

The Nakagami and its related distributions

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Abstract: This paper introduces the Nakagami distribution which is usually used to simulate the ultrasound image. The Nakagami distribution is related to the gamma distribution, the Rayleigh distribution, the weibull distribution, the chi-square distribution and the exponential distribution. Through the gamma distribution, it's much easier to derive the moments of a Nakagami random variable. The confidence interval for the ratio of medians from two independent Nakagami distributions is constructed.

Key-Words: The Nakagami distribution, The gamma distribution, The Rayleigh distribution, The Weibull distribution, The chi-square distribution, The exponential distribution, The ratio of medians.

1 Introduction

The Nakagami distribution is usually used to simulate the ultrasound image. Pavlovic et al. say that the distributions of ratios of random variables are of interest in many areas of the sciences and present the joint probability density function (PDF) and PDF of maximum of ratios $\mu(1) = R_1/r(1)$ and $\mu(2) = R_2/r(2)$ for the cases where $R_1, R_2, r(1)$ and $r(2)$ are Rayleigh, Rician, Nakagami- m , and Weibull distributed random variables [4]. Vegas-Sanchez-Ferrero et al. find that the Generalized Gamma (GG) distribution (which also generalizes the Nakagami distribution) accurately characterize the speckle behavior of blood and myocardial tissue in ultrasonic images [5]. Mekic et al. also mention that the distributions of random variables are of interest in many areas of science and derive the probability density function (PDF) and cumulative distribution function (CDF) of ratio of products of two random variables such as Rayleigh, Nakagami- m , Weibull, and alpha-mu random variables [3]. Bouhleb and Sevestre-Ghalila propose a new Markov random field (MRF) model which combines the Nakagami distribution for the backscattered ultrasonic echo in order to get information about backscatter characteristics, such as the scatterer density, amplitude and spacing [1].

With the shape parameter $m > 0.5$ and the spread parameter $\Omega > 0$, the probability density function of the Nakagami distribution is as follows. Notice that the random variable $N > 0$.

$$f(n) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m n^{2m-1} e^{-\frac{m}{\Omega}n^2}.$$

Huang and Johnson [2] provided theorems to construct a confidence interval for ratio of percentiles from two independent distributions. This paper will construct a confidence interval for ratio of medians from two Nakagami populations to compare the speckle behavior in two tissues.

2 Related distributions of the Nakagami distribution

2.1 Distributions related to the Nakagami distribution

The relationship between the Nakagami distribution and related distribution is summarized in Table 1.

Table 1: The relationship between distributions

Variable N 's dist.	Related distribution
Nakagami(m, Ω)	$G = N^2 \sim \text{gamma}(m, \frac{\Omega}{m})$
Nakagami($1, \Omega$)	$R = N \sim \text{Rayleigh}(\sqrt{\frac{\Omega}{2}})$
Nakagami($1, \Omega$)	$W = N \sim \text{weibull}(2, \Omega)$
Nakagami($\frac{p}{2}, p$)	$S = N^2 \sim \text{chi-square}(p)$
Nakagami($1, 2$)	$X = \frac{\lambda N^2}{2} \sim \text{exponential}(\lambda)$

Proof: If $N \sim \text{Nakagami}(m, \Omega)$, let $G = N^2$. Then $n = g^{1/2}$ and $G = N^2 \sim \text{gamma}(m, \frac{\Omega}{m})$.

$$|J| = \left| \frac{dn}{dg} \right| = \left| \frac{1}{2} g^{-\frac{1}{2}} \right|,$$

$$\begin{aligned}
 f(n) &= \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m n^{2m-1} e^{-\frac{m}{\Omega}n^2} \\
 \Rightarrow f(g) &= \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m g^{\frac{2m-1}{2}} e^{-\frac{m}{\Omega}g} \cdot \frac{1}{2}g^{-\frac{1}{2}} \\
 &= \frac{1}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m g^{m-1} e^{-\frac{m}{\Omega}g}.
 \end{aligned}$$

If $N \sim \text{Nakagami}(1, \Omega)$, let $R = N$. Then $R = N \sim \text{Rayleigh}(\sqrt{\frac{\Omega}{2}})$.

$$\begin{aligned}
 f(n) &= 2\left(\frac{1}{\Omega}\right)n e^{-\frac{1}{\Omega}n^2} \\
 \Rightarrow f(r) &= \frac{r}{\sqrt{\frac{\Omega}{2}}} e^{-\frac{r^2}{2\sqrt{\frac{\Omega}{2}}}}.
 \end{aligned}$$

If $N \sim \text{Nakagami}(1, \Omega)$, let $W = N$. Then $W = N \sim \text{weibull}(2, \Omega)$.

$$\begin{aligned}
 f(n) &= 2\left(\frac{1}{\Omega}\right)n e^{-\frac{1}{\Omega}n^2} \\
 \Rightarrow f(w) &= \frac{2}{\Omega}w^{2-1} e^{-\frac{w^2}{\Omega}}.
 \end{aligned}$$

If $N \sim \text{Nakagami}(\frac{p}{2}, p)$, let $S = N^2$. Then $n = s^{1/2}$ and $S = N^2 \sim \text{chi-square}(p)$.

$$|J| = \left|\frac{dn}{ds}\right| = \left|\frac{1}{2}s^{-\frac{1}{2}}\right|,$$

$$\begin{aligned}
 f(n) &= \frac{2}{\Gamma(\frac{p}{2})} \left(\frac{1}{2}\right)^{\frac{p}{2}} n^{p-1} e^{-\frac{1}{2}n^2} \\
 \Rightarrow f(s) &= \frac{2}{\Gamma(\frac{p}{2})} \left(\frac{1}{2}\right)^{\frac{p}{2}} s^{\frac{p-1}{2}} e^{-\frac{1}{2}s} \cdot \frac{1}{2}s^{-\frac{1}{2}} \\
 &= \frac{1}{\Gamma(\frac{p}{2}) \cdot 2^{\frac{p}{2}}} s^{\frac{p}{2}-1} e^{-\frac{s}{2}}.
 \end{aligned}$$

If $N \sim \text{Nakagami}(1,2)$, let $X = \frac{\lambda N^2}{2}$. Then $n = \sqrt{\frac{2x}{\lambda}}$ and $X = \frac{\lambda N^2}{2} \sim \text{exponential}(\lambda)$.

$$|J| = \left|\frac{dn}{dx}\right| = \left|\frac{1}{\sqrt{2\lambda x}}\right|,$$

and

$$\begin{aligned}
 f(n) &= n e^{-\frac{n^2}{2}} \\
 \Rightarrow f(x) &= \sqrt{\frac{2x}{\lambda}} e^{-\frac{2x}{\lambda}} \cdot \frac{1}{\sqrt{2\lambda x}} \\
 &= \frac{1}{\lambda} e^{-\frac{x}{\lambda}}
 \end{aligned}$$

□

2.2 The gamma distribution

The previous subsection shows that the Rayleigh distribution, the weibull distribution, the chi-square distribution and the exponential distribution merely relate to the Nakagami distribution with some particular value of m or Ω . Therefore, let us focus on the gamma distribution from now on.

With the shape parameter α and the scale parameter β , the probability density function of the gamma distribution is as follows.

$$f(g) = \frac{1}{\Gamma(\alpha)\beta^\alpha} g^{\alpha-1} e^{-\frac{g}{\beta}}.$$

The moment generating function:

$$\begin{aligned}
 M_G(t) &= E(e^{tg}) \\
 &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} g^{\alpha-1} e^{-(\frac{1}{\beta}-t)g} dg \\
 &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} g^{\alpha-1} e^{-\frac{(1-\beta t)g}{\beta}} dg \\
 &= \left(\frac{1}{1-\beta t}\right)^\alpha
 \end{aligned}$$

$$\begin{aligned}
 E(G) &= M'_G(0) \\
 &= -\alpha(1-\beta t)^{-\alpha-1}(-\beta)|_{t=0} = \alpha\beta
 \end{aligned}$$

$$\begin{aligned}
 E(G^2) &= M''_G(0) = \alpha\beta^2(\alpha+1) \\
 &= \alpha\beta(-\alpha-1)(1-\beta t)^{-\alpha-2}(-\beta)|_{t=0}
 \end{aligned}$$

$$\begin{aligned}
 Var(G) &= E(G^2) - E(G)^2 \\
 &= \alpha\beta^2(\alpha+1-\alpha) = \alpha\beta^2
 \end{aligned}$$

$$\alpha = \frac{E(G)^2}{Var(G)}, \quad \beta = \frac{Var(G)}{E(G)}.$$

3 The Nakagami distribution

3.1 When m is less than or equal to 0.5

When m is 0.5, the probability density function of a Nakagami random variable becomes

$$\begin{aligned}
 f(n) &= \frac{2}{\Gamma(0.5)} \left(\frac{0.5}{\Omega}\right)^{0.5} n^0 e^{-\frac{0.5}{\Omega}n^2} \\
 &= \frac{2}{\sqrt{2\pi\Omega}} e^{-\frac{n^2}{2\Omega}}.
 \end{aligned}$$

This distribution is constructed by folding a normal distribution with $\mu = 0$ and $\sigma^2 = \Omega$ to the positive support, $0 < n < \infty$. Therefore, it is no longer Nakagami distributed.

However, the estimated value of $\alpha = m$ from previous section 2.2 is not guaranteed to be greater than

0.5. This problem can only be solved by using the gamma instead of the Nakagami distribution.

3.2 The second moment

From the previous subsection, it can be shown that

$$m = \frac{E(N^2)^2}{Var(N^2)}$$

$$\alpha\beta = m\left(\frac{\Omega}{m}\right) = E(G) \Rightarrow \Omega = E(N^2).$$

If a random sample of $n_1, n_2, \dots, n_{1000}$ is selected from a Nakagami distribution with the shape parameter m and the spread parameter Ω , the spread parameter can be estimated by the following two ways

$$\hat{\Omega}_1 = \bar{g} = \frac{1}{1000} \sum_{i=1}^{1000} n_i^2 \tag{1}$$

and

$$\hat{\Omega}_2 = \sum_{i=1}^{1000} \frac{(n_i - \bar{n})^2}{999} + \bar{n}^2. \tag{2}$$

The estimated spread parameters are listed in Table 2. The value of m does not affect the estimation of Ω .

Table 2: The estimated spread parameters

m	Ω	$\hat{\Omega}_1$	$\hat{\Omega}_2$
1	1	0.998862	0.9990763
1	291848	291515.9	291578.4
0.75	1	0.9986822	0.9989524
0.75	291848	291463.4	291542.3
0.5	1	0.9984054	0.9987683
0.5	291848	291382.6	291488.5
0.25	1	0.9979975	0.9985398
0.25	291848	291263.6	291421.8

The estimated shape parameters are listed in Table 3. The value of Ω does not affect the estimation of m .

3.3 The first moment

The first moment can be found by the direct integration and the relationship with the gamma distribution.

$$E(N) = \int_0^\infty \frac{2n}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m n^{2m-1} e^{-\frac{mn^2}{\Omega}} dn.$$

Table 3: The estimated shape parameters

m	Ω	\hat{m}
1	1	1.003686
1	291848	1.003686
0.75	1	0.7531207
0.75	291848	0.7531207
0.5	1	0.5006322
0.5	291848	0.5006322
0.25	1	0.2493589
0.25	291848	0.2493589

Let $n^2 = g$, then $2ndn = dg$.

$$\begin{aligned} E(N) &= \int_0^\infty \frac{1}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m g^{(m+1/2)-1} e^{-\frac{m}{\Omega}g} dg \\ &= \frac{1}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m \cdot \Gamma\left(m + \frac{1}{2}\right) \cdot \frac{\Omega^{m+\frac{1}{2}}}{m^{m+\frac{1}{2}}} \\ &= \frac{\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(m)} \sqrt{\frac{\Omega}{m}}, \end{aligned}$$

when $m = 1$, $E(N) = \frac{\sqrt{\pi\Omega}}{2}$.

The theoretical and estimated means of the Nakagami random variable are summarized in Table 4.

Table 4: The mean of the Nakagami random variable

m	Ω	$E(N)$	Esti. of $E(N)$
1	1	0.8862269	0.897785
1	291848	478.766	485.01
0.75	1	0.8540959	0.866905
0.75	291848	461.4079	468.3277
0.5	1	0.7978846	0.8122323
0.5	291848	431.0408	438.7919
0.25	1	0.6759782	0.6919073
0.25	291848	365.1834	373.7888

$$\begin{aligned} Var(N) &= E(N^2) - E(N)^2 \\ &= \Omega - \left\{ \frac{\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(m)} \sqrt{\frac{\Omega}{m}} \right\}^2 \\ &= \Omega \left\{ 1 - \frac{1}{m} \left(\frac{\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(m)}\right)^2 \right\} \end{aligned}$$

The theoretical and estimated variances of the

Nakagami random variable are summarized in Table 5.

Table 5: The variance of the Nakagami random variable

m	Ω	$Var(N)$	Esti. of $Var(N)$
1	1	0.2146018	0.2143364
1	291848	62631.12	62553.65
0.75	1	0.2705201	0.2701666
0.75	291848	78950.76	78847.58
0.5	1	0.3633802	0.3628748
0.5	291848	106051.8	105904.3
0.25	1	0.5430534	0.5422986
0.25	291848	158489.1	158268.8

It is not easy to obtain the moments from the moment generating function. Let us consider a special case. When $m = 1$, the probability density function is

$$f(n) = \frac{2}{\Omega} n e^{-\frac{n^2}{\Omega}}.$$

And the moment generating function becomes

$$\begin{aligned} M_N(t) &= E(e^{tn}) = \int_0^\infty \frac{2}{\Omega} n e^{-\frac{n^2}{\Omega}} e^{tn} dn \\ &= \int_0^\infty \frac{2}{\Omega} n e^{-\frac{(n-\frac{t\Omega}{2})^2}{\Omega}} e^{\frac{t^2\Omega}{4}} dn \\ &= \int_{\frac{t\Omega}{2}}^\infty (n - \frac{t\Omega}{2}) e^{-\frac{(n-\frac{t\Omega}{2})^2}{\Omega}} e^{\frac{t^2\Omega}{4}} d(n - \frac{t\Omega}{2}) \\ &\quad + \int_{\frac{t\Omega}{2}}^\infty t e^{-\frac{(n-\frac{t\Omega}{2})^2}{\Omega}} e^{\frac{t^2\Omega}{4}} d(n - \frac{t\Omega}{2}) \\ &= -e^{-\frac{(n-\frac{t\Omega}{2})^2}{\Omega}} e^{\frac{t^2\Omega}{4}} \Big|_{\frac{t\Omega}{2}}^\infty \\ &\quad + \left\{ \int_{\frac{t\Omega}{2}}^\infty \frac{1}{\sqrt{\pi\Omega}} e^{-\frac{(n-\frac{t\Omega}{2})^2}{\Omega}} dn \right\} \sqrt{\pi\Omega} t e^{\frac{t^2\Omega}{4}} \\ &= e^{\frac{t^2\Omega}{4}} + \frac{\sqrt{\pi\Omega} t}{2} e^{\frac{t^2\Omega}{4}}, \\ E(N) &= M'_N(0) = \frac{\sqrt{\pi\Omega}}{2} \\ &= \left\{ \frac{t\Omega}{2} e^{\frac{t^2\Omega}{4}} + \frac{\sqrt{\pi\Omega}}{2} e^{\frac{t^2\Omega}{4}} + \frac{\sqrt{\pi\Omega}\Omega t^2}{4} e^{\frac{t^2\Omega}{4}} \right\} \Big|_{t=0}. \end{aligned}$$

3.4 The Median

The median of a Nakagami random variable is $\sqrt{\Omega}$. In order to compare population medians, the confidence intervals for the ratio of two medians from independent populations are constructed by way of two

rewritten theorems based on nonparametric methods in Huang and Johnson [2]. m and n in the theorems are just sample sizes.

Theorem 1 Let X_1, X_2, \dots, X_m be a random sample from $F_1(\cdot)$; let Y_1, Y_2, \dots, Y_n from $F_2(\cdot)$; and let samples be independent. If the population density function, $F'_i(\cdot)$, is positive and continuous in a neighborhood of the median m_i , for $i = 1, 2$, $\lim_{m,n \rightarrow \infty} m(m+n) = \lambda$, ($0 < \lambda < 1$), and $m_2 \neq 0$, then the $100(1 - \alpha)\%$ confidence interval of $\theta = m_1/m_2$ is

$$\frac{\hat{m}_1}{\hat{m}_2} \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{0.25}{m\hat{m}_2^2[\hat{F}'_1(\hat{m}_1)]^2} + \frac{0.25\hat{m}_1^2}{n\hat{m}_2^4[\hat{F}'_2(\hat{m}_2)]^2}}$$

Theorem 2 Let X_1, X_2, \dots, X_m be a random sample from $F_1(\cdot)$ which has a positive continuous derivative $F'_1(\cdot)$ in a neighborhood of m_1 , let Y_1, Y_2, \dots, Y_n be a random sample from $F_2(\cdot)$ which has a positive continuous derivative $F'_2(\cdot)$ in a neighborhood of m_2 , and let two samples be independent. Let the ratio of medians $\theta = m_1/m_2$ be unknown but finite. Under the condition that $\lim_{m,n \rightarrow \infty} m(m+n) = \lambda$, ($0 < \lambda < 1$), θ values of the intersections of $\pm Z_{\alpha/2}$ and

$$Z^{M,NP}(\theta) = \frac{\hat{m}_1 - \theta\hat{m}_2}{\sqrt{\frac{0.25}{m[\hat{F}'_1(\hat{m}_1)]^2} + \frac{0.25\theta^2}{n[\hat{F}'_2(\hat{m}_2)]^2}}}$$

form the $100(1 - \alpha)\%$ confidence interval.

Medians are estimated by order statistics, and $\hat{F}'_i(\hat{m}_i)$ is estimated by the kernel estimator for $i = 1, 2$. One thousand replications of the random sample $X_{i,1}, X_{i,2}, \dots, X_{i,1000}$ are selected independently from a Nakagami distribution with the shape parameter 1 and the spread parameter 4. One thousand replications of the random sample $Y_{i,1}, Y_{i,2}, \dots, Y_{i,1000}$ are selected independently from a Nakagami distribution with the shape parameter 1 and the spread parameter 9. The true ratio of medians from two populations should be $\sqrt{4/9} = 2/3$. The simulated 95% confidence intervals are listed in Table 6.

Table 6: The simulated 95% confidence interval of the ratio of medians

Theorem	Average 95% CI	True coverage
1	(0.62361,0.70915)	93.5%
2	(0.62492,0.71061)	93.9%

4 Real Data

There are several ultrasound images of breast tumors. Each pixel has an observed Nakagami random variable value on it. The random sample of n_1, n_2, \dots, n_{25} from a 5×5 pixel area can create an estimation of m .

Benign cases 01, 02, 03, 04, 05, 06 and malignant cases 07, 08, 09, 10, 11, 12 are much more typical than other cases, so they are selected to be analyzed. Because the normal area is larger than the tumor area in an ultrasound graph, the areas above and below the tumor are truncated to eliminate number of observations.

In Table 7, the median values of Nakagami m estimators are from 0.409 to 0.530 for tumor area of benign cases and from 0.692 to 0.976 for remained normal area of benign cases. The median values of Nakagami m estimators are from 0.474 to 0.644 for tumor area of malignant cases, and 0.494 to 0.698 for remained normal area of malignant cases.

Table 7: Median values of Nakagami m estimators

	Overall	Normal	Tumor	R. N.	R. O.
01	0.6520	0.6829	0.4159	0.7834	0.6511
02	0.6534	0.6718	0.4089	0.9231	0.8458
03	0.7232	0.7559	0.4881	0.7033	0.6335
04	0.6545	0.6719	0.5299	0.9756	0.9103
05	0.6222	0.6553	0.4198	0.6924	0.5919
06	0.5775	0.5850	0.5168	0.8301	0.7732
07	0.6086	0.6739	0.4913	0.6983	0.5649
08	0.6668	0.7095	0.4739	0.6830	0.5922
09	0.6100	0.6412	0.5249	0.6762	0.6109
10	0.6271	0.6719	0.5119	0.6832	0.5929
11	0.6494	0.6523	0.6439	0.6428	0.6435
12	0.5521	0.5690	0.4774	0.4942	0.4880

To compare Nakagami m estimators between tumor area and remained normal area, 95% confidence intervals of the ratio of median Nakagami m estimator for tumor area to median Nakagami m estimator for remained normal area are illustrated in Table 8.

We can conclude that the Nakagami m estimator of benign tumor area is 42.8% to 71.4% of the Nakagami m estimator of remained normal area, and the Nakagami m estimator of malignant tumor area is 68.2% to 101.7% of the Nakagami m estimator of normal area. Malignant remained normal areas look also dark in the ultrasound graph.

Table 8: 95% confidence interval for ratio of tumor area median m over normal area median m

	Ratio of Medians	Theorem 1	Theorem 2
01	0.416/0.783=0.531	(0.514, 0.547)	(0.514, 0.548)
02	0.409/0.923=0.443	(0.428, 0.458)	(0.428, 0.458)
03	0.488/0.703=0.694	(0.674, 0.714)	(0.674, 0.714)
04	0.530/0.976=0.543	(0.531, 0.556)	(0.531, 0.556)
05	0.420/0.692=0.606	(0.585, 0.628)	(0.585, 0.628)
06	0.517/0.830=0.623	(0.604, 0.641)	(0.604, 0.641)
07	0.491/0.698=0.704	(0.687, 0.720)	(0.687, 0.720)
08	0.474/0.683=0.694	(0.682, 0.706)	(0.682, 0.706)
09	0.525/0.676=0.776	(0.762, 0.792)	(0.762, 0.790)
10	0.512/0.683=0.749	(0.736, 0.762)	(0.736, 0.763)
11	0.644/0.643=1.002	(0.987, 1.017)	(0.987, 1.017)
12	0.477/0.494=0.966	(0.942, 0.990)	(0.942, 0.990)

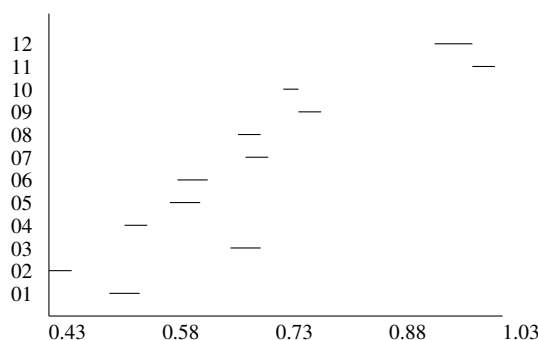


Figure 1: 95% Theorem 1 confidence interval for ratio of tumor area median m over normal area median m

To compare Nakagami m estimators between malignant tumor and benign tumor, 95% confidence intervals of the ratio of median Nakagami m estimator for malignant tumor to median Nakagami m estimator for benign tumor are illustrated in Table 9 and Table 10.

We can conclude that the Nakagami m estimator of malignant tumor is 87.2% to 159.6% of the Nakagami m estimator of benign tumor. There are $6 \times 6 = 36$ confidence intervals for all combinations of malignant and benign cases.

5 Conclusion

In order to make the random variable N follow a Nakagami distribution, the shape parameter m must be greater than 0.5. However, m could be less than 0.5 in the real application. This problem can be solved by simulating the gamma distribution, where m is only need to be positive.

The gamma distribution can be used to derive and estimate the spread parameter of the Nakagami distribution, and to derive and estimate the moments of the Nakagami random variable. The median of the Nakagami random variable is just the square root of the

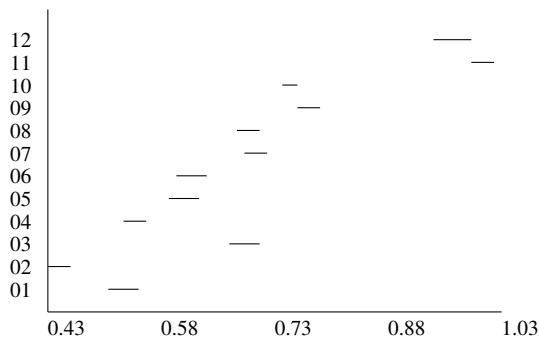


Figure 2: 95% Theorem 2 confidence interval for ratio of tumor area median m over normal area median m

Table 9: 95% Theorem 1 confidence interval for ratio of malignant median m over benign median m

	07	08	09	10	11	12
01	(1.145 1.220)	(1.106 1.176)	(1.225 1.303)	(1.195 1.270)	(1.505 1.596)	(1.110 1.190)
02	(1.162 1.244)	(1.121 1.199)	(1.242 1.328)	(1.212 1.295)	(1.525 1.627)	(1.125 1.212)
03	(0.977 1.038)	(0.942 0.999)	(1.045 1.108)	(1.019 1.080)	(1.283 1.357)	(0.945 1.011)
04	(0.903 0.951)	(0.872 0.917)	(0.966 1.015)	(0.942 0.990)	(1.187 1.244)	(0.874 0.928)
05	(1.132 1.211)	(1.093 1.167)	(1.210 1.293)	(1.181 1.260)	(1.486 1.584)	(1.096 1.180)
06	(0.920 0.981)	(0.888 0.946)	(0.985 1.049)	(0.960 1.021)	(1.209 1.285)	(0.891 0.956)

spread parameter. The confidence interval for the ratio of medians from two independent Nakagami distributions can be constructed. R programs of the Nakagami variable simulation and parameters estimation and the confidence interval construction are in the appendix.

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Appendix

A.1 R program of the Nakagami variable simulation and parameters estimation

```
DistTest=function(Simulated,NM,
BigOmega,WriteTo)
{
  data=read.table(Simulated)
  data.mat=as.matrix(data)
  raw=matrix(0,nrow(data),
ncol(data))
  Be=NM/BigOmega
  for (i in 1:nrow(data))
  {
    for (j in 1:ncol(data))
    {
```

Table 10: 95% Theorem 2 confidence interval for ratio of malignant median m over benign median m

	07	08	09	10	11	12
01	(1.145 1.220)	(1.106 1.176)	(1.225 1.302)	(1.195 1.271)	(1.504 1.595)	(1.110 1.189)
02	(1.161 1.244)	(1.121 1.199)	(1.242 1.328)	(1.212 1.295)	(1.524 1.626)	(1.125 1.212)
03	(0.977 1.037)	(0.943 1.000)	(1.044 1.107)	(1.019 1.080)	(1.282 1.357)	(0.946 1.011)
04	(0.904 0.952)	(0.872 0.917)	(0.966 1.016)	(0.943 0.990)	(1.187 1.244)	(0.874 0.928)
05	(1.131 1.211)	(1.093 1.167)	(1.209 1.292)	(1.181 1.259)	(1.486 1.585)	(1.096 1.179)
06	(0.921 0.982)	(0.889 0.947)	(0.984 1.048)	(0.961 1.022)	(1.209 1.285)	(0.892 0.957)

```
raw[i,j]=qgamma(data.mat[i,j],
  NM, Be)
}
}
nakagami=matrix(0,nrow(data),
ncol(data))
for (i in 1:nrow(data))
{
  for (j in 1:ncol(data))
  {
    nakagami[i,j]=sqrt(raw[i,j])
  }
}
write.table(nakagami,WriteTo,
col.names=FALSE, row.names=FALSE,
sep=" ")
mEsti=matrix(0,nrow(data),4)
for (i in 1:nrow(data))
{
  mEsti[i,1]=mean(raw[i,])^2/
var(raw[i,])
  mEsti[i,2]=mean(raw[i,])
  mEsti[i,3]=(var(nakagami[i,])+
mean(nakagami[i,])^2)
  mEsti[i,4]=var(nakagami[i,])
}
out4=mean(mEsti[,1])
out5=mean(mEsti[,2])
out6=mean(mEsti[,3])
out7=sqrt(var(mEsti[,1]))
out9=mean(nakagami[,1])
out10=(NM>=1&NM<=1)*(sqrt(pi*
BigOmega)/2
)+ (NM>1|NM<1)*(gamma(NM+0.5)*
sqrt(BigOmega)/(gamma(NM)*s
qrt(NM))
)
out11=mean(mEsti[,4])
```

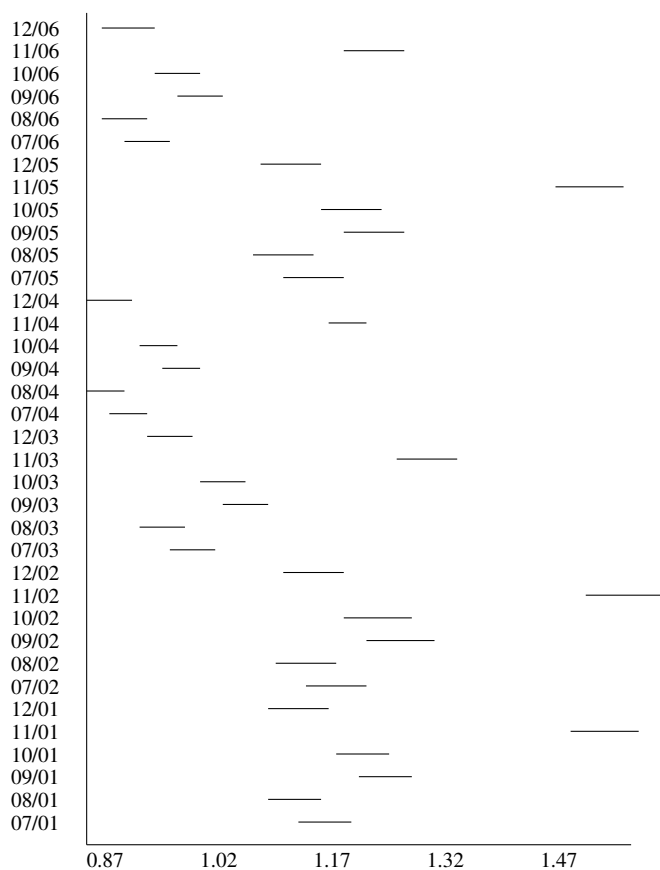


Figure 3: 95% Theorem 1 confidence interval for ratio of malignant median m over benign median m

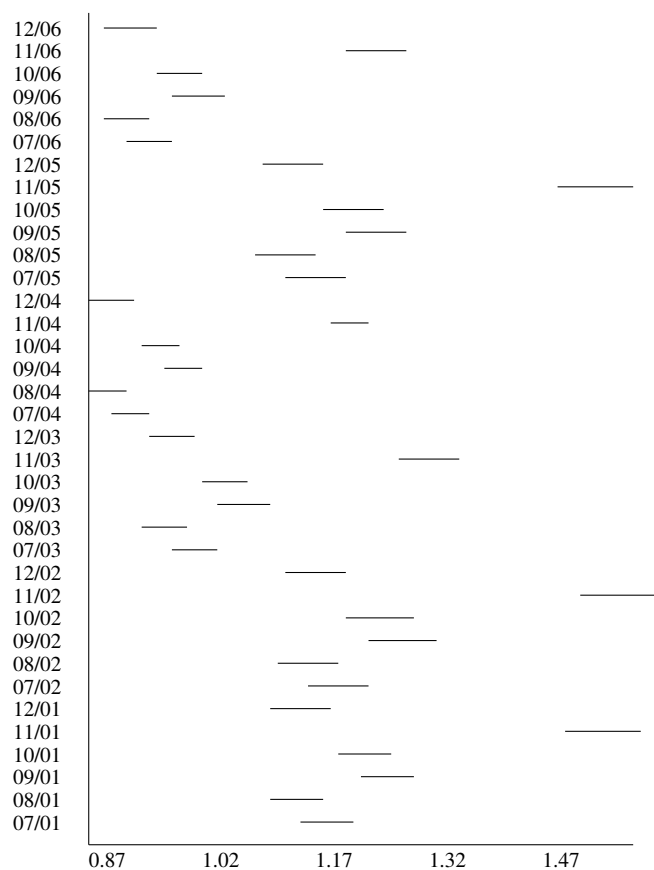


Figure 4: 95% Theorem 2 confidence interval for ratio of malignant median m over benign median m

```

out12=(NM>=1&NM<=1)*((1-pi/4)*
BigOmega
)+(NM>1|NM<1)*(gamma(NM+1)*
BigOmega/(gamma(NM)*NM)
-(gamma(NM+0.5)*sqrt(BigOmega)/(
gamma(NM)*sqrt(NM)
)^2)
list(meanMME=out4,
meanBigOmegaEsti1=out5,
meanBigOmegaEsti2=out6,
sdMME=out7,
meanNakagamiEsti=out9,
meanNakagamiThm=out10,
varNakagamiEsti=out11,
varNakagamiThm=out12
)

```

A.2 R program of the confidence interval construction

```

RoMCI=function(numer, denom
###, Thm1, Thm2
)
{
mal=read.table(numer, header
=FALSE, sep=" ")
ben=read.table(denom, header

```

```

=FALSE, sep=" ")
trueRatio=2/3
malsize=ncol(mal)
bensize=ncol(ben)
npairs=nrow(mal)
malmat=as.matrix(mal)
benmat=as.matrix(ben)

malker=matrix(0,npairs,
1+malsize)
benker=matrix(0,npairs,
1+bensize)
Thm32=matrix(0,npairs,4)
Thm42=matrix(0,npairs,4)
ztheta=rep(0,11)

```

```

for(i in 1:npairs)
{
malker[i,1+malsize]=median(
malmat[i,])
benker[i,1+bensize]=median(
benmat[i,])
sortmal=sort(malmat[i,])
sortben=sort(benmat[i,])
malup=sortmal[ceiling(malsize/

```

```

2)+1:malsize]
mallow=sortmal[1:floor(malsize/
2)]
benup=sortben[ceiling(bensize/
2)+1:bensize]
benlow=sortben[1:floor(bensize/
2)]
a1=c(sqrt(apply(mal[i,],1,var)),
(median(malup[1:floor(malsize/
2)])-median(mallow))/1.34)
A1=min(a1)
a2=c(sqrt(apply(ben[i,],1,var)),
(median(benup[1:floor(bensize/
2)])-median(benlow))/1.34)
A2=min(a2)
h1=0.9*A1*malsize^(-0.2)
h2=0.9*A2*bensize^(-0.2)
for (j in 1:malsize)
{
malker[i,j]=exp(-(malker[i,
1+malsize]-malmat[i,j])/h1)^2/
2)/(sqrt(2*pi)*malsize*h1)
}
for (k in 1:bensize)
{
benker[i,k]=exp(-(benker[i,
1+bensize]-benmat[i,k])/h2)^2/
2)/(sqrt(2*pi)*bensize*h2)
}
Thm32[i,2]=malker[i,1+malsize]/
benker[i,1+bensize]
Thm32[i,1]=Thm32[i,2]-1.96*sqrt(
0.25/(malsize*benker[i,1+
bensize]^2
*sum(malker[i,1:malsize])^2)+
0.25*malker[i,1+malsize]^2/
(bensize*benker[i,1+bensize]^4
*sum(benker[i,1:bensize])^2))
Thm32[i,3]=Thm32[i,2]+(Thm32[
i,2]-Thm32[i,1])
Thm32[i,4]= Thm32[i,1]<
trueRatio & Thm32[i,3]>trueRatio

Thm42[i,2]=Thm32[i,2]

for (l in 1:11)
{
theta=0.1*(1-1)
ztheta[l]=(malker[i,1+malsize]-
theta*benker[i,1+bensize])/
sqrt(0.25/(malsize*sum(malker[i,
1:malsize])^2)+0.25
*theta^2/(bensize*sum(benker[i,
1:bensize])^2))
}

```

```

ltheta=ztheta[ztheta>1.96]
utheta=ztheta[ztheta>-1.96]
lback1=length(ltheta)-1
uback1=length(utheta)-1

for (l in 1:11)
{
theta=0.1*lback1+0.01*(1-1)
ztheta[l]=(malker[i,1+malsize]-
theta*benker[i,1+bensize])/
sqrt(0.25/(malsize*sum(malker[i,
1:malsize])^2)+0.25
*theta^2/(bensize*sum(benker[i,
1:bensize])^2))
}
ltheta=ztheta[ztheta>1.96]
lback2=length(ltheta)-1

for (l in 1:11)
{
theta=0.1*lback1+0.01*lback2+
0.001*(1-1)
ztheta[l]=(malker[i,1+malsize]-
theta*benker[i,1+bensize])/
sqrt(0.25/(malsize*sum(malker[i,
1:malsize])^2)+0.25
*theta^2/(bensize*sum(benker[i,
1:bensize])^2))
}
ltheta=ztheta[ztheta>1.96]
judge= ztheta[length(ltheta)]-
1.96<1.96-ztheta[length(ltheta)+
1]
lback3=length(ltheta)-judge

Thm42[i,1]=0.1*lback1+0.01*
lback2+0.001*lback3

for (l in 1:11)
{
theta=0.1*uback1+0.01*(1-1)
ztheta[l]=(malker[i,1+malsize]-
theta*benker[i,1+bensize])/
sqrt(0.25/(malsize*sum(malker[i,
1:malsize])^2)+0.25
*theta^2/(bensize*sum(benker[i,
1:bensize])^2))
}
utheta=ztheta[ztheta>-1.96]
uback2=length(utheta)-1

for (l in 1:11)
{
theta=0.1*uback1+0.01*uback2+
0.001*(1-1)

```



```

ztheta[1]=(malker[i,1+malsize]-
theta*benker[i,1+bensize])/
sqrt(0.25/(malsize*sum(malker[i,
1:malsize])^2)+0.25
*theta^2/(bensize*sum(benker[i,
1:bensize])^2))
}
utheta=ztheta[ztheta>-1.96]
judge= ztheta[length(utheta)]+
1.96< -1.96-ztheta[length(utheta)+
1]
uback3=length(utheta)-judge

```

```

Thm42[i,3]=0.1*uback1+0.01*uback2+
0.001*uback3

```

```

Thm42[i,4]= Thm42[i,1]<
trueRatio & Thm42[i,3]>trueRatio
}
write.table(Thm32,"Thm32_malben.txt"
,col.names=FALSE, row.names=FALSE,
sep=" ")
write.table(Thm42,"Thm42_malben.txt"
,col.names=FALSE, row.names=FALSE,
sep=" ")
}

```

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