Abstract: A graph $G$ is called $H$-equicoverable if every minimal $H$-covering of $G$ is also a minimum $H$-covering of $G$. In this paper, we investigate the characterization of $P_5$-equicoverable graphs which contain cycles with length at least 5 and give some results of $P_k$-equicoverable graphs.

Key Words: $P_5$-equicoverable, $P_k$-equicoverable, cycle, covering

1 Introduction

A graph $G$ has order $|V(G)|$ and size $|E(G)|$. If vertex $v$ is an endpoint of an edge $e$, then $v$ and $e$ are incident. The degree of vertex $v$ in a graph $G$, written $d_G(v)$ or $d(v)$, is the number of edges incident to $v$. The path and circuit on $k$ vertices are denoted by $P_k$ and $C_k$, respectively. A star is a tree consisting of one vertex adjacent to all the others. The $(n + 1)$-vertex star is the biclique $K_{1,n}$.

A graph $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. Suppose that $E'$ is a nonempty subset of $E$. The subgraph of $G$ whose vertex set is the set of ends of edges in $E'$ and whose edge set is $E'$ is called the subgraph of $G$ induced by $E'$ and is denoted by $G[E']$; $G[E']$ is an edge-induced subgraph of $G$.

The problem that we study stems from the research of $H$-decomposable graphs, randomly decomposable graphs and equipackable graphs. In 2008, Zhang introduced equicoverable graph which is the dual concept of the equipackable graph and characterized all $P_5$-equicoverable graphs. In this paper, we investigate all $P_5$-equicoverable graphs which don't contain 3-cycle or 4-cycle and contain at least one cycle with length at least 5. For further definitions and results, we can refer to [1],[2],[3],[4],[5],[6].

Let $H$ be a subgraph of a graph $G$. An $H$-covering of $G$ is a set $L = H_1, H_2, \ldots, H_k$ of subgraphs of $G$, where each subgraph $H_i$ isomorphic to $H$, and every edge of $G$ appears in at least one member of $L$. A graph is called $H$-coverable if there exists an $H$-covering of $G$. An $H$-covering of $G$ with $k$ copies $H_1, H_2, \ldots, H_k$ is called minimal if, for any $H_j$, $H_j \cup \bigcup_{i=1}^{k} H_i - H_j$ is not an $H$-covering of $G$. An $H$-covering of $H_1, H_2, \ldots, H_k$ is called minimal if there exists no $H$-covering with less than $k$ copies of $H$. A graph is called $H$-equicoverable if every minimal $H$-covering is also a minimum $H$-covering.

Let $C(G; H)$ denote the number of $H$ in the minimal $H$-covering of $G$, or simply $C(G)$ for short and let $c(G; H)$ denote the number of $H$ in the minimum $H$-covering of $G$, or simply $c(G)$ for short. For convenience, we denote by $C_n \cdot P_k$ a graph obtained from a cycle $C_n$ and a path $P_k$ by identifying one vertex of the cycle $C_n$ and an endpoint of the path $P_k$. And we denote by $C_n \cdot K_{1,k}$ a graph obtained from a cycle $C_n$ and a star $K_{1,k}$ by identifying one vertex of the cycle $C_n$ and a leaf of the star $K_{1,k}$.

Then we introduce a definition and a useful proposition:

Definition 1 [6] For a star $K_{1,k}$, we call the vertex of degree $k$ center, and other vertices leaves. A $k$-extendedstar that has one vertex of degree $k$ which is also called center, $k$ vertices of degree 2 and $k$ leaves is a tree obtained by inserting a vertex of degree 2 into each edge of a star $K_{1,k}$. We denote it by $S_k^*$. A second order $k$-extendedstar is a tree obtained by inserting two vertices of degree 2 into each edge of a star $K_{1,k}$, we denote it by $S_k^{*2}$. Similarly, an $n$-th order $k$-extendedstar is a tree obtained by inserting $n$ vertices of degree 2 into each edge of a star $K_{1,k}$, we denote it by $S_k^{*n}$.

In this paper, we denote by $C_n \cdot S_k^{*n}$ a graph obtained from a cycle $C_n$ and an $n$-th order $k$-extendedstar by identifying one vertex of the cycle $C_n$.
and the center of the \( n \)-th order \( k \)-extendedstar. We denote by \( P_n \cdot K_{1,k} \) a graph obtained from a path \( P_n \) and a \( k \)-star by identifying one endpoint of the path \( P_n \) and one leaf of the \( k \)-star.

**Proposition 2** A connected graph \( G \) is \( P_5 \)-equicoverable if and only if it has a subgraph \( P_5 \) except the kind of graphs in Figure 1.

![Figure 1: graphs which are not \( P_5 \)-equicoverable](image)

**Lemma 3** If a connected graph \( G \) can be decomposed into several connected \( P_k \)-equicoverable graphs and at least one component is not \( P_k \)-equicoverable, \( G \) will not be \( P_k \)-equicoverable.

**Theorem 4** [5] Path \( P_n \) is \( P_k \)-equicoverable if and only if \( k \leq n \leq 2k \) or \( n = 3k - 1 \).

**Theorem 5** [5] Cycle \( C_n \) is \( P_k \)-equicoverable if and only if

\[
\begin{align*}
  &k \leq n \leq \frac{3k-1}{2} \text{ or } n = 2k - 1 \text{ if } k \text{ is odd}, \\
  &k \leq n \leq \frac{3k-2}{2} \text{ or } n = 2k - 1 \text{ if } k \text{ is even}.
\end{align*}
\]

**Lemma 6** \( S_k^{n*} \) is \( P_{n+2} \)-equicoverable and \( c(S_k^{n*}; P_{n+2}) = C(S_k^{n*}; P_{n+2}) = k \).

**Proof:** \( S_k^{n*} \) can be obtained by identifying the endpoints of \( k \) copies of \( P_{n+2} \). The \( S_k^{n*} \) contains a path of length at most \( 2n + 2 \), that is, \( P_{2n+3} \).

By Theorem 4, \( P_{2n+3} \) is \( P_{n+2} \)-equicoverable and \( c(P_{2n+3}; P_{n+2}) = C(P_{2n+3}; P_{n+2}) = 2 \). If \( k \) is even, \( c(S_k^{n*}; P_{n+2}) = C(S_k^{n*}; P_{n+2}) = \frac{k}{2} \times 2 = k \); If \( k \) is odd, \( c(S_k^{n*}; P_{n+2}) = C(S_k^{n*}; P_{n+2}) = \frac{k-1}{2} \times 2 + 1 = k \).

2 \( P_5 \)-equicoverable graphs

First, we introduce \( P_5 \)-equicoverable paths and cycles.

**Lemma 7** [5] The path \( P_n \) is \( P_5 \)-equicoverable if and only if \( n = 5, 6, 7, 8, 9, 10, 14 \).

**Proof:** By Theorem 4, we give the results. \( \Box \)

**Lemma 8** [5] The cycle \( C_n \) is \( P_5 \)-equicoverable if and only if \( n = 5, 6, 7, 9 \).

**Proof:** We can refer to Theorem 5. \( \Box \)

**Lemma 9** \( G \) is a connected graph that is not a cycle. If \( G \) doesn’t contain any 3-cycles or 4-cycles and contains a 5-cycle, \( G \) will not be \( P_5 \)-equicoverable unless \( G \) is \( C_5 \cdot S_n^{3*} \) or \( G \) is obtained by adding \( n \) copies of \( P_3 \cdot K_{1,t} (t \geq 3) \) to only one vertex of \( C_5 \).

**Proof:** Case 1: \( G \) is obtained by adding copies of \( P_2 \) to the vertices of \( C_5 \).

1. If each vertex of \( C_5 \) can be added to at most one \( P_2 \), \( G \) can only be one of the seven graphs shown in Figure 2. No matter which graph is in Figure 2, a minimal \( P_5 \)-covering whose covering number \( c(G) \) is greater than the number of the minimum \( P_5 \)-covering \( c(G) \). So the graphs are not \( P_5 \)-equicoverable.

![Figure 2: graphs obtained by adding at most one \( P_2 \) to each vertex of \( C_5 \)](image)

(2) If each vertex of \( C_5 \) can be added to any copies of \( P_2 \), \( G \) is obtained by adding copies of \( P_2 \) to the vertices of the 5-cycle part of \( G_0 \), where \( G_0 \) is one of the graphs in Figure 2. If the number of the copies of \( P_2 \) added is \( n \), we can get a minimal \( P_5 \)-covering whose covering number is \( c(G_0) + n \) (using \( c(G_0) \) copies of \( P_5 \) to cover the \( G_0 \) part and \( n \) copies of \( P_5 \) to cover other parts), while the number of the minimum \( P_5 \)-covering is at most \( c(G_0) + n \). By (1), each of \( G_0 \) is not \( P_5 \)-equicoverable, then \( c(G_0) > c(G_0) \). So \( G \) is not \( P_5 \)-equicoverable.

Case 2: \( G \) is obtained by adding copies of \( P_3 \) to the vertices of \( C_5 \).
Note that we identify the endpoint of each copy of $P_3$ with the vertices of $C_5$, not the center vertex. Otherwise $G$ is the same as one of the graph in Case 1.

(1) If each vertex of $C_5$ can be added to at most one $P_3$, $G$ can only be one of the seven graphs shown in Figure 3. No matter which graph is in Figure 3, a minimal $P_5$-covering whose covering number $C(G)$ is greater than the number of the minimum $P_5$-covering $c(G)$. So the graphs are not $P_5$-equicoverable.

![Figure 3: graphs obtained by adding at most one $P_3$ to each vertex of $C_5$](image1)

(2) If each vertex of $C_5$ can be added to any copies of $P_3$, $G$ is obtained by adding copies of $P_3$ to the vertices of the 5-cycle part of $G_0$, where $G_0$ is one of the graphs in Figure 3. If the number of the copies of $P_3$ added is $n$, we can get a minimal $P_5$-covering whose covering number is $C(G_0) + n$ (using $C(G_0)$ copies of $P_3$ to cover the $G_0$ part and $n$ copies of $P_5$ to cover other parts), while the number of the minimum $P_5$-covering is at most $c(G_0) + n$. By (1), each of $G_0$ is not $P_5$-equicoverable, then $C(G_0) > c(G_0)$. So $G$ is not $P_5$-equicoverable.

Case 3: $G$ is obtained by adding copies of $K_{1,t}$ ($t \geq 3$) to the vertices of $C_5$.

Note that we identify one of leaves of each copy of $K_{1,t}$ with the vertices of $C_5$, not the center vertex. Otherwise $G$ is the same as one of the graph in Case 1.

(1) If each vertex of $C_5$ can be added to at most one $K_{1,t}$, $G$ can only be one of the seven graphs shown in Figure 4. No matter which graph is in Figure 4, a minimal $P_5$-covering whose covering number $C(G)$ is greater than the number of the minimum $P_5$-covering $c(G)$. So the graphs are not $P_5$-equicoverable.

![Figure 4: graphs obtained by adding at most one $K_{1,t}$ to each vertex of $C_5$](image2)
(2) If each vertex of $C_5$ can be added to any copies of $K_{1,1,1}$. $G$ is obtained by adding copies of $K_{1,1,1}$ to the vertices of the 5-cycle part of $G_0$, where $G_0$ is one of the graphs in Figure 4. If the number of the copies of $K_{1,1,1}$ added is $n$, we can get a minimal $P_5$-covering whose covering number is $C(G_0) + n(t - 1)$ (using $C(G_0)$ copies of $P_5$ to cover the $G_0$ part and $n(t - 1)$ copies of $P_5$ to cover other parts), while the number of the minimum $P_5$-covering is at most $c(G_0) + n(t - 1)$. By (1), each of $G_0$ is not $P_5$-equicoverable, then $C(G_0) > c(G_0)$. So $G$ is not $P_5$-equicoverable.

Actually, this case is similar to Case 2.

Case 4: $G$ is obtained by adding copies of $P_2$ and $P_3$ to the vertices of $C_5$.

If only copies of $P_2$ or only copies of $P_3$ are added, $G$ has been discussed in Case 1 or Case 2. Otherwise, we have:

(1) If each vertex of $C_5$ can be added to only one $P_2$ or one $P_3$, $G$ can only be one of the 24 graphs shown in Figure 5. No matter which graph is in Figure 5, a minimal $P_5$-covering whose covering number $C(G)$ is greater than the number of the minimum $P_5$-covering $c(G)$. So the graphs are not $P_5$-equicoverable.

Figure 5: graphs obtained by adding only one $P_2$ or one $P_3$ to each vertex of $C_5$

(2) If each vertex of $C_5$ can be added to any copies of $P_2$ or $P_3$, $G$ is obtained by adding copies of $P_2$ and $P_3$ to the vertices of the 5-cycle part of $G_0$, where $G_0$ is one of the graphs in Figure 5. If the number of the copies of $P_2$ and $P_3$ added is $n$, we can get a minimal $P_5$-covering whose covering number is $C(G_0) + n$ (using $C(G_0)$ copies of $P_5$ to cover the $G_0$ part and $n$ copies of $P_5$ to cover other parts), while the number of the minimum $P_5$-covering is at most $c(G_0) + n$. By (1), each of $G_0$ is not $P_5$-equicoverable, then $C(G_0) > c(G_0)$. So $G$ is not $P_5$-equicoverable.

(3) If each vertex of $C_5$ can be added to at most one $P_2 \cdot P_3$, $G$ can only be one of the seven graphs shown in Figure 6. No matter which graph is in Figure 6, a minimal $P_5$-covering whose covering number $C(G)$ is greater than the number of the minimum $P_5$-covering $c(G)$. So the graphs are not $P_5$-equicoverable; If each vertex of $C_5$ can be added to
any copies of $P_2 \cdot P_3$, $G$ can be decomposed several components which can be $P_5$-coverable. While there is at least one component which is similar to Case 1 or Case 4(2) not $P_5$-equicoverable. $G$ is not $P_5$-equicoverable.

Figure 6: graphs obtained by adding at most one $P_2 \cdot P_3$ to each vertex of $C_5$

Case 5: $G$ is obtained by adding copies of $P_2$ and $K_{1,t}(t \geq 3)$ to the vertices of $C_5$.

The case is similar to Case 4. $G$ is not $P_5$-equicoverable.

Case 6: $G$ is obtained by adding copies of $P_3$ and $K_{1,t}(t \geq 3)$ to the vertices of $C_5$.

The case is similar to Case 2. $G$ is not $P_5$-equicoverable.

Case 7: $G$ is obtained by adding copies of $P_2$ and $P_3$ and $K_{1,t}(t \geq 3)$ to the vertices of $C_5$.

The case is similar to Case 4. $G$ is not $P_5$-equicoverable.

Case 8: $G$ is obtained by adding copies of $P_4$ to the vertices of $C_5$.

(1) If each vertex of $C_5$ can be added to at most one $P_4$, $G$ can only be one of the seven graphs shown in Figure 7. No matter which graph is in Figure 7, a minimal $P_5$-covering whose covering number $C(G)$ is greater than the number of the minimum $P_5$-covering $c(G)$. So the graphs are not $P_5$-equicoverable.

Figure 7: graphs obtained by adding at most one $P_4$ to each vertex of $C_5$

(2) If each vertex of $C_5$ can be added to any copies of $P_4$, $G$ is obtained by adding copies of $P_4$ to the vertices of the 5-cycle part of $G_0$, where $G_0$ is one of the graphs in Figure 7. If the number of the copies of $P_4$ added is $n$, we can get a minimal $P_5$-covering whose covering number is $C(G_0) + n$ (using $C(G_0)$ copies of $P_5$ to cover the $G_0$ part and $n$ copies of $P_5$ to cover other parts), while the number of the minimum $P_5$-covering is at most $c(G_0) + n$. By (1), each of $G_0$ is not $P_5$-equicoverable, then $C(G_0) > c(G_0)$. So $G$ is not $P_5$-equicoverable.

Case 9: $G$ is obtained by adding copies of $P_2$ and $P_4$ to the vertices of $C_5$.

If only copies of $P_2$ or only copies of $P_4$ are added, $G$ has been discussed in Case 1 or Case 8. Otherwise, we have:

(1) If each vertex of $C_5$ can be added to only one $P_2$ or one $P_4$, $G$ can only be one of 24 graphs similar
of the graphs above. If the number of the vertices of the 5-cycle part of $G_0$ is one of the graphs in (1). If the number of the copies of $P_2$ and $P_4$ added is $n$, we can get a minimal $P_5$-covering whose covering number is $C(G_0) + n$ (using $C(G_0)$ copies of $P_5$ to cover the $G_0$ part and $n$ copies of $P_5$ to cover other parts), while the number of the minimum $P_5$-covering is at most $c(G_0) + n$. By (1), each of $G_0$ is not $P_5$-equicoverable, then $C(G_0) > c(G_0)$. So $G$ is not $P_5$-equicoverable.

(3) If each vertex of $C_5$ can be added to at most one $P_2, P_4$, $G$ can only be one of the seven graphs similar to Figure 6. No matter which graph is, a minimal $P_5$-covering whose covering number $C(G)$ is greater than the number of the minimum $P_5$-covering $c(G)$. So the graphs are not $P_5$-equicoverable. 

(2) If each vertex of $C_5$ can be added to any copies of $P_2$ or $P_4$, $G$ is obtained by adding copies of $P_2$ and $P_4$ to the vertices of the 5-cycle part of $G_0$, where $G_0$ is one of the graphs in (1). If the number of the copies of $P_2$ and $P_4$ added is $n$, we can get a minimal $P_5$-covering whose covering number is $C(G_0) + n$ (using $C(G_0)$ copies of $P_5$ to cover the $G_0$ part and $n$ copies of $P_5$ to cover other parts), while the number of the minimum $P_5$-covering is at most $C(G_0) + n$. By (1), each of $G_0$ is not $P_5$-equicoverable, then $C(G_0) > c(G_0)$. So $G$ is not $P_5$-equicoverable.

Case 10: $G$ is obtained by adding copies of $P_3$ and $P_4$ to the vertices of $C_5$.

If only copies of $P_3$ or only copies of $P_4$ are added, $G$ has been discussed in Case 2 or Case 8. Otherwise, we have:

(1) If each vertex of $C_5$ can be added to only one $P_3$ or one $P_4$, $G$ can only be one of the 24 graphs similar to Figure 5. No matter which graph is, a minimal $P_5$-covering whose covering number $C(G)$ is greater than the number of the minimum $P_5$-covering $c(G)$. So the graphs are not $P_5$-equicoverable.

(2) If each vertex of $C_5$ can be added to any copies of $P_3$ or $P_4$, $G$ is obtained by adding copies of $P_3$ and $P_4$ to the vertices of the 5-cycle part of $G_0$, where $G_0$ is one of the graphs above in (1). If the number of the copies of $P_3$ and $P_4$ added is $n$, we can get a minimal $P_5$-covering whose covering number is $C(G_0) + n$ (using $C(G_0)$ copies of $P_5$ to cover the $G_0$ part and $n$ copies of $P_5$ to cover other parts), while the number of the minimum $P_5$-covering is at most $C(G_0) + n$. By (1), each of $G_0$ is not $P_5$-equicoverable, then $C(G_0) > c(G_0)$. So $G$ is not $P_5$-equicoverable.

(3) If each vertex of $C_5$ can be added to at most one $P_3, P_4, G$ can only be one of the seven graphs similar to Figure 6. No matter which graph is, a minimal $P_5$-covering whose covering number $C(G)$ is greater than the number of the minimum $P_5$-covering $c(G)$. So the graphs are not $P_5$-equicoverable. If each vertex of $C_5$ can be added to any copies of $P_3, P_4, G$ can be obtained by adding copies of $P_3$ and $P_4$ to the vertices of the 5-cycle part of $G_0$, where $G_0$ is one of the graphs above. If the number of the copies of $P_3$ and $P_4$ added is $n$, we can get a minimal $P_5$-covering whose covering number is $C(G_0) + 2n$ (using $C(G_0)$ copies of $P_5$ to cover the $G_0$ part and $2n$ copies of $P_5$ to cover other parts), while the number of the minimum $P_5$-covering is at most $C(G_0) + 2n$. Each of $G_0$ is not $P_5$-equicoverable, then $C(G_0) > c(G_0)$. So $G$ is not $P_5$-equicoverable.

Case 11: $G$ is obtained by adding copies of $P_2, P_3$ and $P_4$ to the vertices of $C_5$.

$P_2, P_3$ and $P_4$ are all added to the vertices of $C_5$, otherwise the cases has been discussed.

First, $G$ can be obtained by adding copies of $P_2$ and $P_3$ to the vertices of $C_5$ and we denote it by $G_{23}$. Next we add $P_4$ to $G_{23}$. If the number of the copies of $P_4$ added is $n$, we can get a minimal $P_5$-covering whose covering number is $C(G_{23}) + n$ (using $C(G_{23})$ copies of $P_5$ to cover the $G_{23}$ part and $n$ copies of $P_5$ to cover other parts), while the number of the minimum $P_5$-covering is at most $C(G_{23}) + n$. Each of $G_{23}$ is not $P_5$-equicoverable by Case 4, then $C(G_{23}) > c(G_{23})$. So $G$ is not $P_5$-equicoverable.

Case 12: $G$ is obtained by adding copies of $P_4$ and $K_{1,t}(t \geq 3)$ to the vertices of $C_5$.

The case is similar to Case 10. $G$ is not $P_5$-equicoverable.

Case 13: $G$ is obtained by adding copies of $P_2, P_4$ and $K_{1,t}(t \geq 3)$ to the vertices of $C_5$.

The case is similar to Case 11. $G$ is not $P_5$-equicoverable.

Case 14: $G$ is obtained by adding copies of $P_3, P_4$ and $K_{1,t}$ to the vertices of $C_5$.

The case is similar to Case 10. $G$ is not $P_5$-equicoverable.

Case 15: $G$ is obtained by adding copies of $P_2, P_3, P_4$ and $K_{1,t}(t \geq 3)$ to the vertices of $C_5$.

The case is similar to Case 11. $G$ is not $P_5$-equicoverable.

Case 16: $G$ is obtained by adding copies of $P_5$ to the vertices of $C_5$.

(1) If we add $n$ copies of $P_5$ to only one vertex of $C_5$, both the minimal $P_5$-covering number and the minimum $P_5$-covering number are $n + 2$. So it is $P_5$-equicoverable. We denote the graph by $C_5 \cdot S^3_n$.

(2) If we add $n$ copies of $P_5$ to at least two vertices of $C_5$, there exists a minimal $P_5$-covering number is
119

Lemma 10 \( C_n \cdot P_2(n \geq 6) \) is \( P_5 \)-equicoverable if and only if \( n = 8 \).

**Proof:** (1) If \( C_n \) is \( P_5 \)-equicoverable, we have \( n = 6, 7, 9 \). Because \( C(C_n \cdot P_2; P_5) > c(C_n \cdot P_2; P_3)(n = 6, 7, 9) \), \( C_n \cdot P_2 \) and \( C_7 \cdot P_2 \) and \( C_9 \cdot P_2 \) are not \( P_5 \)-equicoverable.

(2) If \( C_n \) is not \( P_5 \)-equicoverable, we have \( n \neq 6, 7, 9 \). It is easy to find that \( C(C_8 \cdot P_2; P_5) = c(C_8 \cdot P_2; P_5) = 3 \). \( C_8 \cdot P_2 \) is \( P_5 \)-equicoverable. For \( n \geq 10 \), \( C_n \) is not \( P_5 \)-equicoverable. We can use \( C(C_n) \) copies of \( P_5 \) to cover the \( C_n \) part and one copy of \( P_5 \) to cover the else. Also, we can use \( c(C_n) \) copies of \( P_5 \) to cover the \( C_n \) part and one copy of \( P_5 \) to cover the else. While \( c(C_n \cdot P_3) \leq c(C_n) + 1 < C(C_n) + 1 \), \( G \) is not \( P_5 \)-equicoverable.

\( \square \)

Lemma 11 \( C_n \cdot P_3(n \geq 6) \) is \( P_5 \)-equicoverable if and only if \( n = 7 \).

**Proof:** (1) If \( C_n \) is \( P_5 \)-equicoverable, we have \( n = 6, 7, 9 \). Because \( C(C_n \cdot P_3; P_5) > c(C_n \cdot P_3; P_3)(n = 6, 9) \), \( C_6 \cdot P_3 \) and \( C_9 \cdot P_3 \) are not \( P_5 \)-equicoverable. While \( C(C_7 \cdot P_3; P_5) = c(C_7 \cdot P_3; P_5) = 3 \). \( C_7 \cdot P_3 \) is \( P_5 \)-equicoverable.

(2) If \( C_n \) is not \( P_5 \)-equicoverable, we have \( n \neq 6, 7, 9 \). It is easy to find that \( C(C_8 \cdot P_3; P_5) > c(C_8 \cdot P_3; P_3) \). \( C_8 \cdot P_3 \) is \( P_5 \)-equicoverable. For \( n \geq 10 \), \( C_n \) is not \( P_5 \)-equicoverable. We can use \( C(C_n) \) copies of \( P_5 \) to cover the \( C_n \) part and one copy of \( P_5 \) to cover the else. Also, we can use \( c(C_n) \) copies of \( P_5 \) to cover the \( C_n \) part and one copy of \( P_5 \) to cover the else. While \( c(C_n \cdot P_3) \leq c(C_n) + 1 < C(C_n) + 1 \), \( G \) is not \( P_5 \)-equicoverable.

\( \square \)

Lemma 12 \( C_n \cdot P_4(n \geq 6) \) is \( P_5 \)-equicoverable if and only if \( n = 6 \).

**Proof:** (1) If \( C_n \) is \( P_5 \)-equicoverable, we have \( n = 6, 7, 9 \). Because \( C(C_n \cdot P_4; P_5) > c(C_n \cdot P_4; P_3)(n = 7, 9) \), \( C_7 \cdot P_4 \) and \( C_9 \cdot P_4 \) are not \( P_5 \)-equicoverable. While \( C(C_6 \cdot P_4; P_3) = c(C_6 \cdot P_4; P_3) = 3 \). \( C_6 \cdot P_4 \) is \( P_5 \)-equicoverable.

(2) If \( C_n \) is not \( P_5 \)-equicoverable, we have \( n \neq 6, 7, 9 \). It is easy to find that \( C(C_8 \cdot P_4; P_3) > c(C_8 \cdot P_4; P_3) \). \( C_8 \cdot P_4 \) is \( P_5 \)-equicoverable. For \( n \geq 10 \), \( C_n \) is not \( P_5 \)-equicoverable. We can use \( C(C_n) \) copies of \( P_5 \) to cover the \( C_n \) part and one copy of \( P_5 \) to cover the else. Also, we can use \( c(C_n) \) copies of \( P_5 \) to cover the \( C_n \) part and one copy of \( P_5 \) to cover the else. While \( c(C_n \cdot P_4) \leq c(C_n) + 1 < C(C_n) + 1 \), \( G \) is not \( P_5 \)-equicoverable.

\( \square \)

Lemma 13 \( C_n \cdot P_5(n \geq 6) \) is not \( P_5 \)-equicoverable.

Lemma 14 \( C_n \cdot K_{1,t}(n \geq 4, t \geq 3) \) is not \( P_5 \)-equicoverable.

Lemma 15 \( C_n \cdot P_2 \cdot K_{1,t}(n \geq 4) \) is not \( P_5 \)-equicoverable.

Lemma 16 \( C_n \cdot P_3 \cdot K_{1,t}(n \geq 6) \) is not \( P_5 \)-equicoverable.

Lemma 17 \( G \) is a connected graph that is not a cycle. If \( G \) doesn’t contain cycles with length smaller than 6 and contains a 6-cycle, \( G \) is \( P_5 \)-equicoverable if and only if \( G \) is \( C_6 \cdot P_4 \).

**Proof:** Case 1: \( G \) is obtained by adding copies of \( P_2 \) to the vertices of \( C_6 \).

(1) If we add one \( P_2 \) to only one vertex of \( C_6 \), by Lemma 10, it is not \( P_5 \)-equicoverable.

(2) If we add \( n(n \geq 2) \) copies of \( P_2 \) to only one vertex of \( C_6 \), there will be a minimal \( P_5 \)-covering whose covering number is \( n + 2 \). While the number of the minimum \( P_5 \)-covering number is less than or equal to \( n + 1 \).

(3) If we add \( n(n \geq 2) \) copies of \( P_2 \) to at least two vertices of \( C_6 \) and each vertex of \( C_6 \) can be added to at most one \( P_2 \), \( G \) must be one of the eleven graphs
shown in Figure 7. For each graph which contains a 6-cycle, we can blow up a vertex that no \( P_2 \) is added to of \( C_6 \) to two vertices. As a consequence, the original graph with a 6-cycle turns out to be a tree. A blowing up that makes the result tree not \( P_5 \)-equicoverable must exist. So \( G \) is not \( P_5 \)-equicoverable. For example, we blow up \( v_1 \) of the left graph to two vertices \( v_2 \) and \( v_3 \) of the right graph in Figure 8. Obviously, it’s not \( P_5 \)-equicoverable.

![Figure 7: graphs obtained by adding \( n(n \geq 2) \) copies of \( P_2 \) to at least two vertices of \( C_6 \) can be added to at most one \( P_2 \)](image)

Figure 8: \( v_1 \) blown up to two vertices \( v_2 \) and \( v_3 \)

(4) If we add \( n(n \geq 2) \) copies of \( P_2 \) to at least two vertices of \( C_6 \) and each vertex of \( C_6 \) can be added to any copies of \( P_2 \). Without loss of generality, suppose \( G \) is obtained by adding \( m \) copies of \( P_2 \) to \( G_0 \), where \( G_0 \) is one of graphs above in (3). Then there exists a minimal \( P_5 \)-covering whose covering number is \( C(G_0) + m \). We can use \( C(G_0) \) copies of \( P_5 \) to cover the \( G_0 \) part and use \( m \) copies of \( P_5 \) to cover other parts. While the number of the minimum \( P_5 \)-covering number is at most \( c(G_0) + m \). As we all know, for each \( G_0 \), there exists a minimal \( P_5 \)-covering whose \( C(G_0) > c(G_0) \), then it is not \( P_5 \)-equicoverable.

Case 2: \( G \) is obtained by adding copies of \( P_3 \) to the vertices of \( C_6 \).

(1) If we add one \( P_3 \) to only one vertex of \( C_6 \), by Lemma 11, it is not \( P_5 \)-equicoverable.

(2) If we add \( n(n \geq 2) \) copies of \( P_3 \) to only one vertex of \( C_6 \), there will be a minimal \( P_5 \)-covering whose covering number is \( n + 2 \). While the number of the minimum \( P_5 \)-covering number is less than or equal to \( n + 1 \).

(3) If we add \( n(n \geq 2) \) copies of \( P_3 \) to at least two vertices of \( C_6 \) and each vertex of \( C_6 \) can be added to at most one \( P_3 \), \( G \) must be one of the eleven graphs similar to Figure 7. For each graph which contains a 6-cycle, we can blow up a vertex that no \( P_3 \) is added to of \( C_6 \) to two vertices. As a consequence, the original graph with a 6-cycle turns out to be a tree. A blowing up that makes the result tree not \( P_5 \)-equicoverable must exist. So \( G \) is not \( P_5 \)-equicoverable.

(4) If we add \( n(n \geq 2) \) copies of \( P_3 \) to at least two vertices of \( C_6 \) and each vertex of \( C_6 \) can be added to any copies of \( P_3 \). Without loss of generality, suppose \( G \) is obtained by adding \( m \) copies of \( P_3 \) to \( G_0 \), where \( G_0 \) is one of graphs above in (3). Then there exists a minimal \( P_5 \)-covering whose covering number is \( C(G_0) + m \). We can use \( C(G_0) \) copies of \( P_5 \) to cover the \( G_0 \) part and use \( m \) copies of \( P_5 \) to cover other parts. While the number of the minimum \( P_5 \)-covering number is at most \( c(G_0) + m \). As we all know, for each \( G_0 \), there exists a minimal \( P_5 \)-covering whose \( C(G_0) > c(G_0) \), then it is not \( P_5 \)-equicoverable.

Case 3: \( G \) is obtained by adding copies of \( K_{1,t}(t \geq 3) \) to the vertices of \( C_6 \).

Similar to Case 2, \( G \) is not \( P_5 \)-equicoverable.

Case 4: \( G \) is obtained by adding copies of \( P_4 \) to the vertices of \( C_6 \).

(1) If we add one \( P_4 \) to only one vertex of \( C_6 \), by Lemma 12, it is \( P_5 \)-equicoverable.

(2) The following proof is similar to (2), (3), (4) in Case 2, \( G \) is not \( P_5 \)-equicoverable.

Case 5: \( G \) is obtained by adding copies of \( P_2, P_3, P_4, K_{1,t}(t \geq 3) \) to the vertices of \( C_6 \).

There are eleven subcases: \( G \) is obtained by adding copies of at least two of \( P_2, P_3, P_4, K_{1,t}(t \geq 3) \). Similar to the proof process of Case 2, \( G \) is not \( P_5 \)-equicoverable.
Case 6: $G$ is obtained by adding copies of $P_5$ to the vertices of $C_6$.

(1) If we add one $P_5$ to only one vertex of $C_6$, by Lemma 13, it is not $P_5$-equicoverable.

(2) If $G$ is not the graph in (1), $G$ can be decomposed into two connected components: a graph which is not $P_5$-equicoverable and a $P_5$-coverable graph. By Lemma 3, $G$ is not $P_5$-equicoverable.

Case 7: $G$ is obtained by adding copies of $P_4$ and $P_5$ to the vertices of $C_6$.

If only copies of $P_4$ or only copies of $P_5$ are added, $G$ has been discussed in previous. Otherwise, similar to Case 4 of Lemma 9, $G$ is not $P_5$-equicoverable.

Case 8: $G$ is a graph not contained in Case 1-7.

We decompose $G$ into two connected components: a graph $G_0$ contained in Case 1-7 and a graph which is $P_5$-coverable. $G_0$ is not $P_5$-equicoverable, by Lemma 3, $G$ is not $P_5$-equicoverable.

In summary, $G$ is not $P_5$-equicoverable unless it satisfies one of the following: $G \equiv C_6 \cdot P_4$.

Lemma 18 $G$ is a connected graph that is not a cycle.

If $G$ doesn’t contain cycles with length smaller than 7 and contains a 7-cycle, $G$ is $P_5$-equicoverable if and only if $G$ is $C_7 \cdot P_3$.

Lemma 19 $G$ is a connected graph that is not a cycle.

If $G$ doesn’t contain cycles with length smaller than 8 and contains an 8-cycle, $G$ is $P_5$-equicoverable if and only if $G$ is $C_8 \cdot P_2$.

Lemma 20 $G$ is a connected graph that is not a cycle.

If $G$ doesn’t contain cycles with length smaller than 9, $G$ is not $P_5$-equicoverable.

Proof: Case 1: If $G$ is one of the graphs in Lemma 10-Lemma 16, $G$ is not $P_5$-equicoverable.

Case 2: If $G$ is not a graph in Case 1, according to the proof process of Lemma 17, $G$ can be decomposed into connected components: a tree which is not $P_5$-equicoverable and $P_5$-coverable graphs.

In the end, we conclude the main results: A connected graph $G$ is $P_5$-equicoverable if and only if $G$ satisfies one of the following:

Theorem 21 Let $G$ be a connected graph that doesn’t contain 3-cycles or 4-cycles and contains a cycle with length at least 5. Then $G$ is $P_5$-equicoverable if and only if either of the following holds:

1. $G$ is a cycle $C_n$ ($n = 5, 6, 7, 9$);
2. $G$ is $C_5 \cdot S_n^3$ ($n \geq 1$);
3. $G$ is obtained by adding $n$ copies of $P_3 \cdot K_{1,t}$ ($t \geq 3$) to only one vertex of $C_5$.
4. $G$ is $C_6 \cdot P_4$.
5. $G$ is $C_7 \cdot P_3$.
6. $G$ is $C_8 \cdot P_2$.

For disconnected graphs, we have:

Theorem 22 A graph $G$ that doesn’t contain 3-cycles or 4-cycles and contains at least one cycle with length larger than 4 is $P_5$-equicoverable if and only if each component of $G$ is $P_5$-equicoverable.

3 Results of $P_k$-equicoverable graphs

Theorem 23 $C_n \cdot P_2$ is $P_k$-equicoverable if and only if $n = k - 1$ or $n = 2k - 2$.

Proof:

1. When $n \leq k - 2$, $C_n \cdot P_2$ doesn’t contain the subgraph of $P_k$. Then it is not $P_k$-equicoverable.

2. When $n = k - 1$, $C_n \cdot P_2$ is $P_k$-equicoverable and $C(C_n \cdot P_2; P_k) = c(C_n \cdot P_2; P_k) = 2$.

3. When $k - n \leq 2k - 3$, it is easy to find $c(C_n \cdot P_2; P_k) = 2$. Conveniently, denote the edges of $C_n \cdot P_2$ by $e_0, e_1, \cdots e_n$. There exits a minimal $P_k$-covering as following: we denote it by $H = \{H_1, H_2, H_3\}$,

$$H_1 = \{e_0, e_1, e_2, \cdots, e_{k-2}\},$$
$$H_2 = \{e_n, e_1, e_2, \cdots, e_{k-2}\},$$
$$H_3 = \{e_{k-1}, e_k, e_{k+1}, \cdots, e_{n-1}\}.$$

Then $H$ is a minimal $P_k$-covering instead of the minimum $P_k$-covering of $C_n$. It is not $P_k$-equicoverable.

4. When $n = 2k - 2$, $C_n \cdot P_2$ is $P_k$-equicoverable. It is clear that $c(C_n \cdot P_2; P_k) = 3$. We denote the vertices of $C_n \cdot P_2$ by $v_0, v_1, v_2, \cdots, v_{2k-2}$. Generally speaking, suppose that there exists a copy of $P_k$ covering the edge $v_1v_2$, which is denoted by $H_0 = \{v_1v_2, v_2v_3, \cdots, v_{k-1}v_k\}$. Then there also exists a copy of $P_k$ covering the edge $v_kv_k$, which is denoted by $H_i = \{v_{i+1}v_{i+2}, v_{i+2}v_{i+3}, \cdots, v_{i+k-2}v_{i+k-1}\}$ ($0 \leq i \leq k - 1$). Similarly, there must be a copy of $P_k$ covering the edge $v_kv_0$, which is denoted by $H_1 = \{v_{k+1}v_{k+2}, v_{k+2}v_{k+3}, \cdots, v_{2k-3}v_{2k-2}, v_{2k-2}v_{1}, v_{1}v_{0}\}$. And by the definition of the equicoverable, $\{H_0, H_i, H_1\} \leq i \leq k - 1\}$ is the family of the minimal $P_k$-covering of $C_n \cdot P_2$. (or}

$$H_0 = \{v_0v_1, v_1v_2, v_2v_3, \cdots, v_{k-2}v_{k-1}\},$$
$$H_i = \{v_{i+1}v_{i+2}, v_{i+2}v_{i+3}, \cdots, v_{i+k-2}v_{i+k-1}\}$$
$$2 \leq i \leq k - 1,$$
$$H_1 = \{v_{k+1}v_{k+2}, v_{k+2}v_{k+3}, \cdots, v_{2k-3}v_{2k-2},$$
$$v_{2k-2}v_{1}, v_{1}v_{0}\}.$$
As a result, the number of minimal $P_k$-covering of $C_n \cdot P_2$ is only 3. $C_n \cdot P_2$ is $P_k$-equicoverable.

(5) When $n = 3k - 3$, it is easy to find $c(C_n \cdot P_2; P_k) = 4$. We denote the edges of $C_n \cdot P_2$ by $e_0, e_1, \cdots, e_{3k-3}$. There exists a minimal $P_k$-covering as following: we denote it by $H = \{H_1, H_2, H_3, H_4, H_5\}$,

$$
\begin{align*}
H_1 &= \{e_0, e_1, e_2, \cdots, e_{k-2}\}, \\
H_2 &= \{e_1, e_2, \cdots, e_{k-1}\}, \\
H_3 &= \{e_k, e_{k+1}, \cdots, e_{2k-2}\}, \\
H_4 &= \{e_{k+1}, e_{k+2}, \cdots, e_{2k-1}\}, \\
H_5 &= \{e_{2k}, e_{2k+1}, \cdots, e_{3k-3}, e_1\}.
\end{align*}
$$

So it is not $P_k$-equicoverable.

(6) When $2k - 1 \leq n \leq 3k - 4$ and $n \geq 3k - 2$, $C_n \cdot P_2$ is not $P_k$-equicoverable by Theorem 4.

**Corollary 24** $C_n \cdot P_3(n \geq k + 1)$ is $P_k$-equicoverable if and only if $n = 2k - 3$.

**Corollary 25** $C_n \cdot P_4(n \geq k + 1)$ is $P_k$-equicoverable if and only if $n = 2k - 4$.

**Corollary 26** $C_n \cdot P_5(n \geq k + 1)$ is $P_k$-equicoverable if and only if $n = 2k - 5$.

**Theorem 27** $C_n \cdot P_k(n \geq k + 1, k \geq 6)$ is not $P_k$-equicoverable.

**Proof:**

(1) When $k + 1 \leq n \leq 2k - 2$ and $n \geq 2k$, it is easy to come to the conclusion according to Theorem 5.

(2) When $n = 2k - 1$, $c(C_n \cdot P_k; P_k) = 4$. We denote its edges by $e_{p_1}, e_{p_2}, \cdots, e_{p(k-1)}, e_{c_1}, e_{c_2}, \cdots, e_{c(2k-1)}$. There exits a minimal $P_k$-covering as following: we denote it by $H = \{H_1, H_2, H_3, H_4, H_5\}$,

$$
\begin{align*}
H_1 &= \{e_{c_1}, e_{p_1}, e_{p_2}, \cdots, e_{p(k-2)}\}, \\
H_2 &= \{e_{c(2k-1)}, e_{p_1}, \cdots, e_{p(k-2)}\}, \\
H_3 &= \{e_{p_1}, e_{p_2}, \cdots, e_{p(k-1)}\}, \\
H_4 &= \{e_{c_2}, e_{c_3}, \cdots, e_{c_k}\}, \\
H_5 &= \{e_{ck}, e_{c(k+1)}, \cdots, e_{c(2k-2)}\}.
\end{align*}
$$

So it is also not $P_k$-equicoverable.

**Corollary 28** $C_n \cdot K_{1,t}(n \geq k - 1, t \geq 3)$ is not $P_k$-equicoverable.

**Theorem 29** $C_n \cdot S_m^{(k-2)*}$ is $P_k$-equicoverable if and only if $3 \leq n \leq k$ and $c(G) = C(G) = m + 2$.

**Proof:**

(1) When $n \geq k + 1$, it is not $P_k$-equicoverable by Theorem 27.

(2) When $3 \leq n \leq k - 1$, the subgraph $C_n$ doesn’t contain $P_k$. There must be $m$ copies of $P_k$ covering the part of $S_m^{(k-2)*}$; the else can be covered by using only two copies of $P_k$. It is $P_k$-equicoverable and $c(G) = C(G) = m + 2$.

(3) When $n = k$, the $S_m^{(k-2)*}$ part must be covered by $m$ copies of $P_k$. We can only use two copies of $P_k$ to cover the else $C_n$ part. Then the $C_n \cdot S_m^{(k-2)*}$ is $P_k$-equicoverable.

The next comment follows immediately from Theorem 29.

**Corollary 30** $C_n \cdot P_{k-2} \cdot K_{1,t}$ is $P_k$-equicoverable if and only if $3 \leq n \leq k$.

**References:**


