Geometric Modelling of a Class of Sierpinski-type Fractals and Their Geometric Constructions

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Abstract: Study on properties of Sierpinski-type fractals, including dimension, measure, Lipschitz equivalence, etc is very interesting. It is well know that studying fractal theory relies on in-depth observation and analysis to topological structures of fractals and their geometric constructions. But most works of simulating fractals are for graphical goal and often done by non-mathematical researchers. This makes them difficult for most mathematical researchers to understand and application. In [22], the authors simulated a class of Sierpinski-type fractals and their geometric constructions in Matlab environment base on iterative algorithm for the purpose of mathematical research. In this paper, we continue such investigation by adding certain rotation structure. Our results may be used for any graphical goal, not only for mathematical reasons.

Key–Words: Sierpinski-type square, Sierpinski-type triangle, IFS, deterministic algorithm, random iterated algorithm, Matlab

1 Introduction

Geometric modelling of fractal objects not only plays an important role in the study of fractal theory, but also is a difficult process in the field of computer graphics. Computer simulation is based on the basic theory of fractal. But drawing strange and amazing fractal images by computer, simultaneously in turn can provide us the most intuitive discussion and explanation, greatly promoting the development of fractal theory. Computer graphics provides many algorithms to generate fractal images, such as IFS(Iterated Function System), L-system, recursive, time-escaped algorithm, etc. There are various source programs are designed based each algorithm. However, to our knowledge, most algorithm and source programs are for graphical goal and often done by non-mathematical researchers. This makes them are difficult to understand and application for most mathematical researchers because of the lack of professional knowledge of programming language(such as Visual Basic, Visual C++, Delphi, Java, etc). Also, for this reason, the following question aries naturally.

Question 1. *How to use a popular and easy-to-understand way as much as possible, in a real-time information exchange interface, such that the mathematical researchers can well obtain the desired images in their study by modifying a few parameters, without need to know much about the complex computer programming language?*

The present paper does not give a complete answer to Question 1, which is likely to be extremely hard. It does, however, study the question in several important special case that should allow us to gain some deep insight into the problem.

Fixing an integer $n \geq 2$, let $\mathcal{D}_1 = \{0, 1, \cdots, n-1\}^2$ and $\mathcal{D}_2 = \{1, \cdots, n\}^2$. For $A \subset \mathcal{D}_1, B \subset \mathcal{D}_2$, we assume that $1 < \sharp A + \sharp B < n^2$ to exclude the trivial case, where $\sharp A$ and $\sharp B$ denote the cardinalities of A and B respectively. Let $T := T(A, B) \subset \mathbb{R}^2$ be the unique non-empty compact set satisfying the following set equation:

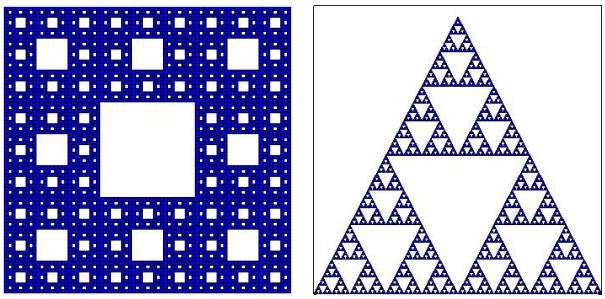
$$T = [(T + A) \cup (B - T)]/n.$$

We shall call $T(A, B)$ a Sierpinski-type square throughout this paper. Let $\alpha = (1, 0)$ and $\beta = (1/2, \sqrt{3}/2)$. If we recast the above sets $\mathcal{D}_1$ and $\mathcal{D}_2$ as follows:

$$\mathcal{D}_1 = \Big\{k_1\alpha + k_2\beta : k_1 + k_2 \leq n-1 \text{ and } k_1, k_2 \in \mathbb{N} \cup \{0\}\Big\}$$

and

$$\mathcal{D}_2 = \Big\{k_1\alpha + k_2\beta : 2 \leq k_1 + k_2 \leq n \text{ and } k_1, k_2 \in \mathbb{N}\Big\},$$



(a) (b)

Figure 1: (a) Sierpinski carpet (b) Sierpinski gasket

then $T(A,B)$ is called a Sierpinski-type triangle. Notice that here A or B may be empty. In general, $T(A,B)$ is of certain rotation structure when $B \neq \emptyset$. A familiar example of a Sierpinski-type square is the Sierpinski carpet(see Figure 1(a)), in which $n = 3, A = \mathcal{D}_1 \setminus \{(1,1)\}\}$, $B = \emptyset$. A familiar example of a Sierpinski-type triangle is the Sierpinski gasket (see Figure 1(b)), in which $n = 2, A = \{(0,0),(1,0),(1/2,\sqrt{3}/2)\}$, $B = \emptyset$.

Let $Q = [0,1]^2$(resp. $\triangle$), where

$$\triangle = \{c_1\alpha + c_2\beta : c_1 + c_2 \leq 1 \text{ and } 0 \leq c_1, c_2 \leq 1\}$$

be the equilateral triangle with lower-left coordinate $(0,0)$ and side 1. We define $T^0(A,B) = Q$,

$$T^1(A,B) = [(Q+A) \cup (B-Q)]/n,$$

and recurrently,

$$T^{k+1}(A,B) = [(T^k(A,B)+A) \cup (B-T^k(A,B))]/n$$

for $k \geq 1$. Then $T^k(A,B)$ is a union of squares (resp. equilateral triangles) of size $1/n^k$ (called them the k-squares (resp. k-triangles)). Clearly $T^{k+1}(A,B) \subset T^k(A,B)$, and

$$T(A,B) = \bigcap_{k=1}^{\infty} T^k(A,B).$$

Recently, many works have been devoted to the study of properties of Sierpinski-type fractals that described as above include, dimension, measure, Lipschitz equivalence, etc, see [8, 10–12, 17, 18, 21–25] and the references in all of these. There has been notable progress on the study of Sierpinski-type fractals. Yet much is still unknown, and this progress has led to more unanswered questions, especially the Sierpinski-type fractals with certain rotation structure. For such fractal, how to achieve the goal described in question 1 has become the most concerned problem for most mathematical researchers at present.

An fundamental concept in fractal geometry is iterated function systems. It is introduced in Hutchinson [9] as a unified way of generating and classifying a broad class of fractals which contains classical Cantor sets, dragon curves, limit sets of Kleinian groups, Sierpinski gaskets, Julia set, and much more. Among all algorithms for generating fractal images, IFS algorithm may be most popular one. There many application on it(see [1,2,4,7,14,16,22] and references there in). IFS algorithm include deterministic algorithm and random iterated algorithm. The mathematical basis of deterministic algorithm was developed in Hutchinson [9]. The random iterated algorithm was developed in Barnsley and Demko [2]. IFS algorithm is very simple, it is easy to implement in any programming language and the results it generates are very spectacular and may be used for any mathematical reason, not only for graphical goal.

Matlab is a high-level technical computing language and interactive environment for algorithm development, data visualization, data analysis, and numerical computation. A proprietary programming language developed by MathWorks. It has a strong audience within the applied mathematics community. Matlab was first adopted by researchers and practitioners in control engineering, Little's specialty, but quickly spread to many other domains. It is now also used in education, in particular the teaching of linear algebra, numerical analysis, and is popular amongst scientists involved in image processing. For further details on Matlab, see [13].

Notice that the powerful advantages of Matlab in numerical calculation and graphic visual ability and its programming language is more easily understood and mastered by researchers in mathematics. In [22], the authors study the geometric modeling of Sierpinski-type with no rotation structure and their geometric constructions in Matlab environment base on IFS algorithm for the purpose of mathematical research. In this paper, we continue such investigation started in [22]. We remove the restriction on no rotation structure and obtain the geometric modeling of $T(A,B)$ and $T^k(A,B)$, which are needed in the study of properties of Sierpinski-type fractal. The objects considered in the present paper are complex and intriguing since the certain rotation structure are allowed. Although the idea used was early developed by others, the analysis is technical and the results obtained are valuable and interesting to the readers.

The paper is organized as follows: In Section 2, we briefly review the iterated function system and corresponding algorithms: deterministic algorithm and random iterated algorithm. We discuss the geometric modeling of $T^k(A,B)$ using deterministic algorithm in Section 3, and give corresponding source program

designed in Matlab programming language. Finally in Section 4, we present the random iterated algorithm implemented in the Matlab programming language to generate $T(A, B)$. Several examples are given in Section 3 and Section 4 to illustrate our results.

2 Preliminary

2.1 Iterated Function System

Let (X, d) be a complete metric space, often Euclidean space $\mathbb{R}^n$. We say that the mapping $S : X \to X$ is a contraction with contraction ratio r if

$$r = \sup_{x \in X, x \neq y} \frac{d(S(x), S(y))}{d(x, y)} < 1.$$

In particular, we say that a contraction S with contraction ratio r is a similitude if $d(S(x), S(y)) = rd(x, y)$ for all $x, y \in X$.

We call a finite family of contractions $\{S_i : i = 1, 2, \cdots, N\}$ are defined in (X, d) an iterated function system or IFS, and denote it by $\{X : S_i, i = 1, 2, \cdots, N\}$. If r_i is contraction ratio of S_i, $i = 1, 2, \cdots, N$, then $r = max\{r_1, r_2, \cdots, r_N\}$ is called contraction ratio of IFS. Let $\mathcal{F}(X)$ denote the class of all non-empty subsets of X. For $A, B \subset \mathcal{F}(X)$, the Hausdorff metric between A and B is defined by

$$d_H(A, B) = max\left[\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\right].$$

By [9], $(\mathcal{F}(X), d_H)$ is a complete metric space when (X, d) is a complete metric space. It is a standard fact that every contraction map (in a complete metric space) has a unique fixed point. Applying the fact to $(\mathcal{F}(X), d_H)$, we can obtain the following contraction principle on $(\mathcal{F}(X), d_H)$, similar description can also be seen in many literatures, such as [5, 6, 9].

Lemma 1. *Let $\{X : S_i, i = 1, 2, \cdots, N\}$ be an iterated function system with contraction ration r and mapping $S : \mathcal{F}(X) \to \mathcal{F}(X)$ be defined by*

$$S(A) = \bigcup_{i=1}^{N} S_i(A), \text{ for any } A \in \mathcal{F}(X).$$

Then S is a contraction with ratio r and there is a unique fixed point(attractor or invariant set) K that satisfies

$$K = S(K) = \bigcup_{i=1}^{N} S_i(K)$$

and for any $A \in \mathcal{F}(X)$

$$K = \lim_{n \to \infty} S^n(A). \quad (1)$$

2.2 Deterministic Algorithm and Random Iterated Algorithm

Considering the iterated function system $\{\mathbb{R}^2 : S_i, i = 1, 2, \cdots, N\}$, where S_i is an affine transformation with the form

$$S_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}, \quad (2)$$

$i = 1, 2, \cdots, N$. According to the Lemma 1, we can obtain two algorithms for generating fractal images on plane. One is deterministic algorithm, the other is random iterated algorithm.

Deterministic algorithm: The mathematical basis of this method is very simple, and it is the result of the Lemma 1. Here is basic process: Choose a non-empty set $A_0 \in \mathcal{F}(\mathbb{R}^2)$. Then compute successively the sequence of sets $\{A_m = S^m(A_0)\}_{m=1}^{\infty}$ by

$$A_{m+1} = S(A_m) = \bigcup_{i=1}^{N} S_i(A_m).$$

It follows from 1 that the sequence $\{A_m\}$ converges to the attractor K of the IFS $\{X : S_i, i = 1, 2, \cdots, N\}$ in the Hausdorff metric. In fact, if m is large enough, then A_m is approximately equal to K, and is basically indistinguishable. Thus the image of attractor drawn by us is actually A_m with large enough m. This approach to generate fractal requires heavy amount of memory, because in each iteration generate some image and to store image generated by affine transformation requires large amount of memory.

Random iterated algorithm: Let $P = \{p_1, p_2, \cdots, p_N\}$ be a set of probability weights, where p_i can be thought of as relative weight for each S_i and $\sum_{i=1}^{N} p_i = 1$. In general, we take

$$p_i \approx \frac{|detA_i|}{\sum_{i=1}^{N} |detA_i|},$$

where

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

and $|detA_i|$ denote the determinant of A_i, $i = 1, 2, \cdots, N$. If $|detA_i| = 0$, we take a small positive number as p_i(For example $p_i = 0.01$) and make appropriate adjustments for other p_k such that $\sum_{i=1}^{N} p_i = 1$. An iterated function system with probabilities consists of an iterated function system $\{\mathbb{R}^2 : S_1, S_2, \cdots, S_N\}$ together with a set $\{p_1, p_2, \cdots, p_N\}$ of probability weights. We often denote such iterated function system by

$$\{\mathbb{R}^2 : S_1, S_2, \cdots, S_N, p_1, p_2, \cdots, p_N\}.$$

This random method is different from the deterministic approach in that the initial set is a singleton point and at each level of iteration, just one of the defining contraction transformations is used to calculate the next level. At each level, the contraction transformation is randomly selected and applied. Points are plotted, except for the early ones, and are discarded after being used to calculate the next value. The random algorithm avoids the need of a large computer memory, it is best suited for the small computers on which one point at a time can be calculated and display on a screen. On the other hand it takes thousand of dots to produce an image in this way that does not appear too skimpy. Here is basic process: Choose an arbitrary point $x_0 \in \mathbb{R}^2$ as start point, we randomly choose a mapping S_i in $\{S_1, S_2, \cdots, S_N\}$ according to probability distribution $\{p_1, p_2, \cdots, p_N\}$, and move to $x_1 = S_i(x_0)$. We then make another random choice of S_j, and move to $x_2 = S_j(x_1)$. This continues indefinitely, we obtain a sequence $\{x_0, x_1, \cdots\}$. When N_{max} is large enough, according to Lemma 1, $\{x_n : n \geq N_{max}\}$ is indistinguishable from K. In particular, we choose $N_{max} = 0$ if $x_0 \in K$.

3 Construction of Sierpinski-type Fractals Using Deterministic Algorithm

According to [3], we can obtain a Sierpinski-type square(resp. Sierpinski-type triangle) $T(A, B)$ described in introduction as follows: we decompose initial pattern $[0, 1]^2$(resp. $\triangle$) into n^2 closed subsquares(resp. subtriangles) in the obvious manner, so that these subsquares(resp. subtriangles) have disjoint interiors and sidelength $1/n$. We choose some subsquares(resp. subtriangles) according to the rule described by A and B, again divide each of these subsquares(resp. subtriangles) into n^2 congruent ones, chose the subsquares(resp. subtriangles) according the same rule and repeat the procedure inductively to the infinity, then we get Sierpinski-type triangle(resp. Sierpinski-type square) $T(A, B)$. For any $k \geq 1$, then $T^k(A, B)$ described in introduction is the union of squares (resp. triangles) that are chosen in the step k. For $a \in A, b \in B$, set

$$S_a(x) = \frac{1}{n}(x + a), S_b(x) = \frac{1}{n}(b - x). \quad (3)$$

Then $T(A, B)$ is unique invariant set of iterated function system $\{S_a\}_{a\in A} \cup \{S_b\}_{b\in B}$.

3.1 Steps for Creating $T^k(A, B)$(resp. $T(A, B)$) Using Deterministic Algorithm

For creating $T^k(A, B)$ using deterministic algorithm steps given below should be considered.

Step 1: Draw an initial pattern($[0, 1]^2$ or $\triangle$) on the plane, and decompose it into n^2 congruent ones in the obvious manner (for example, we can divide a given equilateral triangle into n^2 congruent triangles by drawing $n - 1$ lines, parallel to each edge and dividing the other two edges into n equal parts).

Step 2: Create a $2 \times 4l^k$ (resp. $2 \times 3l^k$) matrix of zeros to store vertex coordinates of all small squares (resp. triangles) chosen after each construction.

Step 3: Initialize the matrix in Step 2 with vertex coordinates of initial pattern which are decribed in the first step.

Step 4: Applying transformations (3) on vertex coordinates of initial pattern and repeat Step 1 for each subpattern obtained after applying transformations.

Step 5: Again apply transformations (3) on the vertex coordinates of all subpattern obtained in Step 4 and repeat Step 1 for each new subpattern.

Step 6: Repeat step 5 again and again, this step 5 can be repeated infinite number of times.

3.2 Source Program for Creating k-squares $T^k(A, B)$ Using Deterministic Algorithm

function Fractal_ square (E, F, n, k)
% FRACTAL_SQUARE: Display the geometric construction of k-squares $T^k(A, B)$.
% Call format: Fractal_square (E, F, n, k)
% E *and* F *are two two dimensional arrays with the form*

$$[c_{11}, c_{12}, \ldots, c_{1m}; c_{21}, c_{22}, \ldots, c_{2m}],$$

where for any $1 \leq i \leq m$,

$$(nc_{1i}, nc_{2i}) \in \begin{cases} A, & \text{when E is of such form;} \\ B, & \text{when F is of such form.} \end{cases}$$

% $1/n$ *is contraction radio.*
% k *is iterated depth.*
% Divide a unit square $[0, 1]^2$ *into* n^2 *congruent squares by drawing* $n - 1$ *lines, parallel to a pair of opposite edges and dividing the other pair of opposite edges into* n *equal parts.*
$t = 0 : 1/n : 1;$
for $i = 1 : (n + 1)$
$tx(1) = t(i); tx(2) = t(i); ty = [0, 1];$
plot$(tx, ty,' b')$
hold on
plot$(ty, tx,' b')$

```
    hold on
end
axis square
d = 1;
if isempty(E)
    LE = 0;
else
    LE =length(E(1,:));
end
if isempty(F)
    LF = 0;
else
    LF =length(F(1,:));
end
l = LE + LF;
M =zeros(2, 4 * l^k); % M is an 2 × 4l^k matrix of
zeros which be used to store vertex coordinates of all
small squares after each iteration.
M(:, 1 : 4) = [0, 1, 1, 0; 0, 0, 1, 1]; % Initialization of
matrix M.
for h = 1 : k
    C = 1/n * M; D = −1/n * M;
    for i = 0 : LE − 1,
        M(:, i * 4 * (l^(h − 1)) + 1 : ((i + 1) * (4 *
(l^(h − 1))))) = C(:, 1 : (4 * (l^(h − 1)))) + E(:
, i + 1) * ones(1, 4 * (l^(h − 1)));
    end
    for j = 0 : LF − 1
        M(:, ((LE + j) * (4 * (l^(h − 1)))) + 1 :
(((LE + j) + 1) * (4 * (l^(h − 1))))) = D(:, 1 :
(4*(l^(h−1)))) + F(:, j+1) * ones(1, 4*(l^(h−1)));
    end
    d = d/n;
% Divide each h−square into n^2 congruent squares
    for m = 1 : l^h
        squaregrid(min(M(1, 4 * m − 3 : 4 *
m)), min(M(2, 4 * m − 3 : 4 * m)), n, d);
    end
end
% Fill each small square in T^k(A, B) with blue.
for i = 1 : l^k
    patch(M(1, 4 * i − 3 : 4 * i), M(2, 4 * i − 3 :
4 * i), ′b′);
end
set(gca,'xtick',[],'xticklabel',[]),
set(gca,'ytick',[],'yticklabel',[]) % Do not dis-
play the coordinate axis.
function squaregrid(x, y, r, s)
a = [x : s/r : x + s]; b = [y : s/r : y + s];
plot(a, meshgrid(b, a), ′b′)
hold on
plot(meshgrid(a, b), b, ′b′)
```

The percent-sign (%) implies that this is a remark statement after it, the text shown in italics following this sign. The remark statement is ignored when running program.

3.3 Source Program for Creating k-triangles $T^k(A, B)$ Using Deterministic Algorithm

```
function Fractal_triangle(E, F, n, k)
% FRACTAL_TRIANGLE: Display the geometric con-
struction process of k−triangles T^k(A, B).
% Call format: Fractal_triangle(E, F, n, k).
% E and F are two two dimensional arrays with the
form
```

$$[c_{11}, c_{12}, \ldots, c_{1m}; c_{21}, c_{22}, \ldots, c_{2m}],$$

where for any $1 \le i \le m$,

$$(nc_{1i}, nc_{2i}) \in \begin{cases} A, & \text{when E is of such form;} \\ B, & \text{when F is of such form.} \end{cases}$$

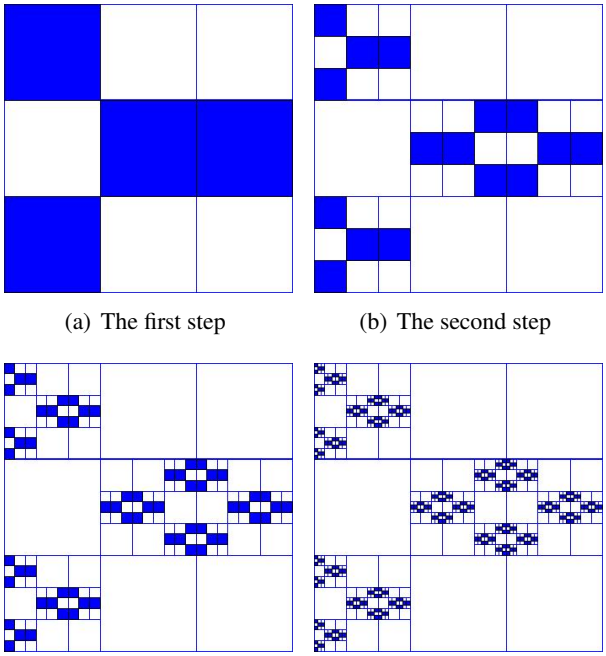
```
% 1/n is contraction radio.
% k is iterated depth.
trianglegrid([0, 1, 1/2], [0, 0, sqrt(3)/2], n); % Draw
an equilateral triangle with vertex coordinates
(0, 0), (1, 0), (1/2, √3/2), and divide it into n^2 con-
gruent triangles.
axis square, hold on
if isempty(E)
    LE = 0;
else
    LE =length(E(1,:));
end
if isempty(F)
    LF = 0;
else
    LF =length(F(1,:));
end
l = LE + LF;
M =zeros(2, 3 * l^k); % M is an 2 × 3l^k matrix of
zeros which be used to store vertex coordinates of all
small triangles after each iteration.
% Initialization of M.
M(:, 1 : 3) = [0, 1, 1/2; 0, 0, sqrt(3)/2];
for h = 1 : k
    C = 1/n * M;
    D = −1/n * M;
    for i = 0 : (LE − 1)
        M(:, i * 3 * (l^(h − 1)) + 1 : ((i + 1) * (3 *
(l^(h − 1))))) = C(:, 1 : (3 * (l^(h − 1)))) + E(:
, i + 1) * ones(1, 3 * (l^(h − 1)));
    end;
    for j = 0 : (LF − 1)
        M(:, ((LE + j) * (3 * (l^(h − 1)))) + 1 :
(((LE + j) + 1) * (3 * (l^(h − 1))))) = D(:, 1 :
(3*(l^(h−1)))) + F(:, j+1) * ones(1, 3*(l^(h−1)));
```

end;
for $m = 1 : l\hat{}h$ % *Divide each equilateral triangle in* $T^h(A, B)$ *into* n^2 *congruent triangles.*
trianglegrid($M(1, 3*m-2 : 3*m), M(2, 3*m-2 : 3*m), n$)
end
end
% *Fill each equilateral triangle in* $T^k(A, B)$ *by blue.*
for $i = 1 : l\hat{}k$
patch($M(1, 3*i-2 : 3*i), M(2, 3*i-2 : 3*i), 'b'$);
end
set(gca,'xtick',[],'xticklabel',[]),
set(gca,'ytick',[],'yticklabel',[]) % *Do not display the coordinate axis.*
function trianglegrid(x, y, n)
plot($[x(1), x(2), x(3), x(1)], [y(1), y(2), y(3), y(1)]$)
hold on
a = [linspace($x(1), x(2), n+1$); linspace($y(1), y(2), n+1$)];
b = [linspace($x(2), x(3), n+1$); linspace($y(2), y(3), n+1$)];
c = [linspace($x(3), x(1), n+1$); linspace($y(3), y(1), n+1$)];
for $i = 2 : n$
plot($[a(1, i), b(1, n+2-i)], [a(2, i), b(2, n+2-i)], 'b'$)
hold on
end
for $j = 2 : n$
plot($[b(1, j), c(1, n+2-j)], [b(2, j), c(2, n+2-j)], 'b'$)
hold on
end
for $k = 2 : n$
plot($[c(1, k), a(1, n+2-k)], [c(2, k), a(2, n+2-k)], 'b'$)
hold on
end

Remark 2. *The difference between here and source program presented in Section 3.3 is that we replace the command of dividing square by the command of dividing triangle in each step construction.*

3.4 Some Examples

Saving the text in a file called Fractal_ square.m (resp. Fractal_triangle.m) in your current directory. We can obtain the fine structure of $k-$squares(resp. $k-$triangles) $T^k(A, B)$ by calling corresponding self-defining function: Fractal_ square(E, F, n, k)(resp.Fractal_triangle(E, F, n, k)) in the Command Window of Matlab.



(a) The first step (b) The second step

(c) The third step (d) The fourth step

Figure 2: The first four steps of constructing Sierpinski-type square $T(A_1, B_1)$.

Example 1. *Let* $n = 3$,

$$A_1 = (0, 0), (0, 2), (2, 1), B_1 = (2, 2).$$

We can obtain the fine structure of the first forth steps of constructing Sierpinski-type square $T(A_1, B_1)$ *by running the following instructions in turn:*

Fractal_square($[0, 0, 2/3; 0, 2/3, 1/3], [2/3; 2/3], 3, k$),

$k = 1, 2, 3, 4$,*see Figure 2.*

Example 2. *Let* $n = 3$,

$$A_2 = \{(1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{3}{2}, \frac{\sqrt{3}}{2})\},$$

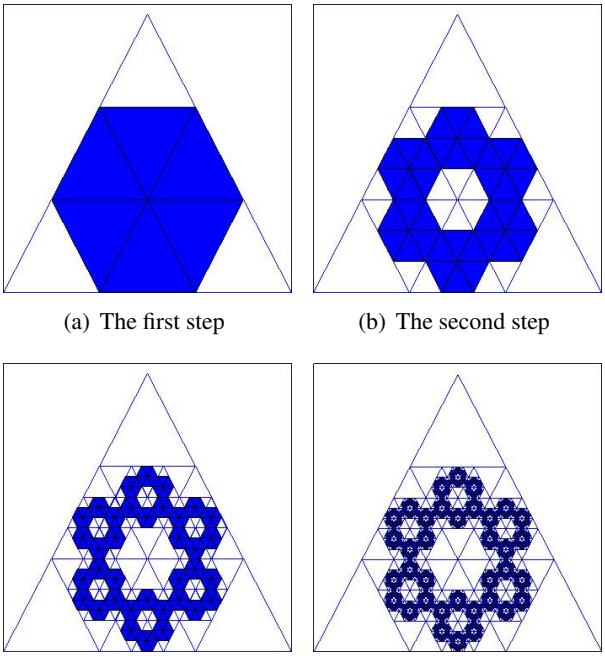
$$B_2 = (\frac{3}{2}, \frac{\sqrt{3}}{2}), (\frac{5}{2}, \frac{\sqrt{3}}{2}), (2, \sqrt{3})\}.$$

We can obtain the fine structure of the first forth steps of constructing Sierpinski-type triangle $T(A_2, B_2)$ *by running the instructions in turn: Fractal_triangle*($E, F, 3, k$)*, where*

$$E = [1/3, 1/6, 1/2; 0, sqrt(3)/6, sqrt(3)/6],$$

$$F = [1/2, 5/6, 2/3; sqrt(3)/6, sqrt(3)/6, sqrt(3)/3],$$

$k = 1, 2, 3, 4$,*see Figure 3.*



(a) The first step (b) The second step

(c) The third step (d) The forth step

Figure 3: The first four steps of constructing Sierpinski-type triangle $T(A_2, B_2)$.

When either of A and B is empty set, we can obtain the fine structure of $k-$squares(resp. $k-$triangles) $T^k(A, B)$ according to the method presented in Example 3(resp. Example 4).

Example 3. *Let* $n = 4$,

$$A_3 = \{(0,0),(3,0),(1,1),(2,1),(1,2),(2,2),$$

$$(0,3),(3,3)\}, B_3 = \emptyset,$$

We can obtain the fine structure of the first forth steps of constructing Sierpinski-type square $T(A_3, B_3)$ *by running the following instructions in turn:*

$$\textit{Fractal\_square}(E, F, 4, k), k = 1, 2, 3, 4,$$

where $E = [0, 3/4, 1/4, 1/2, 1/4, 1/2, 0, 3/4; 0, 0, 1/4, 1/4, 1/2, 1/2, 3/4, 3/4]$, $F = []$, *see Figure 4.*

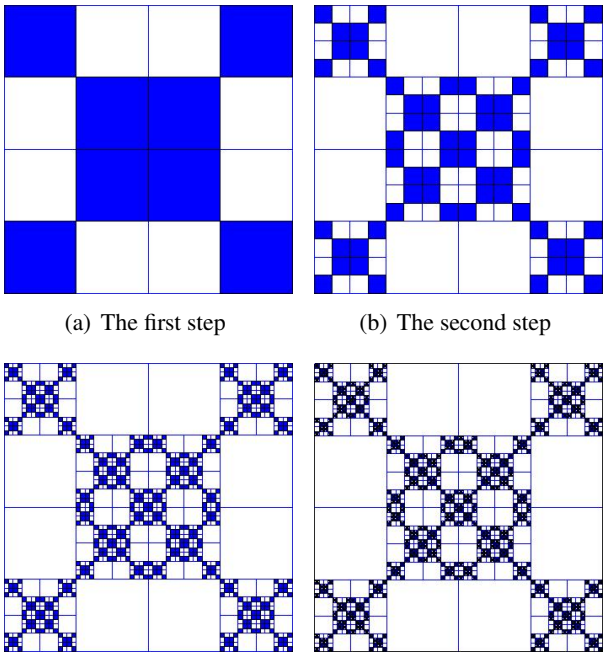
Example 4. *Let* $n = 2$,

$$A_4 = \{(0,0),(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2})\}, B_4 = \emptyset.$$

We can obtain the fine structure of the first forth steps of constructing classic Sierpinski gasket $T(A_4, B_4)$ *by running the instructions in turn:*

$$\textit{Fractal\_triangle}([0, 1/2, 1/4; 0, 0, sqrt(3)/4], [], 2, k),$$

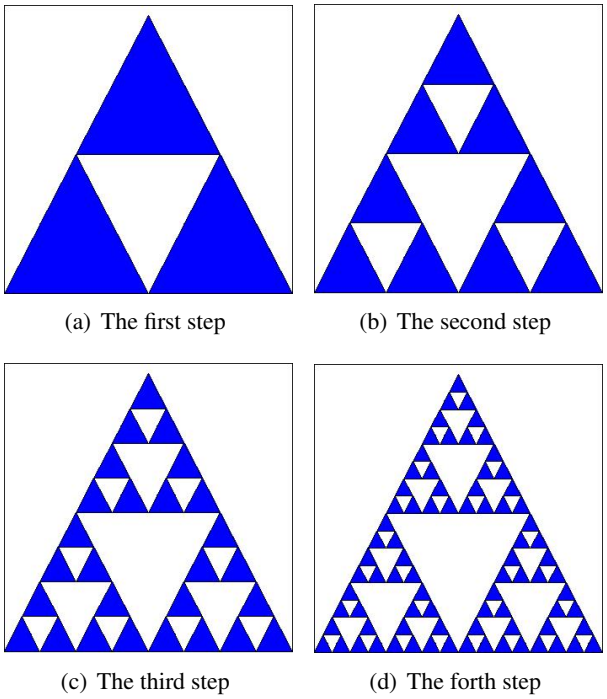
$k = 1, 2, 3, 4$, *see Figure 5.*



(a) The first step (b) The second step

(c) The third step (d) The third step

Figure 4: The first four steps of constructing Sierpinski-type square $T(A_3, B_3)$.



(a) The first step (b) The second step

(c) The third step (d) The forth step

Figure 5: The first four steps of constructing Sierpinski gasket $T(A_4, B_4)$.

Remark 3. *Dividing each square(resp. equilateral triangle) chosen in each step for generating $T(A,B)$ into n^2 congruent squares(resp. triangles) will be helpful for researcher to investigate deeper the connection between higher and lower k-squares (resp. k-triangles). We can make the grid not shown in the figure by deleting the function of drawing grid in source programs presented as above when we don't want grid.*

Remark 4. *Our results in present paper include the results in our previous paper [22].*

4 Applying Random Iterated Algorithm to Generate Sierpinski-type Fractal

Applying random iterated algorithm to generate fractal, the principle is clear and simple, it is easy to implement in any programming language and the results it generates are very spectacular and may be used for any graphical goal, not only for mathematical reasons. The method may be illustrated with the Yuval Fishers special copyrighter(see [20]) which receives as entry an arbitrary image (may be a point) and applies to it the set of affine transformation, generating a new image. The image obtained is transmitted, using a feedback process, on the entry of the copyrighter and the process is repeated for several times. For example, consider that the transformations are those which describe the Sierpinski Triangle. If we test the Yuval Fisher copyrighter for different initial images we can observe that the final image is the same, so it not depends on the initial image but is defined by the affine transforms applied to it. It is one of the most used methods of generating a self-similar fractals. For creating $T(A,B)$ using random iterated algorithm steps given below should be considered.

Step 1: Choose $(x,y)=(0,0)$ as starting point.

Step 2: Let (x_1,y_1) be the point obtained by applying a transformation in the IFS, where each transformation are chosen with probability $1/n$.

Step 3: Repeat the step 2 with (x_1,y_1) as initial point.

Step 4: Repeat step 3 again and again, this step 3 can be repeated infinite number of times.

4.1 Source Program for Creating $T(A,B)$ Using Random Iterated Algorithm

function Sierpinskifractal (k,n,M,P)
% SIERPINSKIFRACTAL: Drawing Sierpinski-type square(resp. triangle) $T(A,B)$ by random iterated algorithm.
% Call format: Sierpinskifractal (k,n,M,P).
% k is the number of iterations.
% n is the number of affine transformations in IFS.
% $M=[a_{11},a_{12},\cdots,a_{16};a_{21},a_{22},\cdots,a_{26};\cdots,a_{n1},a_{n2},\cdots,a_{n6}]$ is a $n\times 6$ array, where

$$[a_{i1},a_{i2},\cdots,a_{i6}]=[a_i,b_i,e_i,c_i,d_i,f_i]$$

satisfying (2).
% $P=(p_1,p_2,\cdots,p_n)$ is a dimension array with $\sum_{i=1}^{n}p_i=1$.
$x=0$; $y=0$; r =rand$(1,k)$; B =zeros$(2,k)$; w =zeros$(1,n)$;$w(1)=P(1)$;
for $i=2:n$
  $w(i)=w(i-1)+P(i)$;
end
$m=1$;
for $i=1:k$
  for $j=1:n$
    if $r_i<w(j)$
      $a=M(j,1);b=M(j,2);e=M(j,3);c=M(j,4);d=M(j,5);f=M(j,6)$;
      break;
    end
  end
  $x=a*x+b*y+e$; $y=c*x+d*y+f$;
  $B(1,m)=x$; $B(2,m)=y$;
  $m=m+1$;
end
plot($B(1,:)$,$B(2,:)$,'.','markersize',0.5)
set(gca,'xtick',[],'xticklabel',[]);
set(gca,'ytick',[],'yticklabel',[])

4.2 Comments and Examples

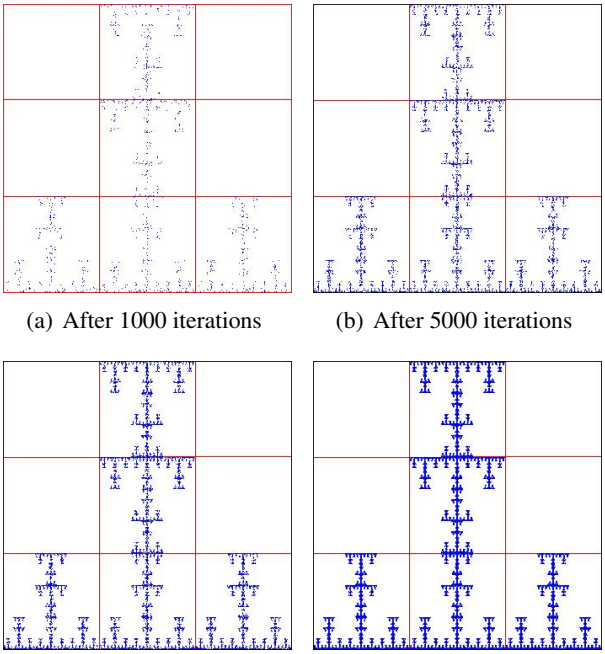
Remark 5. *We can't obtain the fine structure of $k-$squares(resp. $k-$triangles) by using random iterated algorithm. But we can obtain close approximation of Sierpinski-type square(resp. Sierpinski-type triangle) $T(A,B)$ faster than using deterministic algorithm and need lesser computer memory.*

Remark 6. *For facilitating the readers to observe and understand, one may add the command lines which divide $[0,1]^2$(resp. equilateral triangle) into n^2 congruent ones (see source programs in Section 3 or [22] for detail) at the beginning of program, the rest remain the same.*

Example 5. *Let $n=5$,*

$$A_5=\{(0,0),(1,0),((2,0))\},B_5=\{(2,2),(2,3)\}.$$

We can obtain close approximation of Sierpinski-type square $T(A_5,B_5)$ by calling the function: fractal $(k,5,M,P)$ in the Matlab command window prompt, see Figure 6. Table 1 lists corresponding transformations.



(a) After 1000 iterations (b) After 5000 iterations

(c) After 10000 iterations (d) After 50000 iterations

Figure 6: Approximation of $T(A_5, B_5)$ after applying some number of iteration.

Table 1: $T(A_5, B_5)$

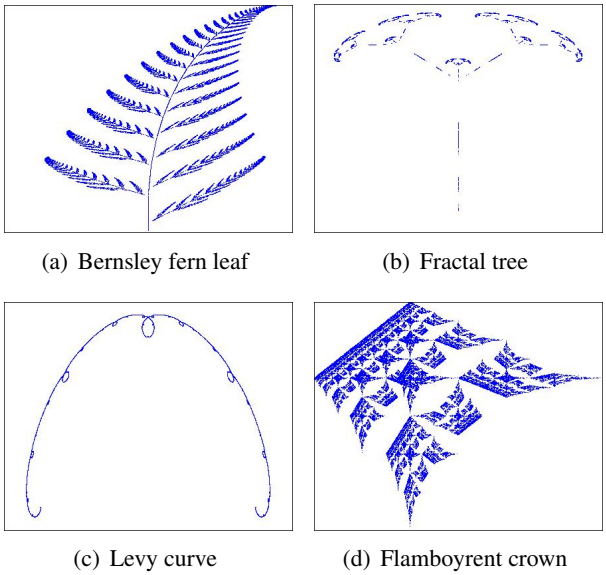
S_i	a	b	c	d	e	f	p
S_1	1/3	0	0	1/3	0	0	1/5
S_2	1/3	0	0	1/3	1/3	0	1/5
S_3	1/3	0	0	1/3	2/3	0	1/5
S_4	-1/3	0	0	-1/3	2/3	2/3	1/5
S_5	-1/3	0	0	-1/3	2/3	1	1/5

Remark 7. *This program can draw more fractals except the Sierpinski-type fractals mentioned in this paper.*

Example 6. *We can obtain four famous fractals as shown in Figure 7, the corresponding transformations are listed in Table 2-Table 5.*

Table 2: Bernsley fern leaf

S_i	a	b	c	d	e	f	p
S_1	0	0	0	0.16	0	0	0.01
S_2	0.85	0.04	-0.04	0.85	0	80	0.85
S_3	0.2	-0.26	0.23	0.22	0	80	0.07
S_4	-0.15	0.28	0.26	0.24	0	20	0.07



(a) Bernsley fern leaf (b) Fractal tree

(c) Levy curve (d) Flamboyrent crown

Figure 7: Examples of Non-Sierpinski-type fractals generated after 50000 iterations

Table 3: Fractal tree

S_i	a	b	c	d	e	f	p
S_1	0	0	0	0.5	0	0	0.05
S_2	0.42	-0.42	0.42	0.42	0	200	0.4
S_3	0.42	0.42	-0.42	0.42	0	200	0.4
S_4	0.1	0	0	0.1	0	200	0.15

Table 4: Levy curve

S_i	a	b	c	d	e	f	p
S_1	0.5	-0.5	0.5	0.5	0	0	0.05
S_2	0.5	0.5	-0.5	0.5	150	150	0.5

Table 5: Flamboyrent crown

S_i	a	b	c	d	e	f	p
S_1	0.25	0	0	0.5	0	0	0.154
S_2	0.5	0	0	0.5	-75	150	0.307
S_3	-0.25	0	0	-0.25	75	300	0.078
S_4	0.5	0	0	0.5	0	225	0.307
S_5	0.5	0	0	-0.25	150	375	0.154

5 Conclusions

Geometric construction of fractal image with IFS begins with original image and successively applying

IFS over the image up to finite number of time. This paper presents the application of theory of IFS in geometric modeling of a class of Sierpinski-type fractals with certain rotation structure and their geometric constructions base on Matlab environment. We can obtain the fine structure of each setp which generates Sierpinski-type fractal by using deterministic algorithm. Using random iteration algorithm, we can obtain close approximation of fractal image faster than using deterministic algorithm and need lesser computer memory. Since IFS can be defined for any dimension, the extension of our techniques to three dimensional objects is the most important next step.

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