Characterization of Lemniscate-like Curves Generated By Linkages

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Abstract: Cassini Ovals is a well-known family of curves characterized by the product of distances from two fixed points called foci. One special case is the Lemniscate of Bernoulli, a figure-8 curve which can be drawn by many ways. This paper focuses on the family of lemniscate-like curves especially generated by three-bar linkage systems where the marker is the midpoint of the middle rod. An algebraic characterization of such family of curves is investigated using the distances from the foci. It turns out that the corresponding Cartesian equation is of the form of the Hippopede defined as the intersection of a torus and a plane. A geometric construction showing the connection between three-bar linkages and lemniscate-like curves leads to another representation using polar equations. A parametric representation is also given to illustrate the family of curves using an EXCEL worksheet. Finally, another interesting family of curves so called skewed lemniscate-like curves is constructed and characterized algebraically as a natural generalization where the marker can be located on any fixed point on the middle rod.

Key–Words: Lemniscate of Bernoulli, lemniscate-like curve, Hippopede, characterization, three-bar linkage system, skewed lemniscate-like curve, Cassini Ovals

1 Introduction

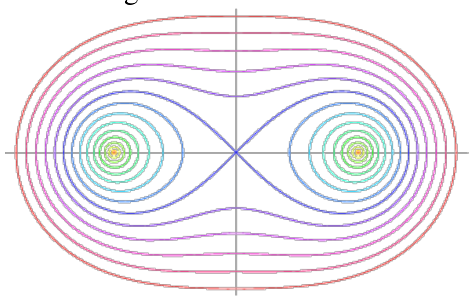
This paper is the result of extended observations originating during an exploration of A. B. Kempe's work "How to Draw a Straight Line; a Lecture on Linkages" [2] by the authors in [3].

Given two fixed points in the plane, Cassini Ovals can be characterized as the set of points so that the product of the distances from the two fixed points (called the foci) is a constant. When foci are $2a$ away, if we collect points with fixed constant b^2 for the product of distances from foci, we obtain either two disconnected loops, the Lemniscate of Bernoulli, or a connected simple loop enclosing both foci depending on the range of b: $b < a$, $b = a$ and $b > a$. See Figure 1 for an illustration[1]. Among many construction methods of the Lemniscate of Bernoulli, linkage systems are of great interest in this paper due to its usefulness in generalization and characterization in several ways.

A linkage system corresponds to a set of bars connected to each other and the plane by articulations (called linkages) typically allowing a marking point on one of the bars to move describing a curve in the plane. More generally a linkage system can have the marking point describe a region as opposed to a curve–sometimes with special geometric effects. The formal terms used to describe the anatomy of a Linkage System throughout the rest of the paper follows.

[1]http://mathworld.wolfram.com/CassiniOvals.html

Figure 1: Cassini Ovals



1. Fixed point/Fixed Pivot: A fixed point of a linkage is a point which does not change its location during the motion of the linkage system.

2. Rod/Bar: A rod or bar is the line segment that connects two distinct points. A rod can take only rigid motions during the motion of the linkage system.

3. Hinge Point/Pivot Point: A hinge point is the linkage point, and thus serves as the connection between two rods.

4. Mover: A mover is the end point on a rod, so called the driver, which rotates around a fixed point. It has one degrees of freedom and, in fact, we can use the angle of rotation to represent the location of the marker which draws the desired curve as a function of mover.

5. Marker: A marker, also known as tracer, is a point on a rod which will draw the whole or part of the desired curve as the mover rotates around a fixed point on a rod.

To draw the families of curves that include the Lemniscate of Bernoulli, one can use a three-bar linkage system. This system corresponds to a chain of three bars linked (hinged) to each other with the outside two bars also linked to the plane at two fixed points. The relative lengths of these bars, the distance between the fixed points, and the position of the marker (along the middle bar) determine the particular curve being drawn. The Lemniscate of Bernoulli is a particular case of the full set of curves that may be drawn in this way. Lemniscate-like curves, skewed lemniscate-like curves, and 'teardrop' curves also appear.

Figure 2: Three-bar linkage system

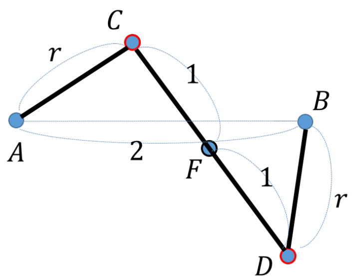


Figure 2 is a description of a three-bar linkage system: A and B are fixed points; AC, CD, DB are rods; C and D are hinge points; AC is the driver and C is the mover rotating around the fixed point A; F is the marker.

It turns out that the trace of the marker of three-bar linkages with specific rod lengths produces a family of lemniscate-like curves and it is natural to ask how we can characterize such family of curves. This paper will focus on answering this question in various ways. In the following sections, we will investigate algebraic characterization, geometric construction, parametric representation, and a generalization to skewed lemniscate-like curves.

2 Algebraic characterization

In this section, we give an algebraic characterization of the family of lemniscate-like curves generated by three-bar linkage systems. See Figure 2 for an illustration. We initially pick the distance $\operatorname{dist}(A, B) = 2$ and $|CD| = 2$ for our computations and later generalize to encompass any scale.

Theorem 1 *If a three-bar linkage system is constructed by two fixed points A, B with $\operatorname{dist}(A, B) = 2$, and three bars AC, BD, CD with $|AC| = |BD| = r$, and $|CD| = 2$, then the family of lemniscate-like curves generated by the marker F, the midpoint of the line segment CD can be characterized as*

$$\left(d_1^2 + 1 - \frac{r^2}{2}\right)\left(d_2^2 + 1 - \frac{r^2}{2}\right) = \left(2 - \frac{r^2}{2}\right)^2 \quad (1)$$

where $d_1 = |AF|$ and $d_2 = |BF|$.

Proof. The proof comes from the combination of the Pappus's theorem and the law of cosines. In fact, the quadrilateral $ACBD$ is an anti-parallelogram and $\angle BCD = \angle ADC$. For the triangles ADC and BCD, the law of cosines gives

$$\frac{|BC|^2 + 2^2 - r^2}{4|BC|} = \cos\angle BCD$$
$$= \cos\angle ADC = \frac{|AD|^2 + 2^2 - r^2}{4|AD|}.$$

Solving this equation leads to the relation

$$|BC| \cdot |AD| = 4 - r^2. \quad (2)$$

On the other hand, we apply the Pappus's theorem to those triangles and get

$$\begin{aligned} r^2 + |AD|^2 &= 2(1^2 + |AF|^2) = 2(1 + d_1^2), \quad (3) \\ r^2 + |BC|^2 &= 2(1^2 + |BF|^2) = 2(1 + d_2^2). \quad (4) \end{aligned}$$

Combining (3), (4), and (2), we have the desired result (1). □

The characterization (1) generalizes of the Lemniscate of Bernoulli. In fact, if $r = \sqrt{2}$, then (1) reduces to

$$d_1 d_2 = 1 = \left(\frac{\operatorname{dist}(A, B)}{2}\right)^2$$

which exactly coincides the definition of the Lemniscate of Bernoulli.

Moreover, we can confirm the relation (1) using an EXCEL worksheets. See Figure 9 and Figure 10 in the Appendix for simulations: $r = \sqrt{2}$ and $r = 1$.

The highlighted column K shows the ratio of the left hand side to the right hand side of (1). The validity of the relation (1) is illustrated by the constant value 1 in column K.

Corollary 2 *For the three-bar linkage system in the theorem 1, if $A = (-1, 0)$ and $B = (1, 0)$, then the Cartesian equation of the marker $F(x, y)$ is given as*

$$\left((x+1)^2 + y^2 + 1 - \frac{r^2}{2}\right)\left((x-1)^2 + y^2 + 1 - \frac{r^2}{2}\right) = \left(2 - \frac{r^2}{2}\right)^2.$$

If the two fixed points are $2a$ away and the connecting rods have lengths r, $2a$, and r, then the trace of the marker can be characterized as

$$\left(d_1^2 + a^2 - \frac{r^2}{2}\right)\left(d_2^2 + a^2 - \frac{r^2}{2}\right) = \left(2a^2 - \frac{r^2}{2}\right)^2. \tag{5}$$

A proof of this relation (5) is essentially identical to that of Theorem 1. If we set the two fixed points as $(-a, 0)$ and $(a, 0)$, then the Cartesian equation of the graph can be simplified to

$$(x^2 + y^2)^2 = r^2x^2 - \left(2a^2 - \frac{r^2}{2}\right)y^2$$

which is exactly the Hippopede[2] of the form

$$(x^2 + y^2)^2 = cx^2 + dy^2$$

where $c > 0$ and $c > d$.

3 Geometric construction and polar equation

A geometric characterization of lemniscate-like curves highlights a connection to three-bar linkage systems. This part is mainly motivated from the work of Akopyan [1].

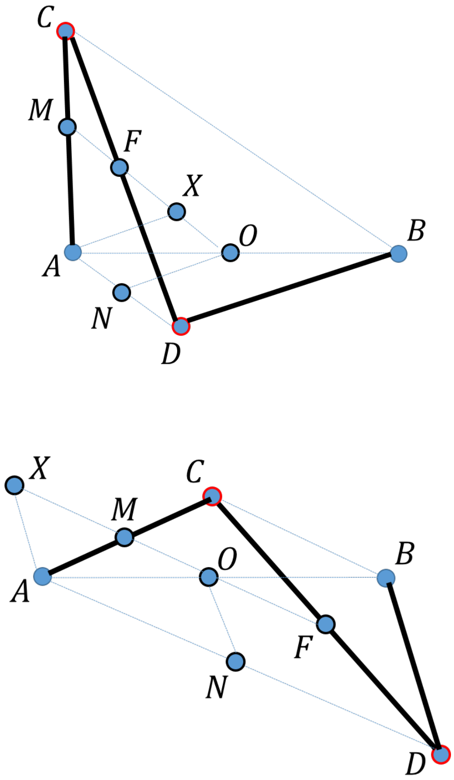
Lemma 3 *Given three-bar linkage system as in Figure 3 where dist(A, B) = $|CD|$ = 2 and $|AC|$ = $|BD| = r$, if M, N and O are midpoints of AC, AD and AB, respectively and $ANOX$ is a parallelogram, then the followings hold.*

(a) Points M, F, X, and O are collinear.

(b) $\overrightarrow{XM} = \overrightarrow{OF}$

(c) $|AM| = |AX| = r/2$

Figure 3: Characterizing the location of F



Proof. Since $ACBD$ is an anti-parallelogram, we have $AD \parallel CB$ and $AMFN$ is also a parallelogram. Therefore M, F, X and O are collinear and $\overrightarrow{FM} = \overrightarrow{OX}$ which lead to $\overrightarrow{XM} = \overrightarrow{OF}$. Finally, $|AX| = |NO| = |BD|/2 = r/2 = |AM|$. □

Lemma 3 holds true for general values of r and we can obtain another characterization of lemniscate-like curves, or the Hippopede. A direct consequence of this lemma can be stated as follows.

Theorem 4 *Given three-bar linkage system as in Figure 4 where dist(A, B) = $|CD|$ = 2, O is the midpoint of AB, $|AC| = |BD| = r$, and the circle with radius $r/2$ and center A, the set of all points F where $|OF| = |MX|$ for any secant MXO of the circle draws a lemniscate-like curve generated by the three-bar linkage system.*

From the triangle AXO, the law of cosine gives

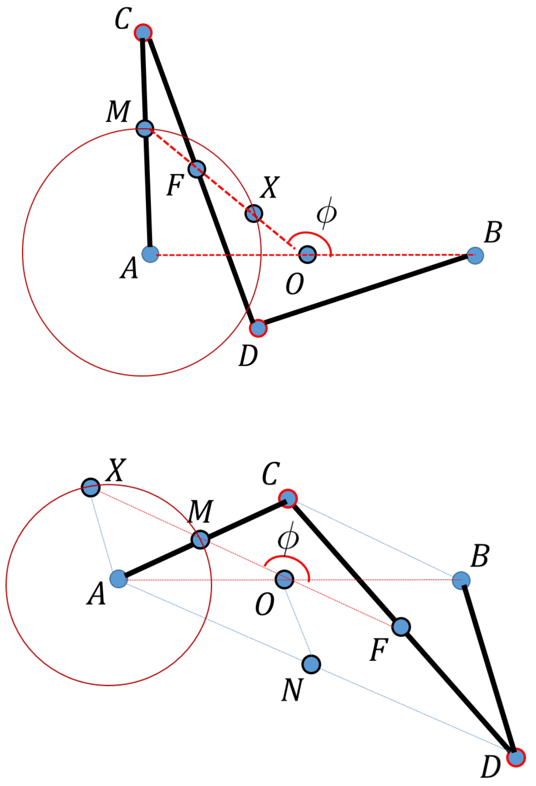
$$\left(\frac{r}{2}\right)^2 = 1 + |OX|^2 - 2|OX|\cos(\pi - \phi)$$

or

$$|OX| = -\cos\phi - \sqrt{\left(\frac{r}{2}\right)^2 - \sin^2\phi}. \tag{6}$$

[2]https://en.wikipedia.org/wiki/Hippopede

Figure 4: Geometric construction



If we apply intersecting secants theorem to the secant OXM, we have

$$\begin{aligned} |OX| \cdot |OM| &= |OX| \cdot (|OX| + |MX|) \\ &= \left(1 - \frac{r}{2}\right)\left(1 + \frac{r}{2}\right) \end{aligned}$$

or

$$|MX| = -|OX| + \frac{1}{|OX|}\left(1 - \frac{r^2}{4}\right). \quad (7)$$

Substituting (6) to (7), we obtain the following result:

Corollary 5 *For a three-bar linkage system with rod lengths r, 2, r and foci at $(-1, 0)$ and $(1, 0)$, the trace of the marker, or the midpoint of the middle rod can be represented by a polar equation:*

$$\rho = 2\sqrt{\left(\frac{r}{2}\right)^2 - \sin^2\phi}$$

Finally, from this polar representation, we can classify the family of curves as follows:

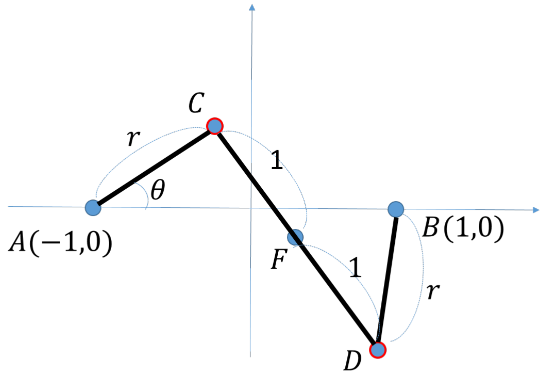
(a) If $0 < r < 2$, then we have a figure-8 curve, or lemniscate-like curve.

(b) If $r = 2$, then we have two unit circles centered at foci.

(c) If $r > 2$, then we have a connected simple loop enclosing both foci.

4 Parametric representation

Figure 5: Three-bar linkage system: Midpoint marker



If we want to construct a graph, it is very helpful to get a parametric representation. To find a parametric representation of the marker F, we assume that $A = (-1, 0)$, $B = (1, 0)$, $|AC| = |BD| = r$, $|CD| = 2$, and F is the midpoint of CD. We measure the angle θ counterclockwise from the ray AB to to the driver AC so that $C = (-1 + r\cos\theta, r\sin\theta)$. From the key fact that the quadrilateral $ACBD$ is an anti-parallelogram, we have $\overrightarrow{CB} = (2 - r\cos\theta, -r\sin\theta)$ and $\overrightarrow{AD} = t\overrightarrow{CB}$ for some constant t. The point $D = (-1, 0) + t(2 - r\cos\theta, -r\sin\theta)$ satisfies $|BD| = r$ which gives a quadratic equation in t:

$$\{t(2 - r\cos\theta) - 2\}^2 + (t \cdot r\sin\theta)^2 = r^2.$$

Since $t = 1$ is a trivial solution, it is not surprising to factor the equation and get the solution set as

$$\left\{\frac{4 - r^2}{4 - 4r\cos\theta + r^2}, 1\right\}.$$

We discard the possibility $t = 1$ to represent D as

$$D = \left(-1 + \frac{(2 - r\cos\theta)(4 - r^2)}{4 - 4r\cos\theta + r^2}, \frac{-(4 - r^2)r\sin\theta}{4 - 4r\cos\theta + r^2}\right),$$

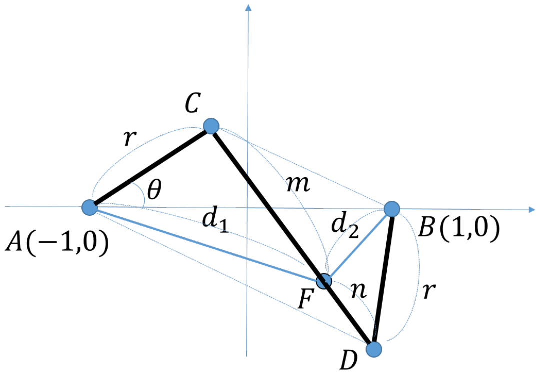
and the marker F as

$$\begin{aligned} F &= \frac{C + D}{2} \\ &= \left(\frac{r(2 - r\cos\theta)(2\cos\theta - r)}{4 - 4r\cos\theta + r^2},\right. \\ &\qquad \left.\frac{r^2\sin\theta(r - 2\cos\theta)}{4 - 4r\cos\theta + r^2}\right). \end{aligned}$$

5 Generalization to skewed lemniscate-like curves

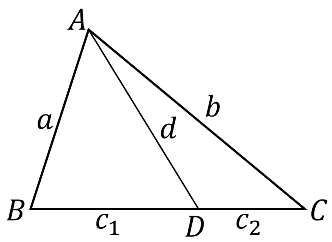
In this section, we generalize the location of the marker so that it can be any fixed point on the middle rod. We observe a family of skewed lemniscate-like curves if $m \neq n$. See Figure 6 where $m + n = 2$. Figure 11 and 12 illustrate the case of $m = 0.6$ and $n = 1.4$ for $r = \sqrt{2}$ and $r = 1.1$.

Figure 6: Three-bar linkage system: Generalized marker



To tackle this problem, we need a generalized version of Pappus's theorem.

Figure 7: Generalized Pappus's Theorem



Lemma 6 *For Figure 7, we have the relation:*

$$c_1 c_2 + d^2 = \frac{c_2}{c_1 + c_2} a^2 + \frac{c_1}{c_1 + c_2} b^2$$

In particular, if $c_1 = c_2 = c$, then the relation reduces to

$$c^2 + d^2 = \frac{1}{2}\left(a^2 + b^2\right).$$

Proof. We may use vectors and inner product to prove the theorem. In fact,

$$\overrightarrow{AD} = \frac{c_2}{c_1 + c_2}\overrightarrow{AB} + \frac{c_1}{c_1 + c_2}\overrightarrow{AC}$$

which leads to

$$\begin{aligned}(c_1+c_2)^2|\overrightarrow{AD}|^2 \\ = c_2^2|\overrightarrow{AB}|^2 + c_1^2|\overrightarrow{AC}|^2 + 2c_1c_2\overrightarrow{AB}\cdot\overrightarrow{AC}.\end{aligned} \tag{8}$$

Since $|\overrightarrow{AC} - \overrightarrow{AB}| = |\overrightarrow{BC}| = c_1 + c_2$, we have

$$\overrightarrow{AB}\cdot\overrightarrow{AC} = \frac{|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 - (c_1 + c_2)^2}{2}.$$

We plug this inner product into (8) and simplify to get the desired result. □

Now we are ready to give a geometric characterization of the family of curves generated by a generalized marker of three-bar linkage systems. See Figure 6.

Theorem 7 *For a three-bar linkage system with two fixed points A and B, and three bars AC, BD, CD such that $dist(A, B) = |CD| = 2$ and $|AC| = |BD| = r$, then the family of skewed lemniscate-like curves generated by the marker F with $|CF| = m$ and $|FD| = n = 2 - m$ can be characterized as*

$$\begin{aligned}\left(d_1^2 + mn - \frac{nr^2}{2}\right)\left(d_2^2 + mn - \frac{mr^2}{2}\right) \\ = \frac{mn\left(4 - r^2\right)^2}{4}\end{aligned} \tag{9}$$

where $d_1 = |AF|$ and $d_2 = |BF|$.

The proof is similar to that of Theorem 1 and left to the reader. Note that the relation (2) is still valid while we need a generalized version of Pappus's theorem (Lemma 6) to replace (3) and (4).

6 Conclusion

Figure 8: Relationship between sets of curves

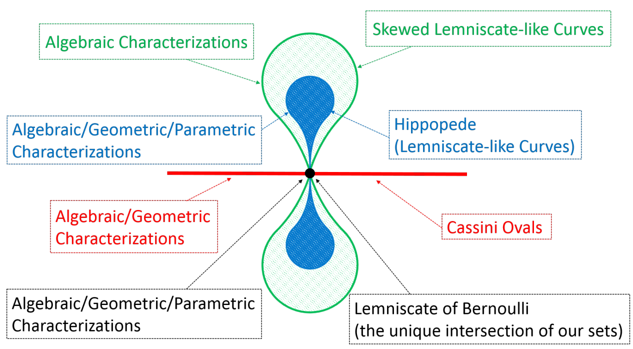


Figure 8 indicates the relationship between the sets made up of the families of curves we have connected. The geometric relationship that characterizes Cassini Ovals leads to an algebraic expression that, in the case of the Lemniscate of Bernoulli, can be connected with a construction using a three-bar linkage system. We verify this connection algebraically for a three-bar linkage system including the scale-independent case. From the three-bar linkage construction, we extend the algebraic relation to lemniscate-like curves, or the Hippopede, and represent these parametrically. This representation is used in EXCEL worksheets to numerically illustrate that the algebraic characterization is correct.

With this we incorporate another geometric characterization (the geometric representation described in Figure 4) for which we construct a polar representation. This allows us to broaden the set of curves for which we have an algebraic characterization connected with a geometric characterization to include all our lemniscate-like curves. We then further expand our algebraic characterization from the three-bar linkage system to include skewed lemniscate-like curves where the marker is located on any fixed point on the middle rod.

References:

[1] Arseniy Akopyan, *The Lemniscate of Bernoulli, without Formulas*, The Mathematical Intelligencer, Volume 36, Issue 4, pp 47-50, 2014.

[2] A. B. Kempe, *How to Draw a Straight Line; a Lecture on Linkages*, London, Macmillan, 1877. Albion, MI: Cambridge, 1952.

[3] Hyung Ju Nam and Kim Christian Jalosjos, *Parametric Representations of Polynomial Curves Using Linkages*, Parabola incorporating Function, Volume 52, Issue 1, 2016, https://www.parabola.unsw.edu.au/2010-2019/volume-52-2016/issue-1.

Appendix

Figure 9: $r = \sqrt{2}$ and midpoint F

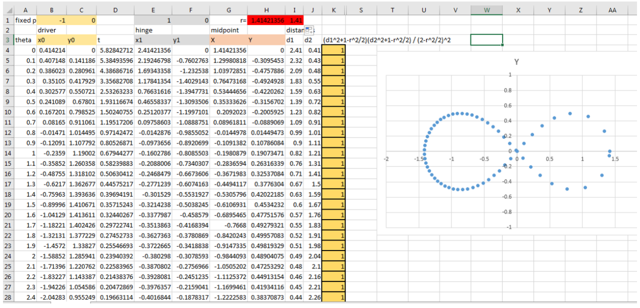


Figure 10: $r = 1.1$ and midpoint F

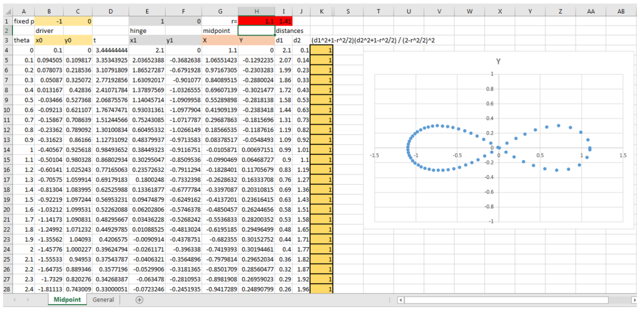


Figure 11: $r = \sqrt{2}, m = 0.6, n = 1.4$

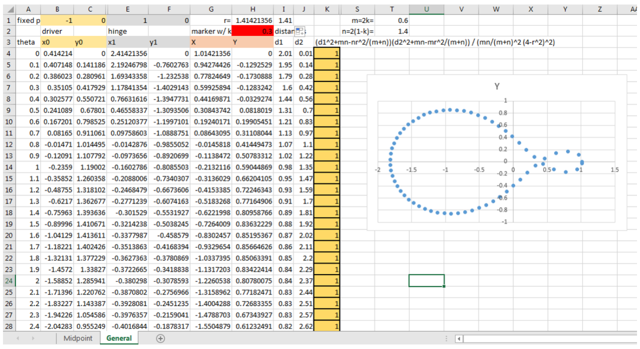


Figure 12: $r = 1.1, m = 0.6, n = 1.4$