On qualitative properties and asymptotic behavior of solutions to higher-order nonlinear differential equations

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Abstract: We discuss the asymptotic behavior of solutions to a higher-order Emden–Fowler type equation with constant potential. Several author's results are presented concerning both positive and oscillatory solutions to equations with regular and singular nonlinearities. We discuss the existence and asymptotic behavior of "blow-up" solutions. Results on the asymptotic behavior of oscillating solutions are formulated. For the third- and forth-order equations an asymptotic classification of all solutions is presented. Some applications of the results obtained are proposed.

Key–Words: Qualitative properties, asymptotic behavior, nonlinear equations, blow-up, oscillatory solutions.

1 Introduction

Consider the equation

$$y^{(n)} = P(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sgn} y, \qquad (1)$$

where $n \geq 2$, $k \in (0,1) \cup (1,\infty)$, P is a positive, continuous and Lipschitz continuous in the last n variables function satisfying the inequalities $0 < P_* < P < P^*$. Consider also a special case of (1), namely

$$y^{(n)} = p_0\, |y|^k \operatorname{sgn} y \qquad (2)$$

with $p_0 > 0,\ k \neq 1$. Hereafter, we put

$$\alpha = \frac{n}{k-1}. \qquad (3)$$

The main purpose of this article is to collect together and to present recent and new author's results on asymptotic properties of solutions to equation (2).

Equation (1) has been investigated by a lot of mathematicians from different points of view because it is a generalization of the well-known Emden–Fowler equation (see, for example, [1]–[2]). The first asymptotic classification of solutions to the Emden–Fowler equation of the second order appears in [3]. Asymptotic classification of solutions to equation (1) in the case $P = P(x)$ is presented in [4]. Generalizations of the equation of higher orders were investigated later in the book [4], [5] (see also references in these books) and in a great number of articles of different authors. In particular, sufficient conditions are given for the existence of some special types of solutions to these equations (see, for example, [4]–[10], [13], [19]). In this article some new results on asymptotic behavior of "blow-up" solution and the results on the existence of oscillatory quasi-periodic solutions are formulated and the methods of their obtaining are done. (See also [23].) Qualitative properties of solutions to third- and fourth-order equations of this type were investigated in [10] – [17]. In [25] asymptotic classification of solutions to equation (2) is done in the case of regular $k > 1$ and singular $0 < k < 1$ nonlinearities as $n = 3, 4$. Here more precise result for the behavior of oscillatory solutions to equation (2) for $n = 3$ is done.

2 Asymptotic behavior of blow-up solutions

Definition 1 *A solution $y(x)$ of equation (1) is said to be* ***n*-positive** *if it is maximally extended in both* *directions and eventually satisfies the inequalities*

$$y(x) > 0,\ y'(x) > 0, \dots,\ y^{(n-1)}(x) > 0.$$

Note that if the above inequalities are satisfied by a solution of (1) at some point x_0, then they are also satisfied at any point $x > x_0$ in the domain of the solution. Moreover, such a solution, if maximally extended, must be a so-called **blow-up solution**, i. e. must have a vertical asymptote at the right endpoint of its domain.

Immediate calculations show that equation (2) has n-positive solutions with exact power-law behavior, namely

$$y(x) = C(x^* - x)^{-\alpha} \tag{4}$$

defined on $(-\infty,\, x^*)$ with

$$C = \left(\frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{p_0} \right)^{\frac{1}{k-1}} \tag{5}$$

and arbitrary $x^* \in \mathbf{R}$.

For $n = 1$ all n-positive solutions of (2) are defined by (4). For $n \in \{2, 3, 4\}$ it is known that any n-positive solution of (2) and even of more general equations (1) is asymptotically equivalent, near the right endpoint of its domain, to the solution defined by (4) with appropriate x^*:

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)),\ \ x \to x^* - 0, \tag{6}$$

where C is defined by (5) with p_0 equal, in the case of equation (1), to the limit of $P(x,\, y_0,\, \dots, y_{n-1})$ as $x \to x^* - 0$, $y_0 \to \infty$, …, $y_{n-1} \to \infty$. See [4] for $n = 2$, and [5], [11], [14], for $n \in \{3, 4\}$.

For equation (1) with some additional assumptions on the function P the existence of solutions with power-law asymptotic behavior (6) is proved. For $5 \le n \le 11$, the existence of an $(n-1)$-parametrical family of such solutions is obtained (see [5]).

2.1 Existence of positive solutions with non-power-law asymptotic behavior

In [4], Problem 16.4, a question was posed about the equivalence to (4), as $x \to x^*$, of all positive blow-up solutions to (2) with the vertical asymptote $x = x^*$. The natural hypothesis that they all satisfy (6) for any $n > 4$ appears to be wrong even for equation (2). It was proved [18] that for any N and $K > 1$ there exist an integer $n > N$ and a real number $k \in (1, K)$ such that equation (2) has a solution of the form

$$y = p_0^{-\frac{1}{k-1}} (x^* - x)^{-\alpha}\, h(\log{(x^* - x)}), \tag{7}$$

where h is a positive periodic non-constant function on $\mathbf{R}$. Still it was not clear how large n should be for the existence of that type of positive solutions.

Theorem 2 *[21] If* $12 \le n \le 14$, *then there exists* $k > 1$ *such that equation (2) has a solution* $y(x)$ *with*

$$y^{(j)}(x) = p_0^{-\frac{1}{k-1}} (x^* - x)^{-\alpha - j}\, h_j(\, \log(x^* - x)\,),$$

$$j = 0, 1, \dots, n-1,$$

where h_j *are periodic positive non-constant functions on* $\mathbf{R}$.

Sketch of the proof of Theorem 2. For investigation of blow-up solutions to equation (2) having the vertical asymptote $x = x^*$, the substitutions

$$x^* - x = e^{-t}, \qquad y = (C + v)\, e^{\alpha t} \tag{8}$$

with C defined by (5) transforms equation (2) with $p_0 = 1$ to another one, which can be reduced to the first-order system

$$\frac{dV}{dt} = A_\alpha V + F_\alpha(V), \tag{9}$$

where A_α is a constant $n \times n$ matrix with eigenvalues satisfying the equation

$$\prod_{j=0}^{n-1} (\lambda + \alpha + j) = \prod_{j=0}^{n-1} (1 + \alpha + j) \tag{10}$$

and F_α is a mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$ satisfying $||F_\alpha(V)|| = O\,(||V||^2)$ and $||F'_{\alpha,V}(V)|| = O\,(||V||)$ as $V \to 0$.

The Hopf Bifurcation theorem [30] provides the existence, for some α, of a periodic non-constant solution to system (9), which can be transformed to the solution needed in Theorem 2.

To apply the Hopf Bifurcation theorem, we need to proof the existence of the family λ_α of complex simple roots of equation (10) such that for some $\tilde\alpha$ we have $\operatorname{Re} \lambda_{\tilde\alpha} = 0$ and

$$\operatorname{Re} \frac{d\lambda_\alpha}{d\alpha}\, (\tilde\alpha) \ne 0. \tag{11}$$

All roots of equation (10) are simple, which can be proved for any $n > 1$.

The existence of pure imaginary roots for some $\tilde\alpha$ can be proved for any $n > 11$. To do this, consider the positive C^1-functions $\rho_n(\alpha)$ and $\sigma_n(\alpha)$ defined for all $\alpha > 0$ via the equations

$$\prod_{j=0}^{n-1} (\, \rho_n(\alpha)^2 + (\alpha + j)^2\,) = \prod_{j=0}^{n-1} (1 + \alpha + j)^2$$

and

$$\sum_{j=0}^{n-1} \arg{(\, \sigma_n(\alpha) i + \alpha + j\,)} = 2\pi$$

supposing $\arg z \in [0, 2\pi)$ for all $z \in \mathbf{C} \setminus \{0\}$.

One can show that $\rho_n(\alpha)/\alpha \to 0$ as $\alpha \to +\infty$, while $\sigma_n(\alpha)/\alpha \to \tan 2\pi/n > 0$, whence for sufficiently large α we have $\rho_n(\alpha) < \sigma_n(\alpha)$.

For sufficiently small $\alpha > 0$, one can prove that $\sigma_{12}(\alpha) < 2 < \rho_{12}(\alpha)$ and, for any $\alpha > 0$, that $\rho_{n+1}(\alpha) > \rho_n(\alpha)$ and $\sigma_{n+1}(\alpha) < \sigma_n(\alpha)$.

So, for any $n \geq 12$ there exists $\tilde{\alpha} > 0$ such that $\rho_n(\tilde{\alpha}) = \sigma_n(\tilde{\alpha})$ producing the pure imaginary root $\lambda_{\tilde{\alpha}} = \rho_n(\tilde{\alpha})\, i$ of equation (10).

As for inequality (11), it was successfully proved only for $n \in \{12, 13, 14\}$, and the greater is n, the more cumbersome the proof turns out.

Remark 3 *Computer calculations give approximate values of k providing the existence of the above-type solutions. They are, with the corresponding values of α, as follows:*

if $n = 12,$ *then* $\alpha \approx 0.56,\ k \approx 22.4;$
if $n = 13,$ *then* $\alpha \approx 1.44,\ k \approx 10.0;$
if $n = 14,$ *then* $\alpha \approx 2.37,\ k \approx 6.9.$

2.2 On power-law asymptotic behavior of solutions to weakly super-linear Emden–Fowler type equations with constant potential

It appears that a weaker version of the I.T. Kiguradze's hypothesis about power-law asymptotic behavior of blow-up solutions for higher-order equations (2) is correct.

Theorem 4 *For any integer $n > 4$ there exists $K > 1$ such that for any real $k \in (1, K)$, all n-positive solutions to equation (2) have the power-law asymptotic behavior (6) near the right endpoint of their domains.*

3 Asymptotic behavior of oscillatory solutions

This section is devoted to the existence of oscillatory quasi-periodic in some sense solutions to a higher-order Emden–Fowler type differential equation

$$y^{(n)} + p_0\, |y|^k \operatorname{sgn} y = 0, p_0 \neq 0. \qquad (12)$$

with $n > 2$ and $k \in (0, 1) \cup (1, \infty)$.

Theorem 5 *For any integer $n > 2$ and real $k > 1$ there exists a periodic oscillatory function h on $\mathbf{R}$ such that for any $p_0 > 0$ and $x^* \in \mathbf{R}$ the function*

$$y(x) = p_0^{-\frac{1}{k-1}} (x^* - x)^{-\alpha}\, h(\log(x^* - x)) \qquad (13)$$

is a solution to equation (12) on $(-\infty, x^*)$.

Definition 6 *A solution having the form (13) is called* **quasi-periodic**.

Sketch of the proof of Theorem 5. For $0 \leq j < n$ put

$$B_j = \frac{nk}{n + j(k-1)} > 1, \qquad \beta_j = \frac{1}{B_j}.$$

For any $q = (q_0, \ldots, q_{n-1}) \in \mathbf{R}^n$ let $y_q(x)$ be the maximally extended solution to the equation

$$y^{(n)}(x) + |y(x)|^k = 0 \qquad (14)$$

with the initial data $y^{(j)}(0) = q_j,\ \ 0 \leq j < n$.

Consider also the function $N : \mathbf{R}^n \to \mathbf{R}$ and the mapping $\tilde{N} : \mathbf{R}^n \setminus \{0\} \to \mathbf{R}^n \setminus \{0\}$ defined by

$$N\,(q) = \sum_{j=0}^{n-1} |q_j|^{B_j}\,, \qquad \tilde{N}(q)_j = N\,(q)^{-\beta_j}\, q_j$$

and satisfying $N\left(\tilde{N}(q)\right) = 1$ for all $q \in \mathbf{R}^n \setminus \{0\}$.

Next, consider the subset $Q \subset \mathbf{R}^n$ consisting of all $q \in \mathbf{R}^n$ satisfying the following conditions:

1) $q_0 = 0,$
2) $q_j \geq 0, \quad 0 < j < n,$
3) $N(q) = 1.$

The restriction of the projection

$$(q_0, \ldots, q_{n-1}) \mapsto (q_1, \ldots, q_{n-2})$$

to the set Q is a homeomorphism of Q onto the convex compact subset of $\mathbf{R}^{n-2}$ consisting of all its points with non-negative coordinates satisfying the inequality $\sum\limits_{j=1}^{n-2} |q_j|^{B_j} \leq 1$.

Lemma 7 *For any $q \in Q$ there exists $a_q > 0$ satisfying $y_q(a_q) = 0$ and $y_q^{(j)}(a_q) < 0$ for $0 < j < n$.*

Note that a_q is not only the first positive zero of $y_q(x)$, but the only positive one.

To continue the proof of Theorem 5, consider the function $\xi : q \mapsto a_q$ taking each $q \in Q$ to the first positive zero of the function y_q. Due to the implicit function theorem, the function ξ is continuous.

Consider the C^1 "solution" mapping

$$S : (q, x) \mapsto= \left(y_q(x), y_q'(x), \ldots, y_q^{(n-1)}(x)\right)$$

defined on a domain including $\mathbf{R}^n \times \{0\}$ and the continuous mapping $\tilde{S} : q \mapsto \tilde{N}\,(-S\,(q, \xi(q)))\,$, which maps Q into itself.

By the Brouwer fixed-point theorem, there exists $\hat{q} \in Q$ such that $\tilde{S}\,(\hat{q}) = \hat{q}$. In other words, there exists a non-negative solution $\hat{y}(x) = y_{\hat{q}}(x)$ to equation (14) defined on a segment $[0, a_1]$ with $a_1 = a_{\hat{q}}$, positive on the open interval $(0, a_1)$, and such that

$$\lambda^{-\beta_j}\, \hat{y}^{(j)}(a_1) = -\hat{y}^{(j)}(0), \quad 0 \le j < n, \tag{15}$$

with

$$\lambda = N\left(S\left(\hat{q}, \xi(\hat{q})\right)\right) = \sum_{j=0}^{n-1} \left|\hat{y}^{(j)}(a_1)\right|^{B_j} > 0.$$

Since $\hat{y}(x)$ is non-negative, it is also a solution to the equation

$$y^{(n)}(x) + |y(x)|^k \operatorname{sgn} y(x) = 0.$$

Due to property (15), the solution $\hat{y}(x)$ can be smoothly extended onto some segment $[a_1, a_2]$, then onto $[a_2, a_3]$, etc., as well as in the opposite direction, with the following relation between the lengths of the neighboring segments and the values of $\hat{y}(x)$ at their points:

$$\frac{a_s - a_{s-1}}{a_{s+1} - a_s} = b = \lambda^{\frac{k-1}{nk}},$$

$$\hat{y}(x) = -b^{\alpha}\, \hat{y}\left(b(x - a_s) + a_{s-1}\right),$$

where $x \in [a_s, a_{s+1}]$, $b(x - a_s) + a_{s-1} \in [a_{s-1}, a_s]$.

It can be proved that $b > 1$ whenever $k > 1$, which yields $a^* = \sum\limits_{s=0}^{\infty} < \infty$, and that the function

$$h(t) = e^{t\alpha}\, \hat{y}\left(a^* - e^t\right)$$

is just a periodic function needed for Theorem 5.

Corollary 8 *For any integer $n > 2$ and real $k > 1$ there exists a periodic oscillatory function h on $\mathbf{R}$ such that for any $p_0 \in \boldsymbol{R}$ satisfying $(-1)^n p_0 > 0$ and any $x^* \in \boldsymbol{R}$ the function*

$$y(x) = |p_0|^{-\frac{1}{k-1}}\, (x - x^*)^{-\alpha}\, h(\log(x - x^*))$$

is a solution to equation (12) on (x^*, ∞).

Theorem 9 *For any integer $n > 2$ and real positive $k < 1$ there exists a non-constant oscillatory periodic function h such that for any p_0 with $(-1)^n p_0 > 0$ and any real x^* the function*

$$y(x) = |p_0|^{\frac{1}{1-k}} (x^* - x)^{|\alpha|} h(\log(x^* - x)),$$

is a solution to equation (12) on $(-\infty,\, x^*)$.

Part of these results are included in [23], its application can be found in [26].

4 Asymptotic classification of solutions to the third- and fourth-order Emden–Fowler type differential equations

For equation (12) in the cases $n = 3$ and $n = 4$, $p_0 > 0$ and $p_0 < 0$, for regular nonlinearity $k > 1$ and singular nonlinearity $0 < k < 1$ asymptotic classification of all solutions is given. Sketch of proof is done. Detailed proofs for some of the cases listed in this classification see in [5], [22], [24].

4.1 Regular nonlinearity ($k > 1$)

Theorem 10 *Suppose $k > 1$ and $p_0 > 0$. Then all non-trivial non-extensible solutions to the equation*

$$y'''(x) + p_0\, |y|^{k-1}\, y(x) = 0 \tag{16}$$

are divided into the following five types according to their asymptotic behavior.

1–2. *Defined on semi-axes $(b, +\infty)$ Kneser (up to the sign) solutions:*

$$y(x) = \pm C_{3k}\, (x - b)^{-\alpha},$$

where

$$C_{3k} = \left|\frac{3(k+2)(2k+1)}{p_0\,(k-1)^3}\right|^{\frac{1}{k-1}}. \tag{17}$$

3. *Defined on semi-axes $(-\infty, b)$ oscillatory, in both directions, solutions having the form*

$$y(x) = (b - x)^{-\alpha}\, h\left(\log(b - x)\right) \tag{18}$$

with some oscillatory periodic function h.

4–5. *Defined on bounded intervals (b', b'') oscillatory near the right boundary and non-vanishing near the left one solutions satisfying*

$$y(x) = \pm(x - b')^{-\alpha}\, (C_{3k} + o(1))$$

as $x \to b' + 0$ and

$$|y(x')| = |b'' - x'|^{-\alpha}\, (\widetilde{C}_{3k} + o(1)), \tag{19}$$

with some $\widetilde{C}_{3k}$ at their local-extremum points x' tending to $b'' - 0$.

Remark 11 *The case $p_0 < 0$ can be reduced to the above one by the substitution $x \mapsto -x$.*

Remark 12 *Theorem 10 gives a more precise description of the behavior of $|y(x')|$ in comparison with [25]. Note that if $y(x)$ is a solution to equation (2), then the function $z(x) = \pm\lambda^{\alpha} y(\lambda x + c)$ with arbitrary constants c and $\lambda > 0$ is also a solution to this equation. So, we can choose among the solutions of (16) oscillating on the whole domain a solution $Y(x)$ with a resonance asymptote at $x = 1$ and a local maximum at $x = 0$. Note that $Y(x)$ is a quasi-periodic solution to (2) and its existence follows from theorem 5. If x' is any other local-extremum point of $Y(x)$, then $|Y(x')| = Y(0)\,(1 - x')^{-\alpha}$. We can prove that the constant $\widetilde{C}_{3k}$ in (19) is equal to $Y(0)$.*

Theorem 13 *Suppose $k > 1$ and $p_0 > 0$. Then all non-trivial non-extensible solutions to the equation*

$$y^{\mathrm{IV}}(x) + p_0\,|y|^{k-1}\,y(x) = 0 \tag{20}$$

are divided into the following three types according to their asymptotic behavior.

1. *Defined on semi-axes $(-\infty, b)$ oscillatory solutions. The distance between their neighboring zeros infinitely increases near the left boundaries of the domains and tends to zero near the right ones. The solutions and their derivatives satisfy the relations*
$$\lim_{x\to-\infty} y^{(j)}(x) = 0,$$
$$\limsup_{x\to b} \left|y^{(j)}(x)\right| = \infty$$
for $j = 0, 1, 2, 3$. At the local-extremum points x' the following estimates hold:
$$C_1\,|x' - b|^{-\alpha} \le |y(x')| \le C_2\,|x' - b|^{-\alpha} \tag{21}$$
with positive constants C_1 and C_2 depending only on k and p_0.

2. *Defined on semi-axes $(b, +\infty)$ oscillatory solutions. The distance between their neighboring zeros tends to zero near the left boundaries of the domains and infinitely increases near the right ones. The solutions and their derivatives satisfy the relations*
$$\lim_{x\to+\infty} y^{(j)}(x) = 0,$$
$$\limsup_{x\to b} \left|y^{(j)}(x)\right| = \infty$$
for $j = 0, 1, 2, 3$. At the local-extremum points x' estimates (21) hold with positive constants C_1 and C_2 depending only on k and p_0.

3. *Defined on bounded intervals (b', b'') oscillatory solutions. All their derivatives $y^{(j)}$, with $j = 0, 1, 2, 3, 4$ satisfy*
$$\limsup_{x\to b'} \left|y^{(j)}(x)\right| = \limsup_{x\to b''} \left|y^{(j)}(x)\right| = \infty.$$
At the local-extremum points x' sufficiently close to any boundary of the domain, estimates (21) hold respectively with $b = b'$ or $b = b''$ and the positive constants C_1 and C_2 depending only on k and p_0.

Theorem 14 *Suppose $k > 1$ and $p_0 < 0$. Then all non-trivial non-extensible solutions to equation (20) are divided into the following thirteen types according to their asymptotic behavior.*

1–2. *Kneser (up to the sign) solutions on semi-axes $(b, +\infty)$:*
$$y(x) = \pm C_{4k}\,(x - b)^{-\alpha},$$
where
$$C_{4k} = \left(\frac{4(k+3)(2k+2)(3k+1)}{|p_0|\;(k-1)^4}\right)^{\frac{1}{k-1}}. \tag{22}$$

3–4. *"Left" Kneser (up to the sign) solutions on semi-axes $(-\infty, b)$:*
$$y(x) = \pm C_{4k}\,(b - x)^{-\alpha}.$$

5. *Periodic oscillatory solutions on $(-\infty, +\infty)$. All of them can be received from one, say $z(x)$, by the relation*
$$y(x) = \lambda^4 z(\lambda^{k-1} x + x_0)$$
with arbitrary $\lambda > 0$ and x_0. So, there exists such a solution with any maximum $h > 0$ and with any period $T > 0$, but not with any pair (h, T).

6–9. *Defined on bounded intervals (b', b'') solutions with the power-law asymptotic behavior near the boundaries of the domain (with the independent signs $\pm$):*
$$y(x) \sim \pm C_{4k}\,(x - b')^{-\alpha},\; x \to b' + 0,$$
$$y(x) \sim \pm C_{4k}\,(b'' - x)^{-\alpha},\; x \to b'' - 0.$$

10–11. *Defined on semi-axes $(-\infty, b)$ solutions which oscillate near $-\infty$ and have the power-law asymptotic behavior near the right boundary of the domain:*
$$y(x) \sim \pm C_{4k}\,(b - x)^{-\alpha}, \quad x \to b - 0.$$
For each solution a finite limit of the absolute values of its local extrema exists as $x \to -\infty$.

12–13. *Defined on semi-axes $(b, +\infty)$ solutions which oscilate near $+\infty$ and have the power-law asymptotic behavior near the left boundary of the domain:*

$$y(x) \sim \pm C_{4k}\,(x-b)^{-\alpha}, \quad x \to b+0.$$

For each solution a finite limit of the absolute values of its local extrema exists as $x \to +\infty$.

4.2 Singular nonlinearity ($0 < k < 1$)

While studying the asymptotic behavior of solutions in the case of regular nonlinearity, $k > 1$, only maximally extended solutions are usually considered because solutions can behave in a special way only near the boundaries of their domains. If $k < 1$, then some special behavior can occur also near internal points of the domains. This is why a notion of *maximally unique (MU)* solutions is introduced.

Definition 15 *A solution u defined on $(a,\, b)$ with $-\infty \le a < b \le +\infty$ to any ordinary differential equation is called a* **MU-solution** *if the following two conditions hold:*
(i) the equation has no other solution equal to u on some subinterval of $(a,\, b)$;
(ii) either there is no solution defined on another interval containing $(a,\, b)$ and equal to u on $(a,\, b)$, or there exist at least two such solutions not equal to each other at points arbitrary close to the boundary of $(a,\, b)$.

Theorem 16 *Suppose $0 < k < 1$ and $p_0 > 0$. Then all MU-solutions to the equation*

$$y'''(x) = p_0\,|y|^{k-1}\,y(x) \tag{23}$$

are divided into the following five types according to their asymptotic behavior.

1–2. *Constant-sign solutions with the power-law behavior on $(b,\, +\infty)$:*

$$y(x) = \pm C_{3k}\,(x-b)^{|\alpha|},$$

3. *Oscillatory, in both directions, solutions on $(-\infty,\, b)$ having the form*

$$y(x) = (b-x)^{|\alpha|}\;h(\log(b-x))$$

with some oscillatory periodic function h.

4–5. *Defined on $(-\infty, +\infty)$ solutions oscillating near $-\infty$, having the asymptotically power-law behavior near $+\infty$:*

$$y(x) = \pm C_{3k}\,x^{|\alpha|}(1+o(1)), \quad x \to +\infty,$$

and having no point x_0 with $y(x_0) = y'(x_0) = y''(x_0) = 0$. At their local-extremum points x' they satisfy

$$|y(x')| = |x'|^{|\alpha|}\,(\widehat{C}_{3k} + o(1)) \tag{24}$$

as $x' \to -\infty$ with some $\widehat{C}_{3k}$.

Remark 17 *The case $p_0 < 0$ can be reduced to the above one by the substitution $x \mapsto -x$.*

Remark 18 *Theorem 10 gives a more precise description of the behavior of $|y(x')|$ in comparison with [25].*

Theorem 19 *Suppose $0 < k < 1$ and $p_0 > 0$. Then all MU-solutions to equation (20) are divided into the following three types according to their asymptotic behavior.*

1. *Oscillatory solutions defined on $(-\infty, b)$. The distance between their neighboring zeros infinitely increases near $-\infty$ and tends to zero near b. The solutions and their derivatives satisfy the relations*

$$\lim_{x\to b} y^{(j)}(x) = 0, \qquad \limsup_{x\to-\infty} \left|y^{(j)}(x)\right| = \infty$$

for $j = 0, 1, 2, 3$. At the local-extremum points x' the following estimates hold:

$$C_1\,|x'-b|^{|\alpha|} \le |y(x')| \le C_2\,|x'-b|^{|\alpha|} \tag{25}$$

with positive constants C_1 and C_2 depending only on k and p_0.

2. *Oscillatory solutions defined on $(b, +\infty)$. The distance between their neighboring zeros tends to zero near b and infinitely increases near $+\infty$. The solutions and their derivatives satisfy the relations*

$$\lim_{x\to b} y^{(j)}(x) = 0, \qquad \limsup_{x\to+\infty} \left|y^{(j)}(x)\right| = \infty$$

for $j = 0, 1, 2, 3$. At the local-extremum points x' estimates (25) hold with the positive constants C_1 and C_2 depending only on k and p_0.

3. *Oscillatory solutions defined on $(-\infty,\, +\infty)$. All their derivatives $y^{(j)}$, $j = 0, \ldots, 4$ satisfy*

$$\limsup_{x\to-\infty} \left|y^{(j)}(x)\right| = \limsup_{x\to+\infty} \left|y^{(j)}(x)\right| = \infty.$$

At the local-extremum points x' the estimates

$$C_1\,|x'|^{|\alpha|} \le |y(x')| \le C_2\,|x'|^{|\alpha|} \tag{26}$$

hold near $-\infty$ or $+\infty$ with the positive constants C_1 and C_2 depending only on k and p_0.

Theorem 20 *Suppose $0 < k < 1$ and $p_0 < 0$. Then all MU-solutions to equation (20) are divided into the following thirteen types according to their asymptotic behavior.*

1–2. *Defined on semi-axes $(-\infty, b)$ solutions with the power-law asymptotic behavior near the boundaries of the domain (with the same signs $\pm$):*

$$y(x) \sim \pm C_{4k}\ |x|^{|\alpha|}, \quad x \to -\infty,$$

$$y(x) \sim \pm C_{4k}\,(b-x)^{|\alpha|}, \quad x \to b-0,$$

where C_{4k} is defined by (22).

3–4. *Defined on $(b, +\infty)$ solutions with the power-law asymptotic behavior near the boundaries of the domain (with the same signs $\pm$):*

$$y(x) \sim \pm C_{4k}\,(x-b)^{|\alpha|}, \quad x \to b+0,$$

$$y(x) \sim \pm C_{4k}\, x^{|\alpha|}, \quad x \to +\infty.$$

5. *Defined on the whole axis periodic oscillatory solutions. All of them can be received from one such solution, say $z(x)$, by the relation*

$$y(x) = \lambda^4 z(\lambda^{k-1}x + x_0)$$

with arbitrary $\lambda > 0$ and x_0. So, there exists such a solution with any maximum $h > 0$ and with any period $T > 0$, but not with any pair (h, T).

6–9. *Defined on $(-\infty, +\infty)$ solutions having the power-law asymptotic behavior near $-\infty$ and $+\infty$ (with all sign combinations admitted):*

$$y(x) \sim \pm C_{4k}\ |x|^{|\alpha|}, \quad x \to \pm\infty.$$

10–11. *Defined on $(-\infty, +\infty)$ solutions which oscillate as $x \to -\infty$ and have the power-law asymptotic behavior near $+\infty$:*

$$y(x) \sim \pm C_{4k}\, x^{|\alpha|}, \quad x \to +\infty.$$

Each solution has a finite limit of the absolute values of its local extrema as $x \to -\infty$.

12–13. *Defined on $(-\infty, +\infty)$ solutions which oscillate as $x \to +\infty$ and have the power-law asymptotic behavior near $-\infty$:*

$$y(x) \sim \pm C_{4k}\ |x|^{|\alpha|}, \quad x \to -\infty.$$

Each solution has a finite limit of the absolute values of its local extrema as $x \to +\infty$.

4.3 Method of Proofs.

To obtain the above results on asymptotic classification of all maximally extended solutions to the equation

$$y^{(n)} + p_0|y|^k \operatorname{sgn} y = 0, \ \ p_0 \neq 0, \tag{27}$$

with $k > 1$ and all MU-solutions to equation (27) with $0 < k < 1$, an auxiliary dynamical system is investigated on the m-dimensional sphere S^m with

$$m = n - 1$$

(see [14]; [5], Ch.5–7; [22] for regular nonlinearity, [24] for singular nonlinearity).

Note that if a function $y(x)$ is a solution to equation (27), the same is true for the function

$$z(x) = B^\alpha y(Bx + C), \tag{28}$$

where $B > 0$, and C are any constants satisfying

$$|A|^{k-1} = B^n. \tag{29}$$

Now, any non-trivial solution $y(x)$ to equation (27) generates the curve $\Big(y(x), y'(x), \ldots, y^{(m)}(x)\Big)$ in $\mathbf{R}^n \setminus \{0\}$. We can define an equivalence relation on $\mathbf{R}^n \setminus \{0\}$ such that all solutions obtained from $y(x)$ by (28)–(29) generate equivalent curves, i.e. curves passing through equivalent points (may be for different x). We assume the points $(y_0, y_1, y_2, \ldots, y_m)$ and $(z_0, z_1, z_2, \ldots, z_m)$ in $\mathbf{R}^n \setminus \{0\}$ to be equivalent if and only if there exists a positive constant λ such that

$$z_j = \lambda^{\alpha+j}\, y_j, \qquad j = 0, 1, \ldots, m.$$

The quotient space obtained is homeomorphic to the m-dimensional sphere $S^m \subset \mathbf{R}^n$ defined by the equation

$$\sum_{j=0}^{m} y_j^2 = 1$$

and having exactly one representative of each equivalence class since the equation

$$\sum_{j=0}^{m} \lambda^{2(\alpha+j)} y_j^2 = 1$$

for any $(y_0, y_1, \ldots, y_m) \in \mathbf{R}^n \setminus \{0\}$ has exactly one positive root λ.

Now, equivalent curves in $\mathbf{R}^n \setminus \{0\}$ generate the same curves in the quotient space. The last ones are trajectories of an appropriate dynamical system, which can be described, in different charts covering the quotient space, by different formulas using different independent variables.

For example, consider the chart that covers all points corresponding to positive values of solutions and has the coordinate functions $u_j,\ j = 1, \dots, m$, defined by

$$u_j = y^{(j)} y^{-\gamma_j}, \qquad \gamma_j = 1 + \frac{j}{\alpha}.$$

On this chart the dynamical system can be written as

$$\begin{cases} \dfrac{du_1}{dt} = u_2 - \gamma_1 u_1^2, \\ \dfrac{du_j}{dt} = u_{j+1} - \gamma_j u_1 u_j, \quad 1 < j < m, \\ \dfrac{du_m}{dt} = -p_0 - \gamma_m u_1 u_m \end{cases}$$

with the independent variable

$$t = \int_{x_0}^{x} y(\xi)^{1/\alpha}\, d\xi.$$

Qualitative properties of the trajectories of the dynamical system on the sphere do not depend essentially on whether k in (27) is greater or less than 1. However, the properties of the related solutions to equation (27) differ according to the case, regular or singular, considered.

Globally, the dynamical system can have some fixed points, which depends on the sign of p_0 and the parity of n. They correspond to the solutions with the power-law behavior, which can be defined by explicit formulas, namely,

$$y(x) = \pm C\, |x - x^*|^{-\alpha}$$

with arbitrary x^* and C defined by (17), (22), or similar formulas for $n > 4$.

In the regular case, these solutions, if maximally extended, have a vertical asymptote at one of their domain boundaries (which is finite) and tend to zero near another one (which is infinite). In the singular case, the related MU-solutions vanish with all their $n-1$ lower-order derivatives at one of their domain boundaries (which is finite) and become unbounded near another one (which is infinite).

The dynamical system on the sphere can also have non-constant periodical trajectories. They are generated by oscillatory solutions to (27) that can be written by using some periodic functions, but can be non-periodic themselves. Their extrema and the lengths of their constant-sign intervals behave in different ways according to the sign of p_0, the parity of n, and the regular or singular case considered.

Investigation of stability of the fixed points and periodical trajectories gives information on the rest of the solutions to equation (27), which appear to have, near the boundaries of their domains, asymptotically the same behavior as that of the solutions mentioned before.

5 Some applications of these results

Equation (2) it is a model of nonlinear equations. Methods developed for its research can be applied to study of more complex non-linear equations of the form (1). (See [4], [5]). Asymptotic properties can be used for researches of boundary value problems to partial differential equations. (See for example [27]–[29].) The equation of type (1) also appears in investigation of some spectral problems (see [5] **IV**).

6 Open problems connected with equation (2)

1. Does positive blow-up solution with non-power law asymptotic behavior exist for $5 \le n \le 11$ and $n \ge 15$? **2.** Does positive blow-up solutions with different from power-law (6) and non-power law (7) behavior exist for $n \ge 4$?

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