Optimal investment problem for an insurer with dependent risks under the constant elasticity of variance (CEV) model

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Abstract: In this paper, we consider the optimal investment problem for an insurer who has n dependent lines of business. The surplus process of the insurer is described by a n-dimensional compound Poisson risk process. Moreover, the insurer is allowed to invest in a risk-free asset and a risky asset whose price process follows the constant elasticity of variance (CEV) model. The investment objective is maximizing the expected utility of the insurer's terminal wealth.Applying dynamic programming approach, we establish the corresponding Hamilton-Jacobi-Bellman (HJB) equation. Optimal investment strategy is obtained explicitly for exponential utility. Finally, we provide a numerical example to analyze the effects of parameters on the optimal strategy.

Key–Words: Optimal investment, Dependent risks, Constant elasticity of variance (CEV) model, Utility maximization, Hamilton-Jacobi-Bellman (HJB) equation

1 Introduction

Recently, optimal investment problem for an insurer has been studied in many literatures. For diffusion risk model, Browne [1] investigated the problem for maximizing the utility of terminal wealth and minimizing the probability of ruin. Hipp and Plum [2] assumed the insurer can invest in a risky asset and obtained the explicit strategy for ruin probability minimization under compound Poisson risk model .Later Liu and Yang [3] extended the research of Hipp and Plum [2] by adding a risk-free asset into the model of Hipp and Plum [2].Yang and Zhang [4] proposed the jump-diffusion risk process and obtained the optimal strategy for ruin probability minimization. Wang et al. [5] applied the martingale approach to investigate the optimal investment problem for an insurer. Bai and Guo [6] supposed that the insurer can invest in multiple risky assets and obtained the optimal strategy to maximize the utility of terminal wealth. Under the objective of minimizing the ruin probability, Li et al. [7] studied the optimal investment problem for both the insurer and the reinsurer when the insurer can purchase proportional reinsurance. Besides, some more researches about the optimal investment problems in different contexts have been extensively studied, see, Chang and Lu [8],Chang et al. [9],Li and Liu [10],and references therein.

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In the above-mentioned literature, they generally assume that the insurance company only has one business line. But in practice, many insurance companies have two or more different lines of business, for instance, auto insurance, third party insurance, casualty insurance, endowment insurance, and so on. What's more, the n lines of insurance business usually have a relation of dependence. The classical example of dependent risks is natural disasters, such as an earthquake, typhoon or tsunami, where usually cause at least two kinds of claims such as death claims, medical claims, etc.

Currently, some researchers began to study optimal reinsurance problem with lines of business.In the static setting, Centeno [11] studied the optimal excess-of-loss reinsurance strategy for two dependent classes of insurance risks. By using martingale central limit theorem, Bai et al.[12] first derived a two-dimensional diffusion approximation for the two-dimensional compound Poisson reserve risk process and studied an optimal excess-of-loss reinsurance problem for the approximated diffusion model. Liang and Yuen [13] considered the optimal proportional reinsurance problem with two dependent risks under the variance premium principle. Yuen et al.[14] extended the research of Liang and Yuen [13] to the case with the reinsurance premium calculated under the expected value principle and the model with multiple dependent classes of insurance business, closedform strategies are derived for both compound Poisson risk model and the diffusion approximation risk model.

In this paper,we consider investment problem for an insurer with dependent risks. The insurer is allowed to invest in a risk-free asset and a risky asset. Furthermore, the price process of the risky asset follows the constant elasticity of variance (CEV) model,which is a natural extension of geometric Brownian motion (GBM).The CEV model was proposed by Cox and Ross [15]. At first, the CEV model was usually used in option pricing research,see e.g., Beckers [16],Davydov and Linetsky [17],Jones [18]. Recently, the CEV model has been introduced to optimal investment research by Darius [19].For the portfolio selection problem, Li et al.[20] considered the optimal investment problem with taxes, dividends and transaction costs under the constant elasticity of variance (CEV) model and obtained the solutions for the logarithmic, exponential and quadratic utility functions. For the optimal reinsurance and investment problem, Gu et al.[21] investigated the optimal reinsurance-investment problem with a Brownian motion risk process under the CEV model, and optimal strategies are derived for insurers with CRRA or CARA utility. For jump-diffusion risk process, Liang et al.[22] adopted the CEV model for studying proportional reinsurance problem. Besides, there are some other literatures studied optimal reinsurance-investment problems under the CEV model, see, Lin and Li [23],Li et al.[24] among others. For the optimal investment problem for the DC pension fund, Xiao et al.[25] began to apply the CEV model to investigate the pension fund investment problem and derived the optimal strategy for logarithm utility function using the technologies of Legendre transform and dual theory. Gao [26], [27] extended the work of Xiao et al.[25] by solving the optimal solutions for CRRA and CARA utility functions.

These papers motivate us to consider the optimal investment problem for an insurer with n dependent lines of business. The surplus process of the insurer is described by a n-dimensional compound Poisson risk process, and the insurer is allowed to invest in a risk-free asset and a risky asset whose price process follows the CEV model. The objective of the insurer is to maximize the expected utility of his/her terminal wealth. By applying dynamic programming approach, we establish the Hamilton-Jacobi-Bellman (HJB) equation associated with the optimal problem and transform it into a partial differential equation. Under some given assumptions, explicit solutions to the problem of expected CARA utility maximization are derived. Furthermore, we present a numerical example to analyze the effects of parameters on the optimal strategy.

This paper is organized as follows. In section 2, we introduce the n-dimensional risk model. Section 3 derives the optimal strategies to maximize the utility of terminal wealth. In section 4,numerical examples are carried out to analyze the effects of parameters on the optimal strategies. Finally, we give conclusions in Section 5.

2 Problem formulation

In this paper, we consider an insurer who has n dependent classes of insurance business,such as health insurance/casualty insurance/third party insurance, and so on. The aggregated claims up to time t in the ith line of business are denoted by$\sum_{j=1}^{N_i(t)+N(t)} X_{ij}$,and c_i stands for the premium rate of the ith line. Then the surplus process of the insurer follows:

$$R(t) = x + \sum_{i=1}^{n} c_i t - \sum_{i=1}^{n} \sum_{j=1}^{N_i(t)+N(t)} X_{ij}, \quad (1)$$

where $\{N_i(t)\}$ and $\{N(t)\}$ are $n+1$ independent Poisson processes with intensities λ_i and λ for $i = 1, 2, \cdots, n$. The claim sizes $\{X_{ij}, j = 1, 2, \cdots\}$ are $i.i.d$ positive random variables with common distribution function for $i = 1, 2, \cdots, n$, and $\{X_{ij}, j = 1, 2, \cdots\}$ are independent of $\{N_i(t)\}$ and $\{N(t)\}$.Besides,$\{X_{ij}, j = 1, 2, \cdots\}$ are independent of $\{X_{kj}, j = 1, 2, \cdots\}$ for $k \neq i, i, k = 1, 2, \cdots, n$. Denote $E(X_i) = a_i$ and $Var(X_i) = b_i$ for $i = 1, 2, \cdots, n$. Suppose the premium is calculated according to the expected value principle, i.e.,

$$c_i = (1 + \eta_i)(\lambda_i + \lambda)a_i, \quad (2)$$

where η_i is the positive safety loading in line i.

According to Bai et al.[12],the approximation risk model of (1) is

$$\mathrm{d}R_\infty(t) = \sum_{i=1}^{n} \mu_i \mathrm{d}t + \sum_{i=1}^{n} \gamma_i \mathrm{d}W^{(i)}(t), \quad (3)$$

where

$$\mu_i = (\lambda + \lambda_i)\eta_i a_i, \ \ \gamma_i = ((\lambda + \lambda_i)E[X_i^2])^{\frac{1}{2}}. \quad (4)$$

$W^{(i)}(t)$ are standard Brownian motions. Furthermore, the correlation coefficient of $W^{(i)}(t)$ and $W^{(k)}(t)$ are denoted by ρ_{ik} for $i \neq k, i, k = 1, 2, \cdots, n$, and

$$\rho_{ik} = \frac{\lambda}{\gamma_i \gamma_k} E(X_i)E(X_k) = \frac{\lambda}{\gamma_i \gamma_k} a_i a_k. \quad (5)$$

We assume that the insurer is allowed to invest in a risk-free asset and a risky asset. The price of risk-free asset is given by

$$\mathrm{d}S_0(t) = r_0 S_0(t)\mathrm{d}t, \tag{6}$$

where $r_0 > 0$ is the interest rate, and the price of risky asset is described by the CEV model:

$$\mathrm{d}S(t) = r_1 S(t)\mathrm{d}t + k(S(t))^{\beta+1}\mathrm{d}W^{(0)}(t), \tag{7}$$

where $r_1 > r_0$ is the appreciation rate of the risky asset, $\{W^{(0)}(t)\}$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, P)$.$\beta \leq 0$ represents the elasticity coefficient and $k(S(t))^{\beta}$ stands for the instantaneous volatility of risky asset. The correlation coefficient of $W^{(0)}(t)$ and $W^{(i)}(t)$ are denoted by ρ_{i0} for $i = 1, 2, \cdots, n$.

Let $\pi(t)$ be the money amount invested in the risky asset at time t by the insurer, Then $X(t) - \pi(t)$ is the money amount invested in the risk-free asset, where $X(t)$ is the wealth of the insurer at time t. For a trading strategy $\pi(t)$,the wealth process $X(t)$ is given by:

$$\begin{aligned}
&\mathrm{d}X(t) = \pi(t)\frac{\mathrm{d}S(t)}{S(t)} + (X(t) - \pi(t))\frac{\mathrm{d}S_0(t)}{S_0(t)} \\
&+\mathrm{d}R_{\infty}(t) \\
&= [(r_1 - r_0)\pi(t) + r_0 X(t) + \textstyle\sum_{i=1}^{n} \mu_i]\mathrm{d}t \\
&+\pi(t)k(S(t))^{\beta}\mathrm{d}W^{(0)}(t) + \textstyle\sum_{i=1}^{n} \gamma_i \mathrm{d}W^{(i)}(t), \\
&X(0) = x.
\end{aligned} \tag{8}$$

If $\pi(t)$ is $\mathscr{F}_t$-progressively measurable and $\int_0^T (\pi(t))^2 \mathrm{d}x < \infty$,$\pi(t)$ is called an admissible strategy. Denote Π the set of all admissible strategies.

Suppose that the insurer has a utility function $u(\cdot)$ which is strictly concave and continuously differentiable on $(-\infty, +\infty)$. The insurer aims to maximize the expected utility of terminal wealth, i.e.,

$$\max_{\pi \in \Pi} E[u(X(T))]. \tag{9}$$

3 Solution to optimal investment problem for an insurer with dependent risk

In this section, we try to find the explicit solutions for optimization problem (9) for exponential utility by using dynamic programming approach.

3.1 General framework

The value function is defined as

$$H(t, s, x) = \sup_{\pi \in \Pi} H_{\pi}(t, s, x), 0 < t < T, \tag{10}$$

$$H_{\pi}(t, s, x) = E[u(X(T))|S(t) = s, X(t) = x]. \tag{11}$$

According to Fleming and Soner [28], if H and $H_t, H_x, H_{xx}, H_s, H_{ss}, H_{sx}$ is continuous, H satisfies the following Hamilton-Jacobi-Bellman(HJB) equation:

$$\begin{aligned}
&H_t + r_1 s H_s + (r_0 x + \sum_{i=1}^{n} \mu_i)H_x + \frac{1}{2}(\sum_{i=1}^{n} \gamma_i^2 \\
&+2\sum_{i,j=1,i\neq j}^{n} \rho_{ij}\gamma_i\gamma_j)H_{xx} + \frac{1}{2}k^2 s^{2\beta+2}H_{ss} \\
&+ks^{\beta+1}(\sum_{i=1}^{n} \gamma_i\rho_{i0})H_{sx} + \sup_{\pi}\{\frac{1}{2}\pi^2 k^2 s^{2\beta}H_{xx} \\
&+[(r_1 - r_0)H_x + (\sum_{i=1}^{n} \gamma_i\rho_{i0})ks^{\beta}H_{xx} \\
&+k^2 s^{2\beta+1}H_{sx}]\pi\} = 0.
\end{aligned} \tag{12}$$

Let

$$\begin{aligned}
A &= \sum_{i=1}^{n} \gamma_i\rho_{i0}, B = \sum_{i=1}^{n} \gamma_i^2 + 2\sum_{i,j=1,i\neq j}^{n} \rho_{ij}\gamma_i\gamma_j \\
&= \sum_{i=1}^{n} \gamma_i^2 + 2\sum_{i,j=1,i\neq j}^{n} \lambda a_i a_j,
\end{aligned}$$

and differentiating with respect to π in (12),we have

$$\pi^* = -\frac{(r_1 - r_0)H_x + Aks^{\beta}H_{xx} + k^2 s^{2\beta+1}H_{sx}}{k^2 s^{2\beta}H_{xx}}. \tag{13}$$

Substituting (13) into (12),we transform (12) into a partial differential equation,

$$\begin{aligned}
&H_t + r_1 s H_s + (r_0 x + \sum_{i=1}^{n} \mu_i)H_x + \frac{1}{2}BH_{xx} \\
&+\frac{1}{2}k^2 s^{2\beta+2}H_{ss} + ks^{\beta+1}AH_{sx} \\
&-\frac{[(r_1 - r_0)H_x + Aks^{\beta}H_{xx} + k^2 s^{2\beta+1}H_{sx}]^2}{2k^2 s^{2\beta}H_{xx}} \\
&= 0.
\end{aligned} \tag{14}$$

Next, we solve optimal problem (9) for common constant absolute risk aversion (CARA) utility function, i.e.,

$$u(x) = -\frac{1}{q}e^{-qx}, q > 0. \tag{15}$$

To solve (14),we try to conjecture a solution in the following form

$$V(t, s, x) = -\frac{1}{q}e^{[-q(d(t)x + g(t,s))]}, \tag{16}$$

with boundary condition given by

$$d(T) = 1, g(T, s) = 0.$$

(16) gives

$$\begin{aligned} &V_t = -q(d_t(t)x + g_t)V, \;\; V_s = -qg_sV, \\ &V_{ss} = (q^2g_s^2 - qg_{ss})V, \;\; V_x = -qd(t)V, \\ &V_{xx} = q^2d(t)^2V, \;\; V_{xs} = q^2d(t)g_sV. \end{aligned}$$

Plugging $V_t, V_s, V_{ss}, V_x, V_{xx}, V_{xs}$ into (14),after simplification,we have

$$\begin{aligned} &(-qd_t(t) - qr_0d(t))x - qg_t - qr_0sg_s \\ &-qd(t)(\sum_{i=1}^{n}\mu_i) + \frac{1}{2}q^2d^2(t)[B - A^2] \\ &-\frac{1}{2}k^2qs^{2\beta+2}g_{ss} - \frac{(r_1 - r_0)^2}{2k^2s^{2\beta}} \\ &+\frac{qd(t)(r_1 - r_0)A}{ks^\beta} = 0. \end{aligned} \tag{17}$$

(17) can be split into two equations

$$-qd_t(t) - qr_0d(t) = 0, \tag{18}$$

and

$$\begin{aligned} &-qg_t - qr_0sg_s - qd(t)(\sum_{i=1}^{n}\mu_i) \\ &+\frac{1}{2}q^2d(t)^2[B - A^2] - \frac{1}{2}k^2qs^{2\beta+2}g_{ss} \\ &-\frac{(r_1 - r_0)^2}{2k^2s^{2\beta}} + \frac{qd(t)(r_1 - r_0)A}{ks^\beta} = 0. \end{aligned} \tag{19}$$

From (18) and $d(T) = 1$, we derive

$$d(t) = e^{r_0(T-t)}. \tag{20}$$

To solve (19),let

$$g(t, s) = m(t, y), \;\; y = s^{-2\beta}, \tag{21}$$

then

$$\begin{aligned} &g_t = m_t, \;\; g_s = -2\beta s^{-2\beta-1}m_y, \\ &g_{ss} = 2\beta(2\beta + 1)s^{-2\beta-2}m_y + 4\beta^2s^{-4\beta-2}m_{yy}. \end{aligned}$$

Introducing these derivatives into (19),we obtain

$$\begin{aligned} &-q[m_t + d(t)(\sum_{i=1}^{n}\mu_i) - 2\beta r_0ym_y] \\ &+\frac{q^2d^2(t)}{2}(B - A^2) - k^2q[\beta(2\beta + 1)m_y \\ &+2\beta^2ym_{yy}] - \frac{1}{2k^2}y(r_1 - r_0)^2 \\ &+k^{-1}qd(t)(r_1 - r_0)Ay^{\frac{1}{2}} = 0. \end{aligned} \tag{22}$$

3.2 The case of $\rho_{i0} = 0$

In this section, we suppose that the financial market and risk model are independent, i.e., $W^{(0)}(t)$ is independent with $W^{(i)}(t)$,i.e.,$\rho_{i0} = 0$,which implies that $A = 0$.

Under the assumption that $A = 0$,(22) becomes:

$$\begin{aligned} &-q[m_t + d(t)(\sum_{i=1}^{n}\mu_i) - 2\beta r_0ym_y] \\ &+\frac{q^2d^2(t)}{2}B - k^2q[\beta(2\beta + 1)m_y \\ &+2\beta^2ym_{yy}] - \frac{1}{2k^2}y(r_1 - r_0)^2 = 0. \end{aligned} \tag{23}$$

We try to find a solution to (23) with the following form:

$$m(t, y) = P(t) + Q(t)y, \tag{24}$$

with the boundary condition $Q(T) = 0$ and $P(T) = 0$. Then

$$m_t = P_t + Q_ty, \;\; m_y = Q(t), m_{yy} = 0.$$

Plugging m_t, m_y, m_{yy} into (23),we obtain:

$$\begin{aligned} &(-qQ_t + 2\beta qr_0Q - \frac{(r_1 - r_0)^2}{2k^2})y - qP_t \\ &-qd(t)(\textstyle\sum_{i=1}^{n}\mu_i) + \frac{q^2d^2(t)}{2}B \\ &-k^2qQ\beta(2\beta + 1) = 0. \end{aligned} \tag{25}$$

Decomposing (25) into two equations, we have

$$-qQ_t + 2\beta qr_0Q - \frac{(r_1 - r_0)^2}{2k^2} = 0, \tag{26}$$

$$\begin{aligned} &-qP_t - qd(t)(\sum_{i=1}^{n}\mu_i) + \frac{q^2d^2(t)}{2}B \\ &-k^2qQ\beta(2\beta + 1) = 0. \end{aligned} \tag{27}$$

Taking the boundary condition $Q(T) = 0$ and $P(T) = 0$ into account, we find the solution to (26) and (27) are

$$Q(t) = \frac{(r_1 - r_0)^2}{2qk^2}\frac{1 - e^{2\beta r_0(t-T)}}{2\beta r_0}, \tag{28}$$

$$\begin{aligned} P(t) = &\frac{\sum_{i=1}^{n}\mu_i}{r_0}[1 - e^{r_0(T-t)}] + \frac{qB}{2}\frac{e^{2r_0(T-t)} - 1}{2r_0} \\ &+\frac{(2\beta + 1)(r_1 - r_0)^2}{4qr_0}\frac{e^{2\beta r_0(t-T)} - 1}{2\beta r_0} \\ &-\frac{(2\beta + 1)(r_1 - r_0)^2}{4qr_0}(t - T). \end{aligned} \tag{29}$$

The following theorem summarizes the above derivation.

Theorem 1. *Suppose that the financial market and risk model are independent, i.e.,$\rho_{i0} = 0$,the optimal strategy for problem (9) under the CARA utility function is given by*

$$\pi_1^* = \frac{e^{r_0(t-T)}}{2s^{2\beta}k^2 q}[2(r_1-r_0)+\frac{(r_1-r_0)^2}{r_0}(1-e^{2\beta r_0(t-T)})]. \tag{30}$$

A solution to HJB equation (23) is given by

$$V(t,s,x) = -\frac{1}{q}e^{[-q(d(t)x+g(t,s))]}, \tag{31}$$

where

$$d(t) = e^{r_0(T-t)}, g(t,s) = P(t) + Q(t)s^{2\beta},$$

$$Q(t) = \frac{(r_1-r_0)^2}{2qk^2}\frac{1-e^{2\beta r_0(t-T)}}{2\beta r_0}, \tag{32}$$

$$\begin{aligned} P(t) = & \frac{\sum_{i=1}^{n}\mu_i}{r_0}[1-e^{r_0(T-t)}] + \frac{qB}{2}\frac{e^{2r_0(T-t)}-1}{2r_0} \\ & + \frac{(2\beta+1)(r_1-r_0)^2}{4qr_0}\frac{e^{2\beta r_0(t-T)}-1}{2\beta r_0} \\ & - \frac{(2\beta+1)(r_1-r_0)^2}{4qr_0}(t-T). \end{aligned} \tag{33}$$

Proof: From (13), (16), (21), (24) and (28), we have

$$\begin{aligned} \pi_1^* &= -\frac{(r_1-r_0)V_x + k^2 s^{2\beta+1}V_{sx}}{k^2 s^{2\beta}V_{xx}} \\ &= -\frac{(r_1-r_0)(-qd(t)V) + k^2 s^{2\beta+1}q^2 d(t) g_s V}{k^2 s^{2\beta} q^2 d^2(t) V} \\ &= -\frac{-(r_1-r_0) + k^2 s^{2\beta+1} q g_s}{k^2 s^{2\beta} q d(t)} \\ &= \frac{e^{r_0(t-T)}}{2s^{2\beta}k^2 q}[2(r_1-r_0) + \frac{(r_1-r_0)^2}{r_0} \\ &\cdot(1-e^{2\beta r_0(t-T)})]. \end{aligned}$$

According to (16), (20), (21) and (24),we derive $V(t,s,x)$ as given in (31). □

Theorem 2. *Let $V(t,s,x)$ be a solution to (23),then the value function is $H(t,s,x) = V(t,s,x)$.For the wealth process $X(t)$ associated with an admissible strategy $\pi(t)$,we have*

$$E[U(X(t))] \le V(0,s,x).$$

In particular, for $\pi_1^*(t)$ given by Theorem 1 and the corresponding wealth process $X_1^*(t)$,

$$E[U(X_1^*(t))] = V(0,s,x).$$

Theorem 2 verifies that the value function coincides with the solution to HJB equation (23) given in Theorem 1 and indicates that the strategy given in Theorem 1 is optimal for problem (9).The prove of Theorem 2 is similar as Zhao et al.[29].

Remark 3. *From Theorem 1,we find that the claim processes have no effect on the optimal strategy. Under the assumption that $W^{(0)}(t)$ is independent with $W^{(i)}(t)$,i.e.,$\rho_{i0} = 0$,the financial market and risk model are independent, then the claim process is independent with the financial market. In reality, the impact of claim process of insurer on the volatility of the financial market is very small.*

3.3 The case of $\beta = 0$

Suppose that $\beta = 0$, then the CEV model reduces to the GBM model, and HJB equation (12) reduces to

$$\begin{aligned} & H_t + \sup_\pi\{[(r_1-r_0)\pi + r_0 x + \textstyle\sum_{i=1}^n \mu_i]H_x \\ & + \frac{1}{2}[k^2\pi^2 + B + 2kA\pi]H_{xx}\} = 0. \end{aligned} \tag{34}$$

Again,we can differentiate with respect to π in the following formula:

$$[r_1-r_0)\pi + r_0 x + \sum_{i=1}^n \mu_i]H_x + \frac{1}{2}[k^2\pi^2 + B + 2kA\pi]H_{xx}.$$

we have

$$\pi_2^* = -\frac{(r_1-r_0)}{k^2}\frac{H_x}{H_{xx}} - \frac{A}{k}. \tag{35}$$

Plugging (35) into (34)

$$\begin{aligned} & H_t + (r_0 x + \textstyle\sum_{i=1}^n \mu_i - \frac{A(r_1-r_0)}{k})H_x \\ & + \frac{1}{2}(B-A^2)H_{xx} - \frac{(r_1-r_0)^2}{2k^2}\frac{H_x^2}{H_{xx}} = 0. \end{aligned} \tag{36}$$

In order to solve (34),we try to find the solution to (36) in the following structure:

$$\begin{aligned} \tilde{V}(t,x) = & -\frac{1}{q}exp\{-qxe^{r_0(T-t)} - \frac{1}{2}(\frac{r_1-r_0}{k})^2 \\ & \cdot(T-t) + h(T-t)\}, \end{aligned} \tag{37}$$

and the boundary condition

$$h(T) = 0.$$

Then we have

$$\tilde{V}_x = -\tilde{V}qe^{r_0(T-t)}, \quad \tilde{V}_{xx} = q^2\tilde{V}e^{2r_0(T-t)},$$

$$\tilde{V}_t = \tilde{V}[xr_0 q e^{r_0(T-t)} + \frac{1}{2}(\frac{r_1-r_0}{k})^2 - h'(T-t)].$$

Introducing $\tilde{V}_x, \tilde{V}_{xx}$ into π_2^*,we get

$$\pi_2^* = \frac{r_1 - r_0}{qk^2} e^{-r_0(T-t)} - \frac{A}{k}.$$

putting $\tilde{V}_x, \tilde{V}_{xx}, \tilde{V}_t$ into (36),we can get

$$\begin{aligned} &h'(T-t) = -qe^{r_0(T-t)}(\textstyle\sum_{i=1}^n \mu_i - \frac{A(r_1 - r_0)}{k}) \\ &+\frac{1}{2}q^2(B - A^2)e^{2r_0(T-t)}, \end{aligned}$$

with the boundary condition $h(T) = 0$ and after integrating, we derive

$$\begin{aligned} h(T-t) &= \frac{1}{2}q^2(B - A^2)\tfrac{e^{2r_0(T-t)}-1}{2r_0} \\ -q&\frac{e^{r_0(T-t)} - 1}{r_0}(\textstyle\sum_{i=1}^n \mu_i - \frac{A(r_1 - r_0)}{k}). \end{aligned} \tag{38}$$

According to the above derivation, we have the following theorem.

Theorem 4. *In the case that the risky asset's price follows the GBM model, the optimal strategy and the corresponding value function are as follows:*

$$\pi_2^* = \frac{r_1 - r_0}{qk^2} e^{-r_0(T-t)} - \frac{A}{k}, \tag{39}$$

$$\begin{aligned} &\tilde{V}(t,x) = -\frac{1}{q}exp\{-qxe^{r_0(T-t)} - \frac{1}{2}(\frac{r_1 - r_0}{k})^2 \\ &+h(T-t)\}, \end{aligned} \tag{40}$$

where

$$\begin{aligned} h(T-t) &= \frac{1}{2}q^2(B - A^2)\tfrac{e^{2r_0(T-t)}-1}{2r_0} \\ -q&\frac{e^{r_0(T-t)} - 1}{r_0}(\textstyle\sum_{i=1}^n \mu_i - \frac{A(r_1 - r_0)}{k}), \end{aligned}$$

$$\begin{aligned} &A = \sum_{i=1}^n \gamma_i \rho_{i0}, B = \sum_{i=1}^n \gamma_i^2 + 2\sum_{i,j=1, i\neq j}^n \rho_{ij}\gamma_i\gamma_j \\ &= \sum_{i=1}^n \gamma_i^2 + 2\sum_{i,j=1, i\neq j}^n \lambda a_i a_j, \\ &\gamma_i = ((\lambda + \lambda_i)E[X_i^2])^{\frac{1}{2}}. \end{aligned}$$

Proof: (35) and (37) implies that

$$\begin{aligned} \pi_2^* &= -\frac{(r_1 - r_0)}{k^2}\frac{\tilde{V}_x}{\tilde{V}_{xx}} - \frac{A}{k} \\ &= -\frac{(r_1 - r_0)}{k^2}\frac{-\tilde{V}qe^{r_0(T-t)}}{\tilde{V}q^2e^{2r_0(T-t)}} - \frac{A}{k}. \\ &= \frac{r_1 - r_0}{qk^2}e^{-r_0(T-t)} - \frac{A}{k}. \end{aligned}$$

From (37) and (38), we derive $\tilde{V}(t,x)$ as given in (40). □

The following theorem verifies that the value function coincides with the solution to HJB equation (34) given in Theorem 4 and indicates that the strategy given in Theorem 4 is optimal for problem (9).

Theorem 5. *$\tilde{V}(t,x)$ is the solution of (34), then the value function is $H(t,x) = \tilde{V}(t,x)$. For the wealth process $X(t)$ associated with an admissible strategy $\pi(t)$,we have*

$$E[U(X(t))] \leq \tilde{V}(0,x).$$

In particular, for $\pi_2^*(t)$ given by Theorem 4 and the corresponding wealth process $X_2^*(t)$,

$$E[U(X_2^*(t))] = \tilde{V}(0,x).$$

Remark 6. *If there is only one business line in the risk model, i.e., $i = 1$,(39) reduces to the optimal strategy derived by Yang and Zhang [4].*

From (39), we find that the optimal investment strategy under the GBM model depends not only on the time, interest rate, risk aversion coefficient, appreciation rate and volatility of the risky asset, but also on counting processes and the correlation coefficients between risky asset and claim processes. We can also find this property in Section 4 numerical examples.

Remark 7. *Suppose there are two dependent lines of business, the impact of the claim processes on the optimal investment strategy is given as follows.*

(1)If $\rho_{10} > -\frac{(a_2^2 + b_2)\sqrt{(\lambda + \lambda_1)(a_1^2 + b_1)}}{(a_1^2 + b_1)\sqrt{(\lambda + \lambda_2)(a_2^2 + b_2)}}\rho_{20}$,the optimal investment strategy decreases as λ increases.

(2)If $\rho_{10} < -\frac{(a_2^2 + b_2)\sqrt{(\lambda + \lambda_1)(a_1^2 + b_1)}}{(a_1^2 + b_1)\sqrt{(\lambda + \lambda_2)(a_2^2 + b_2)}}\rho_{20}$,the optimal investment strategy increases as λ increases.

Proof: For $i = 2$,the optimal investment strategy is

$$\begin{aligned} &\pi_2^* = \frac{r_1 - r_0}{qk^2}e^{-r_0(T-t)} - \frac{\sqrt{(\lambda + \lambda_1)(a_1^2 + b_1)}\rho_{10}}{k} \\ &-\frac{\sqrt{(\lambda + \lambda_2)(a_2^2 + b_2)}\rho_{20}}{k}. \end{aligned}$$

Differentiating π_2^* with λ,

$$\begin{aligned} \frac{\partial \pi_2^*}{\partial \lambda} &= -\frac{\rho_{10}(a_1^2 + b_1)}{2k\sqrt{(\lambda + \lambda_1)(a_1^2 + b_1)}} \\ &-\frac{\rho_{20}(a_2^2 + b_2)}{2k\sqrt{(\lambda + \lambda_2)(a_2^2 + b_2)}}. \end{aligned}$$

when $\rho_{10} > -\dfrac{(a_2^2+b_2)\sqrt{(\lambda+\lambda_1)(a_1^2+b_1)}}{(a_1^2+b_1)\sqrt{(\lambda+\lambda_2)(a_2^2+b_2)}}\rho_{20}$,
$\dfrac{\partial \pi_2^*}{\partial \lambda} > 0$,
while $\rho_{10} < -\dfrac{(a_2^2+b_2)\sqrt{(\lambda+\lambda_1)(a_1^2+b_1)}}{(a_1^2+b_1)\sqrt{(\lambda+\lambda_2)(a_2^2+b_2)}}\rho_{20}$,
$\dfrac{\partial \pi_2^*}{\partial \lambda} < 0$. □

Remark 7 shows the impact of the counting processes and the correlation coefficients between risky asset and the claim processes on the optimal investment strategy for $i = 2$.The idea and technique shown here are still useful for $i > 2$.

4 Numerical analysis

In this section, we provide some numerical simulations to illustrate our results. Throughout the numerical analysis, according to Li et al. [20],the basic parameters are given by: $r_0 = 0.03$, r_1=0.12, k=16.16, T=10, β=-0.12, s=67, q=0.05, $\alpha_1 = 2$, $\alpha_2 = 2$, $\lambda = 3$, $\lambda_1 = 1$, $\lambda_2 = 5$, $\rho_{10} = -0.5$, $\rho_{20} = -0.1$.

4.1 Numerical analysis for the case of $\rho_{i0} = 0$

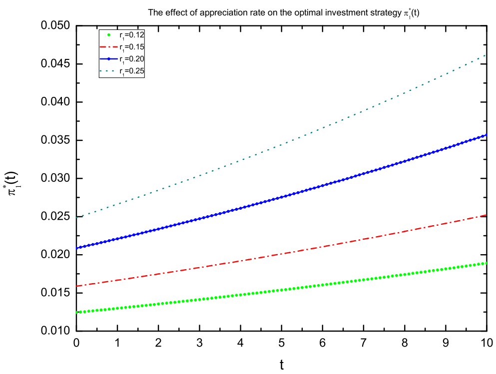

Figure 1: Sensitivity of the optimal strategy w.r.t r_1

Figure 1 shows that the amount invested in risky asset increases with the appreciation rate of risky asset r_1.It is because that as r_1 increases, the insurer will get more profits from risky asset. Therefore, the insurer would like to put more money in the risky asset to gain more profits.

In Figure 2, we plot the impact of interest rate r_0 on the optimal strategy. The optimal strategy decreases with r_0.As r_0 increases, the risk-free asset is more attractive, the insurer will invest more money in the risk-free asset. Thus, the money invested in the risky asset becomes less.

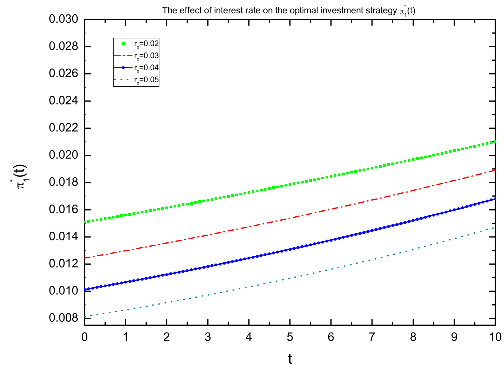

Figure 2: Sensitivity of the optimal strategy w.r.t r_0

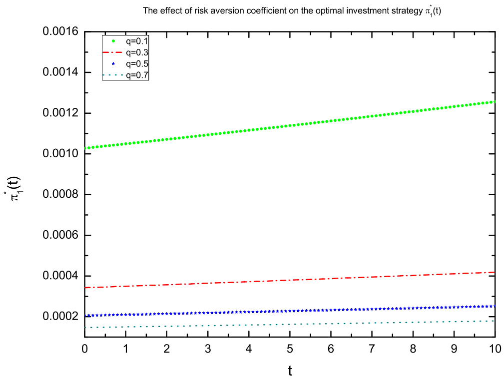

Figure 3: Sensitivity of the optimal strategy w.r.t q

From Figure 3, we find that the risk aversion coefficient q exerts a negative effect on the optimal strategy. The insurer is risk averse and they will invest less in risky asset as the risk aversion coefficient becomes higher.

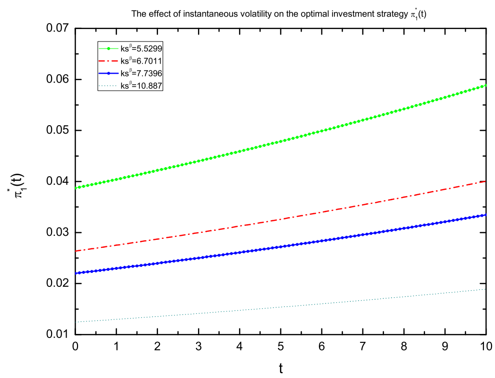

Figure 4: Sensitivity of the optimal strategy w.r.t $ks(0)^{\beta}$

In Figure.4, the optimal investment strategy is a decreasing function of the instantaneous volatility rate.As $ks(0)^{\beta}$ increases, the volatility of risky asset becomes bigger. Thus, it is not appropriate to carry out large-scale investment on risky asset. In order to reduce the impact of the volatility, the insurer will reduce investment in the risky asset investment.

4.2 Numerical analysis for the case of $\beta = 0$

According to Liang and Yuen [13], we assume that there are two business lines and the claim sizes X_{1j} and X_{2j} are exponentially distributed with parameters α_1 and α_2,respectively.Then $a_1 = \frac{1}{\alpha_1}$, $a_2 = \frac{1}{\alpha_2}$, $\gamma_1 = \frac{\sqrt{2(\lambda+\lambda_1)}}{\alpha_1}$, $\gamma_2 = \frac{\sqrt{2(\lambda+\lambda_2)}}{\alpha_2}$.

According to (4) and (5), the correlation coefficient of the two lines of business satisfies

$$\rho_{12} = \frac{\lambda}{\sqrt{2(\lambda+\lambda_1)}\sqrt{2(\lambda+\lambda_2)}}.$$

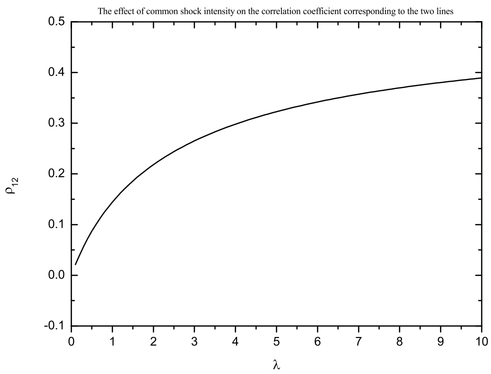

Figure 5: Sensitivity of the correlation coefficient corresponding to the two business lines w.r.t λ

Figure 5 shows the effect of common shock intensity λ on the correlation coefficient corresponding to the two dependent business lines. The correlation coefficient ρ_{12} is an increasing function of the common shock intensity λ. The correlation between line 1 and line 2 increases as λ increases.

Figure 6 shows the effect of common shock intensity λ on the optimal investment strategy π_2^*. From Figure 6, we can see that when $\rho_{10} = 1$, $\rho_{20} = 1$, the optimal investment strategy decreases with λ. Under the assumption that $\rho_{10} = 1$,$\rho_{20} = 1$, the correlation between the underwriting risk of line i for $i = 1, 2$ and investment risk is perfect positive correlated. As λ increases, the underwriting risk becomes larger. In order to reduce overall risk, the insurer will put less money in the risky asset. On the contrary, in the case

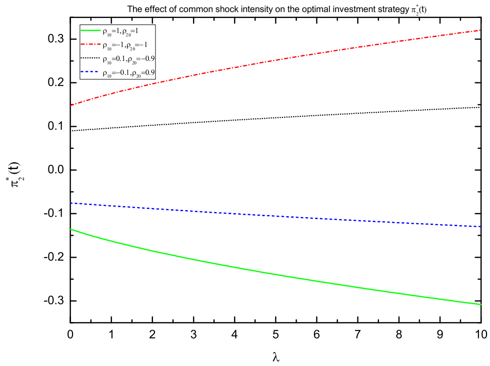

Figure 6: Sensitivity of the optimal strategy w.r.t λ

that $\rho_{10} = -1$,$\rho_{20} = -1$,the underwriting risk of line i and investment risk has negative correlation. In fact,π_2^* decreases with λ when $\rho_{i0} > 0$ for $i = 1, 2$ and increases with λ when $\rho_{i0} < 0$. From Figure 6, we find that in the case $\rho_{10} = -0.1$,$\rho_{20} = 0.9$,the positive correlation $\rho_{20} = 0.9$ plays a significant roles on π_2^*,then π_2^* decreases with λ while when $\rho_{10} = 0.1$,$\rho_{20} = -0.9$,the negative correlation $\rho_{20} = -0.9$ plays a significant roles on π_2^*,thus,π_2^* is an increasing function of λ.

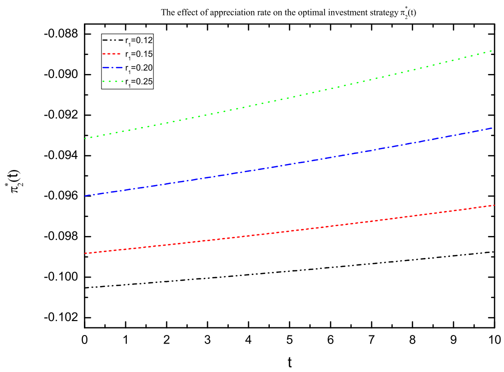

Figure 7: Sensitivity of the optimal strategy w.r.t r_1

Figures 7-10 show the effects of the appreciation rate of risky asset r_1,the interest rate r_0,the volatility of the risky asset k and the risk aversion coefficient q on the optimal strategy respectively for the case that the risky asset's price follows the GBM model. As shown in these figures, we find that the effects are similar to those under the $\rho_{i0} = 0$ cases.

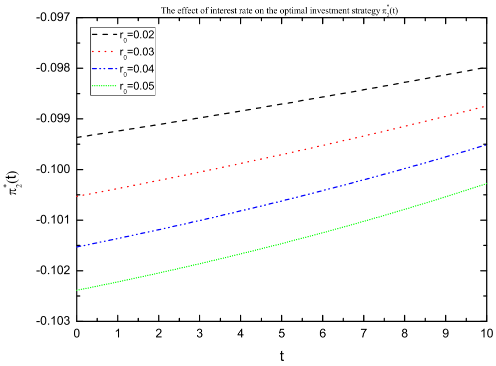

Figure 8: Sensitivity of the optimal strategy w.r.t r_0

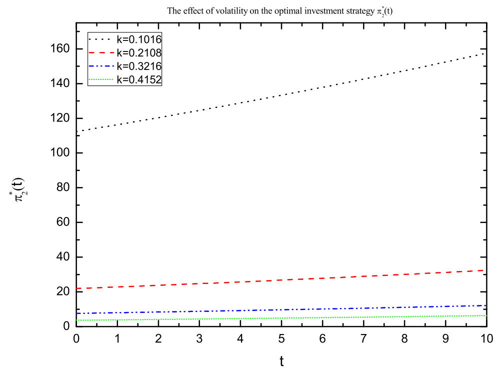

Figure 9: Sensitivity of the optimal strategy w.r.t k

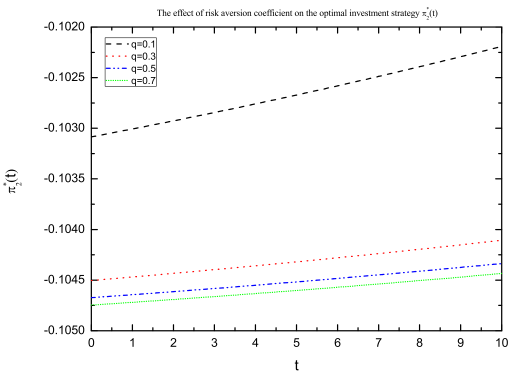

Figure 10: Sensitivity of the optimal strategy w.r.t q

5 Conclusion

Optimal investment problem for an insurer has been around in many literatures. In this paper,we study a more general optimal investment problem for an insurer. We consider an insurer who has n dependent classes of insurance business, and adopt the constant elasticity of variance (CEV) model to describe the dynamic of the risky asset's price process. By applying dynamic programming approach, we establish the corresponding Hamilton-Jacobi-Bellman (HJB) equation. For the objective of maximizing the expected utility of terminal wealth, we obtain explicit solutions for the exponential utility functions under some given assumptions. Finally, a numerical simulation is presented to analyze the properties of the optimal investment strategy. Some interesting results are found: (1) Under the case of $\rho_{i0} = 0$,the financial market and risk model are independent, we find that the claim processes have no effect on the optimal strategy. In practice, the impact of claim process of insurer on the volatility of the financial market is very small. (2) In the case of $\beta = 0$,the CEV model reduces to the GBM model, if $\rho_{i0} \neq 0$,the optimal investment strategy depends on counting processes and the correlation coefficients between risky asset and claim processes. (3) From the numerical simulation, we find that for both the case of $\rho_{i0} = 0$ and the case of $\beta = 0$,the appreciation rate of risky asset has a positive effect on the optimal strategies, while interest rate, volatility of the risky asset and risk aversion coefficient exert negative effects on the optimal strategies. Under the GBM model, the correlation corresponding to two lines of business increases as the common shock intensity λ increases.

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