, by differentiation we get

$$\begin{aligned}z^{'}(t) &= b^{'}(t)\int_0^t h(s)\frac{u(s)}{a(s)}ds+\\ &b(t)h(t)\frac{u(t)}{a(t)}+f(\alpha(t))\alpha^{'}(t)\int_0^{\alpha(t)} g(s)\frac{u(s)}{a(s)}ds\\ &\leq b^{'}(t)\int_0^t h(s)z(s)ds+b(t)h(t)z(t)+\\ &f(\alpha(t))\alpha^{'}(t)\int_0^{\alpha(t)} g(s)z(s)ds\\ &\leq z(t)[b^{'}(t)\int_0^t h(s)ds+b(t)h(t)+\\ &f(\alpha(t))\alpha^{'}(t)\int_0^{\alpha(t)} g(s)ds].\end{aligned}$$

by integration we get

$$\begin{aligned}z(t) &\leq z(0)\exp[b(t)\int_0^t h(s)ds+\\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds].\end{aligned}$$

Since $\frac{u(t)}{a(t)} \leq z(t)$, it follows that

$$\begin{aligned}u(t) &\leq a(t)\exp[b(t)\int_0^t h(s)ds+\\ &\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds].\end{aligned}$$

□

Proposition 21 *Let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t) \leq t$ on R_0^+, f continuous on $(R_0^+ \times R_0^+, R_0^+)$, g, $h \in C(R_0^+, R_0^+)$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$u(t) \leq a(t)+\int_0^t h(s)u(s)ds+\int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)u(\tau)d\tau ds, \tag{6}$$

then we have:

$$u(t) \leq a(t)\exp\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)d\tau ds\right].$$

Proof: Since $a(t)$ is positive and nondecreasing, from (3.7) we have

$$\begin{aligned}\frac{u(t)}{a(t)} &\leq 1+\int_0^t h(s)\frac{u(s)}{a(t)}ds+\\ &\int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)\frac{u(\tau)}{a(t)}d\tau ds\\ &\leq 1+\int_0^t h(s)\frac{u(s)}{a(s)}ds+\\ &\int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)\frac{u(\tau)}{a(\tau)}d\tau ds.\end{aligned}$$

We define a function $z(t)$ on R_0^+ by

$$z(t) = 1+\int_0^t h(s)\frac{u(s)}{a(s)}ds+\int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)\frac{u(\tau)}{a(\tau)}d\tau ds$$

then $z(t)$ is positive and nondecreasing. We have $z(0)=1$, $\frac{u(t)}{a(t)} \leq z(t)$, $t \in R_0^+$, by differentiation we get

$$\begin{aligned}z^{'}(t) &= h(t)\frac{u(t)}{a(t)}+f(t,\alpha(t))\alpha^{'}(t)\int_0^{\alpha(t)} g(s)\frac{u(s)}{a(s)}ds+\\ &\int_0^{\alpha(t)} \partial_t f(t,s)\int_0^s g(\tau)\frac{u(\tau)}{a(\tau)}d\tau ds\\ &\leq h(t)z(t)+f(t,\alpha(t))\alpha^{'}(t)\int_0^{\alpha(t)} g(s)z(s)ds+\\ &\int_0^{\alpha(t)} \partial_t f(t,s)\int_0^s g(\tau)z(\tau)d\tau ds\\ &\leq z(t)[h(t)+f(t,\alpha(t))\alpha^{'}(t)\int_0^{\alpha(t)} g(s)ds+\\ &\int_0^{\alpha(t)} \partial_t f(t,s)\int_0^s g(\tau)d\tau ds].\end{aligned}$$

By integration, we get

$$z(t) \leq z(0) \exp \left[\int_0^t h(s)ds + \int_0^{\alpha(t)} f(t,s) \int_0^s g(\tau)d\tau ds \right]$$

Since $\frac{u(t)}{a(t)} \leq z(t)$, it follows that

$$u(t) \leq a(t) \exp \left[\int_0^t h(s)ds + \int_0^{\alpha(t)} f(t,s) \int_0^s g(\tau)d\tau ds \right]$$

□

Proposition 22 *Let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t) \leq t$ on R_0^+, f, g, $h \in C(R_0^+, R_0^+)$. Let ω a nondecreasing continuous function such that $\omega \geq 1$, $\omega(x) \geq x$, suppose that $a(0) \geq 1$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$\begin{aligned} u(t) &\leq a(t) + \int_0^t h(s)\omega(u(s))ds + \\ &\int_0^{\alpha(t)} f(s) \int_0^s g(\tau)u(\tau)d\tau ds, t \in R_0^+, \end{aligned}$$

then we have:

$$\begin{aligned} W(u(t)) &\leq W(a(0)) + a(t) - a(0) + \int_0^t h(s)ds + \\ &\int_0^{\alpha(t)} f(s) \int_0^s g(\tau)d\tau ds, t \in \mathbb{R}_+, \end{aligned}$$

where W is the function defined by $W(t) = \int_0^t \frac{1}{\omega(s)}ds$.

Proof: Define a function $z(t)$ by

$$\begin{aligned} z(t) &= a(t) + \int_0^t h(s)\omega(u(s))ds + \\ &\int_0^{\alpha(t)} f(s) \int_0^s g(\tau)u(\tau)d\tau ds, t \in \mathbb{R}_+, \end{aligned}$$

then $u(t) \leq z(t)$ and $z(0) = a(0)$. Differentiating (3.8) and using the fact that $z(t)$ is monotone nondecreasing for $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned} z'(t) &= a'(t) + h(t)\omega(u(t)) + \\ &f(\alpha(t))\alpha'(t) \int_0^{\alpha(t)} g(s)u(s)ds \\ &\leq a'(t) + h(t)\omega(z(t)) + \\ &f(\alpha(t))\alpha'(t) \int_0^{\alpha(t)} g(s)z(s)ds \\ &\leq a'(t) + h(t)\omega(z(t)) + \\ &f(\alpha(t))\alpha'(t)z(t) \int_0^{\alpha(t)} g(s)ds. \end{aligned}$$

Then

$$\frac{z'(t)}{\omega(z(t))} \leq a'(t) + h(t) + f(\alpha(t))\alpha'(t) \int_0^{\alpha(t)} g(s)ds,$$

by integration, we get

$$\begin{aligned} W(z(t)) &\leq W(z(0)) + a(t) - a(0) + \int_0^t h(s)ds + \\ &\int_0^{\alpha(t)} f(s) \int_0^s g(\tau)d\tau ds, t \in \mathbb{R}_+. \end{aligned}$$

Finally, it comes that

$$\begin{aligned} W(u(t)) &\leq W(a(0)) + a(t) - a(0) + \int_0^t h(s)ds + \\ &\int_0^{\alpha(t)} f(s) \int_0^s g(\tau)d\tau ds, t \in \mathbb{R}_+. \end{aligned}$$

□

Proposition 23 *Let $f(t,s)$, $g(t,s)$, $h(t,s)$ be continuous on $(R_0^+ \times R_0^+, R_0^+)$ and nondecreasing in t for every s fixed. Moreover, let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t) \leq t$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$\begin{aligned} u(t) &\leq a(t) + \int_0^t h(t,s)u^n(s)ds + \int_0^{\alpha(t)} f(t,s)(u(s) + \\ &\int_0^s g(s,\tau)u(\tau)d\tau)ds, t \in R_0^+, \end{aligned}$$

then we have:

$u(t) \leq$

$$\frac{\exp \int_0^{\alpha(t)} f(t,s)\left(1+\int_0^s g(s,\tau)d\tau\right)ds}{\left\{1-\int_0^t (n-1)h(t,s)a^{n-1}(s)\left[\exp(n-1)\int_0^{\alpha(s)} f(s,r)\left(1+\int_0^r g(r,\tau)d\tau\right)dr\right]ds\right\}}$$

Under the assumption, for $t \in \mathbb{R}_+$

$$\begin{aligned} &1 - \int_0^t (n-1)h(t,s)a^{n-1}(s) \\ &\left[\exp(n-1) \int_0^{\alpha(s)} f(s,r) \left(1 + \int_0^r g(r,\tau)d\tau\right) dr\right] ds > 0 \end{aligned}$$

Proof: The function u satisfies

$$\begin{aligned} u(t) &\leq a(t) + \int_0^t h(t,s)u^{n-1}(s)u(s)ds + \\ &\int_0^{\alpha(t)} f(t,s) \left(u(s) + \int_0^s g(s,\tau)u(\tau)d\tau\right) ds, \end{aligned}$$

using (2.2) it follows that

$$u(t) \le a(t)\left(\exp\int_0^{\alpha(t)} f(t,s)\left(1+\int_0^s g(s,\tau)d\tau\right)ds\right) \exp\int_0^t h(t,s)u^{n-1}(s)ds,$$

then

$$u^{n-1}(t) \le a^{n-1}(t)\exp(n-1)\int_0^{\alpha(t)} f(t,s)(1+ \int_0^s g(s,\tau)d\tau)ds\exp(n-1)\int_0^t h(t,s)u^{n-1}(s)ds,$$

using (3.1), we get the desired estimation.

□

Proposition 24 *Let* $a \in C(R_0^+, R^+)$ *and* $\alpha_i \in C^1(R_0^+, R_0^+)$ *be nondecreasing with* $\alpha_i(t) \le t$ *on* R_0^+, f_i, g_i, $h \in C(R_0^+, R_0^+)$. *If* $u \in C(R_0^+, R_0^+)$ *satisfies*

$$u(t) \le a(t)+\int_0^t h(s)u^m(s)ds + \sum_{i=1}^n \int_0^{\alpha_i(t)} f_i(s)\int_0^s g_i(\tau)u(\tau)d\tau ds, t\in R_0^+,$$

then we have:
$u(t) \le$

$$\frac{a(t)\exp\sum_{i=1}^n\int_0^{\alpha_i(t)} f_i(s)\int_0^s g_i(\tau)d\tau ds}{\left\{1-\int_0^t (m-1)h(s)a^{m-1}(s)\exp(m-1)\sum_{i=1}^n\int_0^{\alpha_i(s)} f_i(r)\int_0^r g_i(\tau)d\tau dr ds\right\}^{\frac{1}{m-1}}}.$$

Under the assumption, for $t \in \mathbb{R}_+$

$$1-\int_0^t (m-1)h(s)a^{m-1}(s)\exp(m-1)\sum_{i=1}^n\int_0^{\alpha_i(s)} f_i(r) \int_0^r g_i(\tau)d\tau dr ds > 0.$$

Since the proof of proposition (3.4) follows by the similar argument as in the proof of proposition (3.2) and proposition (3.3), we omit the details.

4 Applications

4.1 Example 1

Let functions f, g, h, a, α be as in proposition (3.2). Suppose $u \in C(R_0^+, R_0^+)$ is a solution to volterra integral equation

$$u(t) = a(t)+\int_0^t h(s)u^n(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)u(\tau)d\tau ds, t\in R_0^+.$$

If $a(t)$ is bounded and

$$\int_0^{\alpha(\infty)} f(s)ds, \int_0^\infty g(s)ds, \int_0^\infty g(s)ds < \infty,$$

then u is bounded.

4.2 Example 2

We calculate the explicit bound on the solution of the nonlinear integral equation of the form:

$$u(t) = 1+\int_0^t u^2(s)ds + \int_0^t s\int_0^s \tau u(\tau)d\tau ds \quad (7)$$

and we assume that every solution of (4.1) exists on $t \in \mathbb{R}_+$. Then using Proposition 3.9 we get

$$u(t) \le \frac{1}{1-t-\frac{t^4}{9}}.$$

5 Conclusion

In this article, some new retarded integral inequalities of Gronwall type are obtained. Two examples are given to show the applicability of the main results for integral equations.

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