Almost Periodic Solution for a Predator-Prey System with Impulses on Time Scales

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Abstract: This paper is concerned with a predator-prey system with impulses on time scales. Based on the theory of calculus on time scales and the properties of almost periodic functions as well as Razumikhin type theorem, sufficient conditions which guarantee the existence of a unique uniformly asymptotic stable almost periodic solution of the system are obtained, by the relation between the solutions of impulsive system and the corresponding non-impulsive system. Finally, an example and numerical simulations are presented to illustrate the feasibility and effectiveness of the results.

Key–Words: Permanence; Almost periodic solution; Uniformly asymptotic stable; Impulse; Time scale.

1 Introduction

It is well known that, in the real word, lots of dynamic systems have variable structures subject to some abrupt changes. Differential equations with impulses provide an adequate mathematical model of many evolutionary processes that suddenly change their states at certain moments. The theory of impulsive differential equations has become an important aspect of differential equations.

In the past years, predator-prey system with impulse received more researchers' special attention; see, for example [1]-[5] and the references therein. However, in the natural world, there are many species whose developing processes are both continuous and discrete. Hence, using the only differential equation or difference equation cannot accurately describe the law of their developments; see, for example, [6, 7]. Therefore, there is a need to establish correspondent dynamic models on new time scales.

Recently, different types of ecosystems with periodic coefficients on time scales have been studied extensively; see, for example, [8]-[13] and the references therein. However, upon considering long-term dynamical behaviors, the periodic parameters often turn out to experience certain perturbations, that is, parameters become periodic up to a small error. Thus one has to consider the ecosystems to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Therefore, if we consider the effects of the environmental factors (e.g. seasonal effects of weather, food supplies, mating habits, and harvesting), the assumption of almost periodicity is more realistic, more important and more general. To the best of the authors' knowledge, there are few papers on the existence of almost periodic solution of ecosystems on time scales.

Motivated by the above, in the present paper, we shall study an almost periodic predator-prey system with impulses on time scales as follows:

$$\begin{cases} u^{\Delta}(t) &= u(t)[r(t)-a_1(t)u(t) \\ &\quad -b_1(t)u^{\sigma}(t)-c_1(t)v(t)], \\ v^{\Delta}(t) &= -\eta(t)v(t)+g_1(t)u(t),\ t\neq t_k, \\ u(t_k^+) &= (1+h_{1k})u(t_k), \\ v(t_k^+) &= (1+h_{2k})v(t_k),\ k=1,2,\cdots, \end{cases} \tag{1}$$

where $t\in\mathbb{T}$, $\mathbb{T}$ is an almost time scale. All the coefficients $r(t), a_1(t), b_1(t), c_1(t), \eta(t), g_1(t)$ are continuous, almost periodic functions. $u(t_k^+), u(t_k^-)$ represent the right and left limit of $u(t)$ in the sense of time scales, respectively, and $v(t_k^-)=v(t_k), u(t_k^-)=u(t_k)$, for all $\{t_k\}$, where $\{t_k\}$ is a sequence of real number such that $0<t_1<t_2<\cdots<t_k\to+\infty$ as $k\to+\infty$.

The initial condition of system (1) in the form

$$u(t_0)=u_0, v(t_0)=v_0, t_0\in\mathbb{T}, u_0>0, v_0>0. \tag{2}$$

For convenience, we introduce the notation

$$f^u=\sup_{t\in\mathbb{T}} f(t),\ f^l=\inf_{t\in\mathbb{T}} f(t),$$

where f is a positive and bounded function. Throughout this paper, we assume that the coefficients of the almost periodic system (1) satisfy

$$\min\{r^l, a_1^l, b_1^l, c_1^l, \eta^l, g_1^l\} > 0,$$
$$\max\{r^u, a_1^u, b_1^u, c_1^u, \eta^u, g_1^u\} < +\infty.$$

and there exist positive constants h_i^l, h_i^u such that

$$h_i^l \le \Pi_{t_0<t_k<t}(1 + h_{ik}) \le h_i^u$$

with $1 + h_{ik} \ge 0$, for $t \ge t_0, i = 1, 2$.

The aim of this paper is, based on the theory of calculus on time scales and the properties of almost periodic functions as well as Razumikhin type theorem, by using the relation between the solutions of impulsive system and the corresponding non-impulsive system, to obtain sufficient conditions for the existence of a unique uniformly asymptotic stable almost periodic solution of the system (1).

The relevant definitions and the properties of almost periodic functions, see [14, 15]. In this paper, for each interval $\mathbb{I}$ of $\mathbb{T}$, we denote by $\mathbb{I}_{\mathbb{T}} = \mathbb{I} \cap \mathbb{T}$.

2 Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$
$$\mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

The basic theories of calculus on time scales, one can see [16].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall\, t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p \oplus q = p + q + \mu p q,$$
$$\ominus p = -\frac{p}{1+\mu p},$$
$$p \ominus q = p \oplus (\ominus q).$$

Lemma 1. *[16] If $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*

(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$;
(vi) $(e_p(t, s))^{\Delta} = p(t)e_p(t, s)$.

Lemma 2. *[17] Assume that $a > 0, b > 0$ and $-a \in \mathcal{R}^+$. Then*

$$y^{\Delta}(t) \ge (\le) b - ay(t),\ y(t) > 0,\ t \in [t_0, \infty)_{\mathbb{T}}$$

implies

$$y(t) \ge (\le) \frac{b}{a}[1+(\frac{ay(t_0)}{b}-1)e_{(-a)}(t, t_0)],\ t \in [t_0, \infty)_{\mathbb{T}}.$$

Lemma 3. *[17] Assume that $a > 0, b > 0$. Then*

$$y^{\Delta}(t) \le (\ge) y(t)(b - ay(\sigma(t))),\ y(t) > 0,\ t \in [t_0, \infty)_{\mathbb{T}}$$

implies

$$y(t) \le (\ge) \frac{b}{a}[1+(\frac{b}{ay(t_0)}-1)e_{\ominus b}(t, t_0)],\ t \in [t_0, \infty)_{\mathbb{T}}.$$

Let $\mathbb{T}$ be a time scale with at least two positive points, one of them being always one: $1 \in \mathbb{T}$, there exists at least one point $t \in \mathbb{T}$ such that $0 < t \neq 1$. Define the natural logarithm function on the time scale $\mathbb{T}$ by

$$L_{\mathbb{T}}(t) = \int_1^t \frac{1}{\tau}\Delta\tau,\ t \in \mathbb{T} \cap (0, +\infty).$$

Lemma 4. *[18] Assume that $x : \mathbb{T} \to \mathbb{R}^+$ is strictly increasing and $\widetilde{\mathbb{T}} := x(\mathbb{T})$ is a time scale. If $x^{\Delta}(t)$ exists for $t \in \mathbb{T}^k$, then*

$$\frac{\Delta}{\Delta t} L_{\mathbb{T}}(x(t)) = \frac{x^{\Delta}(t)}{x(t)}.$$

Lemma 5. *[16] Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$, then $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) \\ &= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).\end{aligned}$$

Let $C = C([-\tau, 0]_{\mathbb{T}}, \mathbb{R}^n), H^* \in \mathbb{R}^+$. Denote

$$C_{H^*} = \{\varphi, \varphi \in C, \|\varphi\| < H^*\},$$

$$S_{H^*} = \{x, x \in \mathbb{R}^n, \|x\| < H^*\}$$

and $\|\varphi\| = \sup_{\theta \in [-\tau, 0]_{\mathbb{T}}} |\varphi(\theta)|$.

Consider the system

$$x^{\Delta} = f(t, x), \tag{3}$$

where $f(t, \phi)$ is continuous in $(t, \phi) \in \mathbb{R} \times C$ and almost periodic in t uniformly for $\phi \in C_{H^*} \subset C$. $\forall \alpha > 0, \exists L(\alpha) > 0$ such that $|f(t, \phi)| \le L(\alpha)$, as $t \in \mathbb{T}, \phi \in C_{\alpha}$.

In order to investigate the almost periodic solution of system (3), we introduce the associate product system of system (3)

$$x^{\Delta} = f(t, x), \quad y^{\Delta} = f(t, y). \tag{4}$$

Lemma 6. *[19] Assume that there exists a Lyapunov function $V(t, x, y)$ defined on $[0, +\infty)_{\mathbb{T}} \times S_{H^*} \times S_{H^*}$, which satisfies the following conditions:*

(1) $\alpha(|x-y|) \le V(t, x, y) \le \beta(|x-y|)$, where $\alpha(s)$ and $\beta(s)$ are continuous, increasing and positive definite;

(2) $|V(t, x_1, y_1) - V(t, x_2, y_2)| \le \omega(|x_1 - x_2| + |y_1 - y_2|)$, where $\omega > 0$ is a constant;

(3) $V^{\Delta}_{(4)}(t, x, y) \le -\lambda V(t, x, y)$, where $\lambda > 0$ is a constant.

Moreover, assumes that (3) has a solution $\xi(t)$ such that $\|\xi\| \le H < H^*$ for $t \in [t_0, +\infty)_{\mathbb{T}}$. Then system (3) has a unique almost periodic solution which is uniformly asymptotic stable.

Let $\mathbb{D} = \{\{t_k\} \in \mathbb{T} : t_k < t_{k+1},\ k \in \mathbb{Z}, \lim_{k \to \pm\infty} t_k = \pm\infty\}$, we denote the set of all sequences that are unbounded and strictly increasing.

Definition 7. *[20] The set of sequences $\{t_k^j\}, t_k^j = t_{k+j} - t_k, k, j \in \mathbb{Z}, \{t_k\} \in \mathbb{D}$ is said to be uniformly almost periodic if for arbitrary $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods common for any sequences.*

Definition 8. *[20] The piecewise continuous function $\varphi : \mathbb{T} \to \mathbb{R}$ with discontinuity of first kind at the point t_k is said to be almost periodic, if the following hold:*

(i) *The The set of sequences $\{t_k^j\}, t_k^j = t_{k+j} - t_k, k, j \in \mathbb{Z}, \{t_k\} \in \mathbb{D}$ is uniformly almost periodic.*

(ii) *For any $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $\varphi(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|\varphi(t') - \varphi(t'')| < \varepsilon$.*

(iii) *For any $\varepsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, then $|\varphi(t + \tau) - \varphi(t)| < \varepsilon$ for all $t \in \mathbb{T}$ satisfying the condition $|t - t_k| > \varepsilon$, $k \in \mathbb{Z}$.*

Consider the following system

$$\begin{cases} x^{\Delta}(t) &= x(t)[r(t) - a(t)x(t) \\ &\quad -b(t)x^{\sigma}(t) - c(t)y(t)], \\ y^{\Delta}(t) &= -\eta(t)y(t) + g(t)x(t), \end{cases} \tag{5}$$

where

$$a(t) = a_1(t) \prod_{t_0 < t_k < t} (1 + h_{1k}),$$
$$b(t) = b_1(t) \prod_{t_0 < t_k < t} (1 + h_{1k}),$$
$$c(t) = c_1(t) \prod_{t_0 < t_k < t} (1 + h_{2k}),$$
$$g(t) = g_1(t) \prod_{t_0 < t_k < t} (1 + h_{1k})(1 + h_{2k})^{-1}.$$

The initial condition of system (5) in the form

$$x(t_0) = x_0, y(t_0) = y_0, t_0 \in \mathbb{T}, x_0 > 0, y_0 > 0. \tag{6}$$

Lemma 9. *From systems (1) and (5), we have*

(i) if $(x(t), y(t))$ is a solution of system (5) then

$$(u(t), v(t)) = \left(\prod_{t_0 < t_k < t} (1 + h_{1k})x(t), \prod_{t_0 < t_k < t} (1 + h_{2k})y(t) \right)$$

is a solution of system (1);

(ii) if $(u(t), v(t))$ is a solution of system (1) then

$$(x(t), y(t)) = \left(\prod_{t_0 < t_k < t} (1 + h_{1k})^{-1}u(t), \prod_{t_0 < t_k < t} (1 + h_{2k})^{-1}v(t) \right)$$

is a solution of system (5).

Proof. (i) Suppose that $(x(t), y(t))$ is a solution of (5), then for any $t \neq t_k, k = 1, 2, \cdots$, by substituting

$$(x(t), y(t)) = \left(\prod_{t_0 < t_k < t} (1 + h_{1k})^{-1}u(t), \prod_{t_0 < t_k < t} (1 + h_{2k})^{-1}v(t) \right)$$

into system (5), one can see that the first two equations of system (1) hold.

For $t = t_k, k = 1, 2, \cdots$, we have

$$\begin{aligned} u(t_k^+) &= \lim_{t\to t_k^+} \prod_{t_0<t_j<t} (1+h_{1j})x(t) \\ &= \prod_{t_0<t_j\le t_k} (1+h_{1j})x(t_k) \\ &= (1+h_{1k}) \prod_{t_0<t_j<t_k} (1+h_{1j})x(t_k) \\ &= (1+h_{1k})u(t_k). \end{aligned}$$

Similarly, we can get $v(t_k^+) = (1 + h_{2k})v(t_k)$. So, the last two equations of system (1) hold. Thus, $(u(t), v(t))$ is a solution of system (1).

(ii) Suppose that $(u(t), v(t))$ is a solution of system (1). Firstly, we show that $(x(t), y(t))$ is continuous. In fact, it is easy to see that $(x(t), y(t))$ is continuous on the interval $(t_k, t_{k+1}]$. Now, we shall check the continuity of $(x(t), y(t))$ at the impulse points $t_k, k = 1, 2, \cdots$. Since

$$\begin{aligned} x(t_k^+) &= \prod_{t_0<t_j\le t_k} (1+h_{1j})^{-1}u(t_k^+) \\ &= (1+h_{1k})^{-1}u(t_k) = x(t_k), \\ y(t_k^-) &= \prod_{t_0<t_j<t_k} (1+h_{2j})^{-1}v(t_k^-) = y(t_k). \end{aligned}$$

Thus, $(x(t), y(t))$ is continuous on $[t_0, +\infty)_{\mathbb{T}}$.

For any $t \neq t_k, k = 1, 2, \cdots$, by substituting

$$\begin{aligned} (u(t), v(t)) = \Big(&\Pi_{t_0<t_k<t}(1+h_{1k})x(t), \\ &\Pi_{t_0<t_k<t}(1+h_{2k})y(t)\Big) \end{aligned}$$

into system (1), one can see that system (5) hold. Therefore, $(x(t), y(t))$ is a solution of system (5). The proof is completed. □

Remark 10. *System (1) with the initial condition (2) and system (5) with the initial condition (6) have the same dynamic behaviors.*

3 Main results

Assume that the coefficients of (5) satisfy

(H_1) $\qquad r^l > a^u M_1 + c^u M_2$.

Lemma 11. *Let $(x(t), y(t))$ be any positive solution of system (5) with initial condition (6). If (H_1) hold, then system (5) is permanent, that is, any positive solution $(x(t), y(t))$ of system (5) satisfies*

$$m_1 \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M_1, \quad (7)$$

$$m_2 \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le M_2, \quad (8)$$

especially if $m_1 \le x_0 \le M_1$, $m_2 \le y_0 \le M_2$, then

$$m_1 \le x(t) \le M_1,\ m_2 \le y(t) \le M_2,\ t \in [t_0, +\infty)_{\mathbb{T}},$$

where

$$M_1 = \frac{r^u}{b^l},\ M_2 = \frac{g^u M_1}{\eta^l},$$

$$m_1 = \frac{r^l - a^u M_1 - c^u M_2}{b^u},\ m_2 = \frac{g^l m_1}{\eta^u}.$$

Proof. Assume that $(x(t), y(t))$ be any positive solution of system (5) with initial condition (6). From the first equation of system (5), we have

$$x^{\Delta}(t) \le x(t)(r^u - b^l x(\sigma(t))). \quad (9)$$

By Lemma 3, we can get

$$\limsup_{t\to+\infty} x(t) \le \frac{r^u}{b^l} := M_1.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$x(t) < M_1 + \varepsilon,\ \forall t \in [T_1, +\infty]_{\mathbb{T}}.$$

From the second equation of system (5), when $t \in [T_1, +\infty)_{\mathbb{T}}$,

$$y^{\Delta}(t) < -\eta^l y(t) + g^u(M_1 + \varepsilon).$$

Let $\varepsilon \to 0$, then

$$y^{\Delta}(t) \le -\eta^l y(t) + g^u M_1. \quad (10)$$

By Lemma 2, we can get

$$\limsup_{t\to+\infty} y(t) = \frac{g^u M_1}{\eta^l} := M_2.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_2 > T_1$ such that

$$y(t) < M_2 + \varepsilon,\ \forall t \in [T_2, +\infty]_{\mathbb{T}}.$$

On the other hand, from the first equation of system (5), when $t \in [T_2, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} x^{\Delta}(t) > \ & x(t)[r^l - a^u(M_1+\varepsilon) - b^u x(\sigma(t)) \\ & -c^u(M_2+\varepsilon)]. \end{aligned}$$

Let $\varepsilon \to 0$, then

$$x^{\Delta}(t) \ge x(t)[r^l - a^u M_1 - b^u x(\sigma(t)) - c^u M_2]. \quad (11)$$

By Lemma 3, we can get

$$\liminf_{t\to+\infty} x(t) = \frac{r^l - a^u M_1 - c^u M_2}{b^u} := m_1.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_3 > T_2$ such that

$$x(t) > m_1 - \varepsilon,\ \forall t \in [T_3, +\infty]_{\mathbb{T}}.$$

From the second equation of system (5), when $t \in [T_3, +\infty)_{\mathbb{T}}$,

$$y^{\Delta}(t) > -\eta^u y(t) + g^l(m_1 - \varepsilon).$$

Let $\varepsilon \to 0$, then

$$y^{\Delta}(t) \geq -\eta^u y(t) + g^l m_1. \tag{12}$$

By Lemma 2, we can get

$$\liminf_{t\to+\infty} y(t) = \frac{g^l m_1}{\eta^u} := m_2.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_4 > T_3$ such that

$$y(t) > m_2 - \varepsilon,\ \forall t \in [T_4, +\infty]_{\mathbb{T}}.$$

In special case, if $m_1 \leq x_0 \leq M_1$, $m_2 \leq y_0 \leq M_2$, by Lemma 2 and Lemma 3, it follows from (9)-(12) that

$$m_1 \leq x(t) \leq M_1,\ m_2 \leq y(t) \leq M_2,\ t \in [t_0, +\infty)_{\mathbb{T}},$$

This completes the proof. □

Let $S(\mathbb{T})$ be the set of all solutions $(x(t), y(t))$ of system (5) satisfying $m_1 \leq x(t) \leq M_1$, $m_2 \leq y(t) \leq M_2$ for all $t \in \mathbb{T}$.

Lemma 12. $S(\mathbb{T}) \neq \emptyset$.

Proof. By Lemma 11, we see that for any $t_0 \in \mathbb{T}$ with $m_1 \leq x_0 \leq M_1$, $m_2 \leq y_0 \leq M_2$, system (5) has a solution $(x(t), y(t))$ satisfying $m_1 \leq x(t) \leq M_1$, $m_2 \leq y(t) \leq M_2, t \in [t_0, +\infty)_{\mathbb{T}}$. Since $r(t)$, $a(t)$, $b(t)$, $c(t)$, $\eta(t)$, $g(t)$, $\sigma(t)$ are almost periodic, there exists a sequence $\{t_n\}$, $t_n \to +\infty$ as $n \to +\infty$ such that $r(t+t_n) \to r(t), a(t+t_n) \to a(t), b(t+t_n) \to b(t), c(t+t_n) \to c(t), \eta(t+t_n) \to \eta(t), g(t+t_n) \to g(t), \sigma(t+t_n) \to \sigma(t)$ as $n \to +\infty$ uniformly on $\mathbb{T}$.

We claim that $\{x(t+t_n)\}$ and $\{y(t+t_n)\}$ are uniformly bounded and equi-continuous on any bounded interval in $\mathbb{T}$.

In fact, for any bounded interval $[\alpha, \beta]_{\mathbb{T}} \subset \mathbb{T}$, when n is large enough, $\alpha + t_n > t_0$, then $t + t_n > t_0, \forall t \in [\alpha, \beta]_{\mathbb{T}}$. So, $m_1 \leq x(t+t_n) \leq M_1$, $m_2 \leq y(t+t_n) \leq M_2$ for any $t \in [\alpha, \beta]_{\mathbb{T}}$, that is, $\{x(t+t_n)\}$ and $\{y(t+t_n)\}$ are uniformly bounded. On the other hand, $\forall t_1, t_2 \in [\alpha, \beta]_{\mathbb{T}}$, from the mean value theorem of differential calculus on time scales, we have

$$\begin{aligned} & |x(t_1+t_n) - x(t_2+t_n)| \\ \leq\ & M_1[r^u + (a^u + b^u)M_1 + c^u M_2] \\ & \times |t_1 - t_2|, \end{aligned} \tag{13}$$

$$\begin{aligned} & |y(t_1+t_n) - y(t_2+t_n)| \\ \leq\ & (\eta^u M_2 + g^u M_1)|t_1 - t_2|. \end{aligned} \tag{14}$$

The inequalities (13) and (14) show that $\{x(t+t_n)\}$ and $\{y(t+t_n)\}$ are equi-continuous on $[\alpha, \beta]_{\mathbb{T}}$. By the arbitrary of $[\alpha, \beta]_{\mathbb{T}}$, the conclusion is valid.

By Ascoli-Arzela theorem, there exists a subsequence of $\{t_n\}$, we still denote it as $\{t_n\}$, such that

$$x(t+t_n) \to p(t), y(t+t_n) \to q(t),$$

as $n \to +\infty$ uniformly in t on any bounded interval in $\mathbb{T}$. For any $\theta \in \mathbb{T}$, we can assume that $t_n + \theta \geq t_0$ for all n, and let $t \geq 0$, integrate both equations of system (5) from $t_n + \theta$ to $t + t_n + \theta$, we have

$$\begin{aligned} & x(t+t_n+\theta) - x(t_n+\theta) \\ =\ & \int_{t_n+\theta}^{t+t_n+\theta} x(s)[r(s) - a(s)x(s) - b(s)x(\sigma(s)) \\ & -c(s)y(s)]\Delta s \\ =\ & \int_{\theta}^{t+\theta} x(s+t_n)[r(s+t_n) \\ & -a(s+t_n)x(s+t_n) - b(s+t_n)x(\sigma(s+t_n)) \\ & -c(s+t_n)y(s+t_n)]\Delta s, \end{aligned}$$

and

$$\begin{aligned} & y(t+t_n+\theta) - y(t_n+\theta) \\ =\ & \int_{t_n+\theta}^{t+t_n+\theta} [-\eta(s)y(s) + g(s)x(s)]\Delta s \\ =\ & \int_{\theta}^{t+\theta} [-\eta(s+t_n)y(s+t_n) \\ & +g(s+t_n)x(s+t_n)]\Delta s. \end{aligned}$$

Using the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} p(t+\theta) - p(\theta) &= \int_{\theta}^{t+\theta} x(s)[r(s) - a(s)x(s) \\ & \quad -b(s)x(\sigma(s)) - c(s)y(s)]\Delta s, \\ q(t+\theta) - q(\theta) &= \int_{\theta}^{t+\theta} [-\eta(s)y(s) \\ & \quad +g(s)x(s)]\Delta s. \end{aligned}$$

This means that $(p(t), q(t))$ is a solution of system (5), and by the arbitrary of θ, $(p(t), q(t))$ is a solution of system (5) on $\mathbb{T}$. It is clear that

$$m_1 \le p(t) \le M_1,\ m_2 \le q(t) \le M_2,\ \forall t \in \mathbb{T}.$$

This completes the proof. $\square$

Theorem 13. *In addition to the condition (H_1), assume further that the coefficients of system (5) satisfy the following conditions:*

(H_2) $a^l - g^u > 0$;

(H_3) $\eta^l - c^u > 0$.

Then system (5) has a unique positive almost periodic solution which is uniformly asymptotic stable.

Proof. Consider the associated product system of (5),

$$\begin{cases} x_1^{\Delta}(t) &=& x_1(t)[r(t) - a(t)x_1(t) \\ && -b(t)x_1(\sigma(t)) - c(t)y_1(t)], \\ y_1^{\Delta}(t) &=& -\eta(t)y_1(t) + g(t)x_1(t), \\ x_2^{\Delta}(t) &=& x_2(t)[r(t) - a(t)x_2(t) \\ && -b(t)x_2(\sigma(t)) - c(t)y_2(t)], \\ y_2^{\Delta}(t) &=& -\eta(t)y_2(t) + g(t)x_2(t). \end{cases} \tag{15}$$

Let $z(t) = (z_1(t), z_2(t))$ be a positive solution of product system (15), where

$$z_1(t) = (x_1(t), y_1(t)), z_2(t) = (x_2(t), y_2(t)).$$

It follows from (7)-(8) that for sufficient small positive constant ε_0 $(0 < \varepsilon_0 < \min\{m_1, m_2\})$, there exists a $T > 0$ such that

$$\begin{aligned} m_1 - \varepsilon_0 < x_i(t) < M_1 + \varepsilon_0, \\ m_2 - \varepsilon_0 < y_i(t) < M_2 + \varepsilon_0, \end{aligned} \tag{16}$$

where $t \in [T, +\infty)_{\mathbb{T}}, i = 1, 2$.

Since $x_i(t), i = 1, 2$ are positive, bounded and differentiable functions on $\mathbb{T}$, then there exists a positive, bounded and differentiable function $m(t), t \in \mathbb{T}$, such that $x_i(t)(1+m(t)), i = 1, 2$ are strictly increasing on $\mathbb{T}$. By Lemmas 4 and 5, we have

$$\begin{aligned} & \frac{\Delta}{\Delta t} L_{\mathbb{T}}(x_i(t)[1 + m(t)]) \\ = & \frac{x_i^{\Delta}(t)[1 + m(t)] + x_i(\sigma(t))m^{\Delta}(t)}{x_i(t)[1 + m(t)]} \\ = & \frac{x_i^{\Delta}(t)}{x_i(t)} + \frac{x_i(\sigma(t))m^{\Delta}(t)}{x_i(t)[1 + m(t)]},\ i = 1, 2. \end{aligned}$$

Here, we can choose a function $m(t)$ such that $\frac{|m^{\Delta}(t)|}{1+m(t)}$ is bounded on $\mathbb{T}$, that is, there exist two positive constants $\zeta > 0$ and $\xi > 0$ such that $0 < \zeta < \frac{|m^{\Delta}(t)|}{1+m(t)} < \xi$, $\forall t \in \mathbb{T}$.

Set

$$\begin{aligned} & V(t, z_1(t), z_2(t)) \\ = & |e_{-\delta}(t, T)|(|L_{\mathbb{T}}(x_1(t)(1 + m(t))) \\ & -L_{\mathbb{T}}(x_2(t)(1 + m(t)))| + |y_1(t) - y_2(t)|). \end{aligned}$$

where $\delta \ge 0$ is a constant (if $\mu(t) = 0$, then $\delta = 0$; if $\mu(t) > 0$, then $\delta > 0$). It follows from the mean value theorem of differential calculus on time scales for $t \in [T, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} & \frac{1}{M_1 + \varepsilon_0}|x_1(t) - x_2(t)| \\ \le & |L_{\mathbb{T}}(x_1(t)(1 + m(t))) - L_{\mathbb{T}}(x_2(t)(1 + m(t)))| \\ \le & \frac{1}{m_1 - \varepsilon_0}|x_1(t) - x_2(t)|, \end{aligned} \tag{17}$$

then

$$\begin{aligned} & \min\{\frac{1}{M_1 + \varepsilon_0}, 1\}|e_{-\delta}(t, T)|(|x_1(t) - x_2(t)| \\ & +|y_1(t) - y_2(t)|) \\ \le & V(t, z_1(t), z_2(t)) \\ \le & \max\{\frac{1}{m_1 - \varepsilon_0}, 1\}|e_{-\delta}(t, T)|(|x_1(t) - x_2(t)| \\ & +|y_1(t) - y_2(t)|), \end{aligned}$$

that is

$$\begin{aligned} & \min\{\frac{1}{M_1 + \varepsilon_0}, 1\}|e_{-\delta}(t, T)|(|z_1(t) - z_2(t)|) \\ \le & V(t, z_1(t), z_2(t)) \\ \le & \max\{\frac{1}{m_1 - \varepsilon_0}, 1\}|e_{-\delta}(t, T)|(|z_1(t) - z_2(t)|). \end{aligned}$$

Therefore, condition (1) in Lemma 6 is satisfied.

Since

$$\begin{aligned} & |V(t, z_1(t), z_2(t)) - V(t, \widetilde{z}_1(t), \widetilde{z}_2(t))| \\ = & |e_{-\delta}(t, T)|\big||L_{\mathbb{T}}(x_1(t)(1 + m(t))) \\ & -L_{\mathbb{T}}(x_2(t)(1 + m(t)))| + |y_1(t) - y_2(t)| \\ & -|L_{\mathbb{T}}(\widetilde{x}_1(t)(1 + m(t))) \\ & -L_{\mathbb{T}}(\widetilde{x}_2(t)(1 + m(t)))| - |\widetilde{y}_1(t) - \widetilde{y}_2(t)|\big| \\ \le & |L_{\mathbb{T}}(x_1(t)(1 + m(t))) \\ & -L_{\mathbb{T}}(\widetilde{x}_1(t)(1 + m(t)))| + |y_1(t) - \widetilde{y}_1(t)| \\ & +|L_{\mathbb{T}}(x_2(t)(1 + m(t))) \\ & -L_{\mathbb{T}}(\widetilde{x}_2(t)(1 + m(t)))| + |y_2(t) - \widetilde{y}_2(t)| \\ \le & \max\{\frac{1}{m_1 - \varepsilon_0}, 1\}(|x_1(t) - \widetilde{x}_1(t)| \\ & +|y_1(t) - \widetilde{y}_1(t)| \\ & +|x_2(t) - \widetilde{x}_2(t)| + |y_2(t) - \widetilde{y}_2(t)|) \\ = & \max\{\frac{1}{m_1 - \varepsilon_0}, 1\}(|z_1(t) - \widetilde{z}_1(t)| \\ & +|z_2(t) - \widetilde{z}_2(t)|). \end{aligned}$$

Therefore, condition (2) in Lemma 6 holds.

Next, we shall prove condition (3) in Lemma 6 holds. For convenience, We divide the proof into two cases. Let $\gamma=\min\{(m_1-\varepsilon_0)(a^l-g^u),\eta^l-c^u\}$.

Case I. If $\mu(t)>0$, set $\delta>\max\{(b^u+\frac{\xi}{m_1})M_1,\gamma\}$ and $1-\mu(t)\delta<0$. Calculating the upper right derivatives of $V(t)$ along the solution of system (5), it follows from (16), (17), (H_2) and (H_3) that for $t\in[T,+\infty)_{\mathbb{T}}$,

$$
\begin{aligned}
&D^+V^\Delta(t,z_1(t),z_2(t))\\
=\;& |e_{-\delta}(t,T)|\mathrm{sgn}(x_1(t)-x_2(t))\bigg[\frac{x_1^\Delta(t)}{x_1(t)}-\frac{x_2^\Delta(t)}{x_2(t)}\\
&+\frac{m^\Delta(t)}{1+m(t)}\bigg(\frac{x_1(\sigma(t))}{x_1(t)}-\frac{x_2(\sigma(t))}{x_2(t)}\bigg)\bigg]\\
&-\delta|e_{-\delta}(t,T)||L_{\mathbb{T}}(x_1(\sigma(t))(1+m(\sigma(t))))\\
&-L_{\mathbb{T}}(x_2(\sigma(t))(1+m(\sigma(t))))|\\
&+|e_{-\delta}(t,T)|\mathrm{sgn}(y_1(t)-y_2(t))(y_1^\Delta(t)-y_2^\Delta(t))\\
&-\delta|e_{-\delta}(t,T)||y_1(\sigma(t))-y_2(\sigma(t))|\\
=\;& |e_{-\delta}(t,T)|\mathrm{sgn}(x_1(t)-x_2(t))\bigg[-a(t)(x_1(t)\\
&-x_2(t))-b(t)(x_1(\sigma(t))-x_2(\sigma(t)))\\
&-c(t)(y_1(t)-y_2(t))]\\
&+\frac{m^\Delta(t)}{1+m(t)}\frac{x_1(\sigma(t))x_2(t)-x_1(t)x_2(\sigma(t))}{x_1(t)x_2(t)}\bigg]\\
&-\delta|e_{-\delta}(t,T)||L_{\mathbb{T}}(x_1(\sigma(t))(1+m(\sigma(t))))\\
&-L_{\mathbb{T}}(x_2(\sigma(t))(1+m(\sigma(t))))|\\
&+|e_{-\delta}(t,T)|\mathrm{sgn}(y_1(t)-y_2(t))\\
&\times[-\eta(t)(y_1(t)-y_2(t))+g(t)(x_1(t)-x_2(t))]\\
&-\delta|e_{-\delta}(t,T)||y_1(\sigma(t))-y_2(\sigma(t))|\\
=\;& |e_{-\delta}(t,T)|\mathrm{sgn}(x_1(t)-x_2(t))\\
&\times\bigg[-a(t)(x_1(t)-x_2(t))\\
&-b(t)(x_1(\sigma(t))-x_2(\sigma(t)))\\
&-c(t)(y_1(t)-y_2(t))]\\
&+\frac{m^\Delta(t)}{1+m(t)}\frac{x_1(\sigma(t))(x_2(t)-x_1(t))}{x_1(t)x_2(t)}\bigg]\\
&+\frac{m^\Delta(t)}{1+m(t)}\frac{x_1(\sigma(t))-x_2(\sigma(t))}{x_2(t)}\bigg]\\
&-\delta|e_{-\delta}(t,T)||L_{\mathbb{T}}(x_1(\sigma(t))(1+m(\sigma(t))))\\
&-L_{\mathbb{T}}(x_2(\sigma(t))(1+m(\sigma(t))))|\\
&+|e_{-\delta}(t,T)|\mathrm{sgn}(y_1(t)-y_2(t))\\
&\times[-\eta(t)(y_1(t)-y_2(t))+g(t)(x_1(t)-x_2(t))]\\
&-\delta|e_{-\delta}(t,T)||y_1(\sigma(t))-y_2(\sigma(t))|\\
\le\;& -|e_{-\delta}(t,T)|\bigg[a(t)-g(t)\\
&+\frac{|m^\Delta(t)|}{1+m(t)}\frac{x_1(\sigma(t))}{x_1(t)x_2(t)}\bigg]|x_1(t)-x_2(t)|\\
&-|e_{-\delta}(t,T)|\bigg[\frac{\delta}{M_1+\varepsilon_0}-b(t)\\
&-\frac{|m^\Delta(t)|}{1+m(t)}\frac{1}{x_2(t)}\bigg]|x_1(\sigma(t))-x_2(\sigma(t))|\\
&-|e_{-\delta}(t,T)|(\eta(t)-c(t))|y_1(t)-y_2(t)|\\
&-\delta|e_{-\delta}(t,T)||y_1(\sigma(t))-y_2(\sigma(t))|\\
\le\;& -|e_{-\delta}(t,T)|(a^l-g^u)|x_1(t)-x_2(t)|\\
&-|e_{-\delta}(t,T)|(\eta^l-c^u)|y_1(t)-y_2(t)|\\
\le\;& -|e_{-\delta}(t,T)|((m_1-\varepsilon_0)(a^l-g^u)\\
&\times|L_{\mathbb{T}}(x_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(x_2(t)(1+m(t)))|\\
&+(\eta^l-c^u)|y_1(t)-y_2(t)|)\\
\le\;& -\gamma|e_{-\delta}(t,T)|(|L_{\mathbb{T}}(x_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(x_2(t)(1+m(t)))|+|y_1(t)-y_2(t)|)\\
=\;& -\gamma V(t,z_1(t),z_2(t)). \qquad (18)
\end{aligned}
$$

Case II. If $\mu(t)=0$, set $\delta=0$, then $\sigma(t)=t$, $e_{-\delta}(t,T)=1$. Calculating the upper right derivatives of $V(t)$ along the solution of system (5), it follows from (16), (17), (H_2) and (H_3) that for $t\in[T,+\infty)_{\mathbb{T}}$,

$$
\begin{aligned}
&D^+V^\Delta(t,z_1(t),z_2(t))\\
=\;& \mathrm{sgn}(x_1(t)-x_2(t))\bigg(\frac{x_1^\Delta(t)}{x_1(t)}-\frac{x_2^\Delta(t)}{x_2(t)}\bigg)\\
&+\mathrm{sgn}(y_1(t)-y_2(t))(y_1^\Delta(t)-y_2^\Delta(t))\\
=\;& \mathrm{sgn}(x_1(t)-x_2(t))[-(a(t)\\
&+b(t))(x_1(t)-x_2(t))-c(t)(y_1(t)-y_2(t))]\\
&+\mathrm{sgn}(y_1(t)-y_2(t))[-\eta(t)(y_1(t)-y_2(t))\\
&+g(t)(x_1(t)-x_2(t))]\\
\le\;& -(a(t)+b(t)-g(t))|x_1(t)-x_2(t)|\\
&-(\eta(t)-c(t))|y_1(t)-y_2(t)|\\
\le\;& -((m_1-\varepsilon_0)(a^l+b^l-g^u)\\
&\times|L_{\mathbb{T}}(x_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(x_2(t)(1+m(t)))|\\
&+(\eta^l-c^u)|y_1(t)-y_2(t)|)\\
\le\;& -\widehat{\gamma}(|L_{\mathbb{T}}(x_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(x_2(t)(1+m(t)))|+|y_1(t)-y_2(t)|)\\
\le\;& -\gamma V(t,z_1(t),z_2(t)), \qquad (19)
\end{aligned}
$$

where $\widehat{\gamma}=\min\{(m_1-\varepsilon_0)(a^l+b^l-g^u),\eta^l-c^u\}$.

Together with (18) and (19), one can see that condition (3) in Lemma 6 is satisfied.

From the above discussion, we can see that all conditions in Lemma 6 hold. Together with Lemma 11 and Lemma 12, system (5) has a unique positive almost periodic solution which is uniformly asymptotic stable. This completes the proof. □

Theorem 14. *Under the conditions (H_1)-(H_3), it follows from Remark 10 that system (1) with the initial condition (2) has a unique positive almost periodic solution which is uniformly asymptotic stable.*

4 Example and simulations

Consider the following system on time scales

$$\begin{cases} u^{\Delta}(t) &= u(t)[0.8 + 0.2\sin\sqrt{2}t \\ & \quad -(0.045 + 0.005\sin t)u(t) \\ & \quad -u(\sigma(t)) - 0.2v(t)], \\ v^{\Delta}(t) &= -(0.4 + 0.1\cos\sqrt{3}t)v(t) \\ & \quad +(0.015 + 0.005\sin\sqrt{2}t)u(t), \\ & \qquad t \neq t_k, \\ u(t_k^+) &= 0.5u(t_k), \\ v(t_k^+) &= 0.5v(t_k),\ k = 1, 2, \cdots, 20. \end{cases} \tag{20}$$

By a direct calculation, we can get

$$\begin{aligned} & r^u = 1, r^l = 0.6, a^u = 0.0452, a^l = 0.0362, \\ & b^u = b^l = 0.9046, c^u = c^l = 0.1809, \\ & \eta^u = 0.5, \eta^l = 0.3, g^u = 0.02, g^l = 0.01, \\ & M_1 = 1.2055, M_2 = 0.0737, \\ & m_1 = 0.4355, m_2 = 0.0107, \end{aligned}$$

then,

$$\begin{aligned} & r^l - (a^u M_1 + c^u M_2) = 0.9367 > 0, \\ & a^l - g^u = 0.0162 > 0, \\ & \eta^l - c^u = 0.1192 > 0, \end{aligned}$$

that is, the conditions $(H_1) - (H_3)$ hold. According to Theorem 14, system (20) has a unique positive almost periodic solution which is uniformly asymptotic stable.

Dynamic simulations of system (20) with $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, see Figures 1 and 2, respectively.

5 Conclusion

This paper is concerned with a predator-prey system with impulses on time scales. By the relation between the solutions of impulsive system and the corresponding non-impulsive system, based on the theory of calculus on time scales and the properties of almost periodic functions as well as Razumikhin type theorem, sufficient conditions which guarantee the existence of a unique uniformly asymptotic stable almost periodic solution of the system are obtained.

The results obtained in this paper can be applied to the analysis of the periodic (and almost periodic)

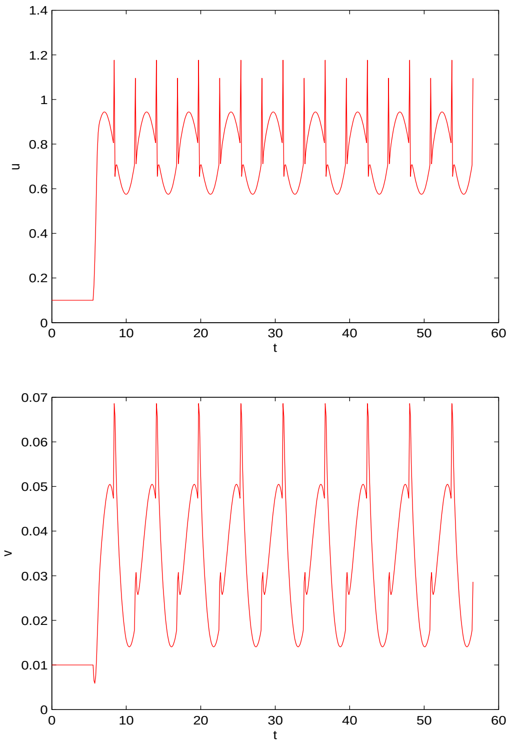

Figure 1: $\mathbb{T} = \mathbb{R}$. Dynamics behavior of system (20) with initial condition $(x(0), y(0)) = (0.5, 0.08)$.

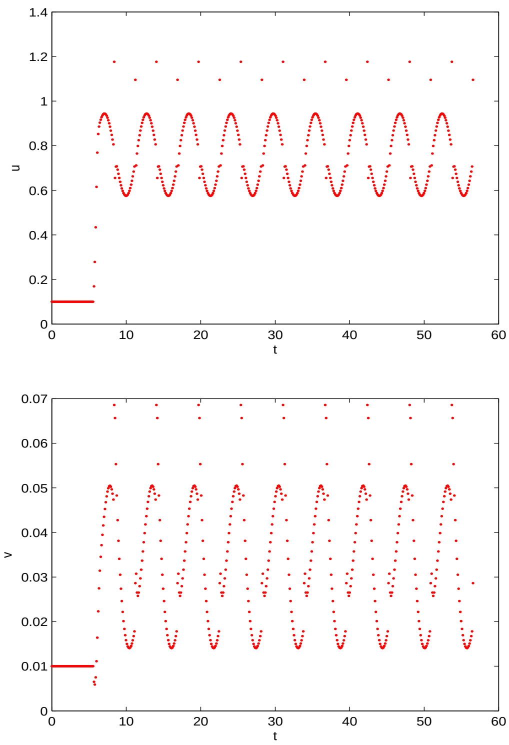

Figure 2: $\mathbb{T} = \mathbb{Z}$. Dynamics behavior of system (20) with initial condition $(x(1), y(1)) = (0.5, 0.05)$.

dynamical regimes into the dynamical systems with strange attractors [21], and to non-autonomous solutions' analysis of non-autonomous gyrostats' systems [22]. Also, one may consider many other systems, see [23]-[30].

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