Existence and iteration of positive solutions for a mixed-order four-point boundary value problem with p-Laplacian

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Abstract: In this paper, we firstly obtain the existence of the monotone positive solutions and establish a corresponding iterative scheme for the following mixed-order four-point boundary value problem with p-Laplacian

$$(\phi_p(D^{\alpha}_{0+}u(t)))' + a(t)f(t,u(t),u'(t)) = 0, \quad 0 < t < 1,$$
$$u'(0) - \beta u(\xi) = 0, \ \ u''(0) = 0, \ \ u'(1) + \gamma u(\eta) = 0.$$

Unlike many other fractional boundary value problem with p-Laplacian, the nonlinear term involves the first-order derivative explicitly, so it is hard to get positive solutions for the problem. The main tool used here is the monotone iterative technique. By the fixed point theorem due to Avery and Peterson, we obtained some sufficient conditions that guarantee the existence of at least three positive solutions to the above boundary value problem. Meanwhile, we give an example to demonstrate the use of the main results of this paper.

Key–Words: Iteration; Mixed-order four-point boundary value problem; Positive solutions; p-Laplacian.

1 Introduction

The equation with p-Laplacian arises from the modeling of different physical and natural phenomena, nonlinear elasticity and glaciology, non-Newtonian mechanics, population biology, combustion theory, nonlinear flow laws and so on, so the existence of positive solutions for integer-order nonlinear boundary value problems with p-Laplacian received wide attention (see, for instance, [1-5] and the references therein). Boundary value problems for nonlinear fractional differential equations arise in the study of models of viscoelasticity, porous media, control, electrochemistry, electromagnetic, etc [6-8]. Therefore, fractional differential equations have become a very important and useful area of mathematics over the last few decades. For details, see [19-40] and the references therein. Among them, the existence of positive solutions for boundary value problem of a nonlinear fractional differential equation with p-Laplacian has gained much importance and attention. For details, see [9-12] and the references therein.

G. Chai [12] investigated the existence and multiplicity of positive solutions for the boundary value problem of fractional differential equation with p-Laplacian operator

$$D^{\beta}_{0+}(\phi_p(D^{\alpha}_{0+}u))(t) + f(t,u(t)) = 0, \ 0 < t < 1,$$

$$u(0) = 0, \ u(1) + \sigma D^{\gamma}_{0+}u(1) = 0, \ D^{\alpha}_{0+}u(0) = 0,$$

where D^{β}_{0+}, D^{α}_{0+} and D^{γ}_{0+} are the standard Riemann-Liouville fractional derivative with $1 < \alpha \le 2, \ 0 < \beta \le 1, \ 0 < \gamma \le 1, \ 0 \le \alpha - \gamma - 1$, σ is a positive number.

Z. Liu and L. Lu [10] studied the boundary value problem for nonlinear fractional differential equations with p-Laplacian operator

$$D^{\beta}_{0+}(\phi_p(D^{\alpha}_{0+}u))(t) = f(t,u(t),D^{\alpha}_{0+}u(t)), 0 < t < 1,$$

$$u(0) = \mu\int_0^1 u(s)ds + \lambda u(\xi), \ D^{\alpha}_{0+}u(0) = kD^{\alpha}_{0+}u(\eta),$$

where $0 < \alpha,\beta \le 1, \ 1 < \alpha+\beta \le 2, \ \mu, \ \lambda, \ k \in R, \ \xi,\eta \in [0,1]$, D^{α}_{0+} denotes the Caputo fractional derivative of order α.

F. Torres [35] considered the existence of single and multiple positive solutions to nonlinear mixed-order three-point boundary value problem for p-Laplacian

$$(\phi_p(D^{\alpha}_{0+}u(t)))' + a(t)f(t,u(t)) = 0, \ 0 < t < 1,$$

$$D^{\alpha}_{0+}u(0) = u(0) = u''(0) = 0, \quad u'(1) = \gamma u'(\eta),$$

where $\eta, \gamma \in (0,1),\ \alpha \in (2,3],\ D^{\alpha}_{0+}$ is the Caputo's derivative.

However, to the best knowledge of the authors, there is less literature available on paper concerned with the mixed-order (both Caputo's fractional-order derivative and integer-order derivative are included in the equation) boundary value problem with p-Laplacian. On the other hand, the monotone iterative technique has been successfully applied to integer order boundary value problem, see for example [13-17] and the references therein, but the research on the existence and iteration of monotone positive solutions for fractional order boundary value problem is proceeding very slowly.

Inspired by the above works, in section 3, we consider the existence and monotone positive solutions for the following mixed-order four-point boundary value problem with p-Laplacian

$$(\phi_p(D^{\alpha}_{0+}u(t)))' + a(t)f(t, u(t), u'(t)) = 0,\ 0 < t < 1, \tag{1}$$

$$u'(0) - \beta u(\xi) = 0,\ u''(0) = 0,\ u'(1) + \gamma u(\eta) = 0, \tag{2}$$

where $\phi_p(s) = |s|^{p-2}s, p > 1, \phi_q = (\phi_p)^{-1}, \frac{1}{p} + \frac{1}{q} = 1, 2 < \alpha \leq 3, 0 \leq \xi \leq \eta \leq 1, 0 \leq \beta, \gamma \leq 1, D^{\alpha}_{0+}$ is the Caputo's fractional derivative. By the application of the monotone iterative technique, we can get the solutions by constructing a corresponding iterative scheme. Furthermore, the nonlinear term involves the first-order derivative explicitly.

In [18], the authors considered the existence of triple positive pseudo-symmetric solutions of the form

$$(\phi_p(u'))'(t) + q(t)f(t, u(t), u'(t)) = 0, \quad t \in (0,1),$$

$$u(0) - \beta u'(\xi) = 0,\ u(\xi) - \delta u'(\eta) = u(1) + \delta u'(1 + \xi - \eta),$$

by means of a fixed point theorem due to Avery and Peterson. This fixed point theorem is used as a classical method for getting triple positive solutions for the differential equations of integer order which the lower order derivatives of unknown function is involved in the nonlinear term explicitly, but it cannot be used directly to obtain the existence of triple positive solutions of the differential equations of fractional order. The main difficulty is that we cannot get the concavity or convexity of function $u(t)$ by the sign of its fractional order derivative. In section 4, by obtaining some new inequalities of the unknown function, we get the existence of at least three positive solutions for the boundary value problem of fractional order.

2 Preliminaries and Lemmas

For the sake of convenience, we formulate the following conditions.

(H_1) : $f(t, x, y) \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+)$, $a(t)$ is a nonnegative continuous function defined on $(0,1)$ and $a(t) \not\equiv 0$ on any subinterval of $(0,1)$. Moreover, $\int_0^1 a(t)dt < +\infty$;

(H_2) : $\Lambda < (\alpha - 1)(1 - \beta\xi)$, where $\Lambda = \gamma(1 - \beta\xi) + \beta(1 + \eta\gamma)$.

In this paper, a positive solution $u(t)$ of boundary value problem (1), (2) means a solution $u(t)$ of (1) and (2) satisfying $u(t) > 0, 0 < t < 1$.

Definition 1 *[38] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2 *[38] The Caputo's derivative of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is defined by*

$$D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{f^{(n)}(s)ds}{(t - s)^{\alpha + 1 - n}},$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 3 *Let $\alpha > 0$ and $u \in C(0,1) \cap L^1(0,1)$. Then the fractional differential equation*

$$D^{\alpha}_{0+}u(t) = 0$$

has

$$u(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},$$

where $c_i \in \mathbb{R},\ \ i = 1, 2, \cdots, n$ and $n = [\alpha] + 1$ as unique solution.

Lemma 4 [38] *Let $\alpha > 0$, then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}$$

for some $c_i \in \mathbb{R},\ \ i = 1, 2, \cdots, n$, and $n = [\alpha] + 1$.

Lemma 5 *For any $h \in C[0,1], 2 < \alpha \leq 3$, the unique solution of*

$$(\phi_p(D^{\alpha}_{0+}u(t)))' + h(t) = 0, \quad 0 < t < 1, \tag{3}$$

$$u'(0)-\beta u(\xi)=0,\ \ u''(0)=0,\ \ u'(1)+\gamma u(\eta)=0, \tag{4}$$

is given by

$$u(t)=\int_0^1 G(t,s)\phi_q\Big(\int_0^s h(\tau)d\tau\Big)ds, \tag{5}$$

where

$$G(t,s)=\begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta(1+\gamma\eta-\gamma t)(\xi-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)} \\ +\frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Lambda\Gamma(\alpha-1)} \\ +\frac{\gamma(1-\beta\xi+\beta t)(\eta-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)}, \quad s\le\xi, s\le t, \\ \frac{\beta(1+\gamma\eta-\gamma t)(\xi-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)} \\ +\frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Lambda\Gamma(\alpha-1)} \\ +\frac{\gamma(1-\beta\xi+\beta t)(\eta-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)}, \quad s\le\xi, t\le s, \\ -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Lambda\Gamma(\alpha-1)} \\ +\frac{\gamma(1-\beta\xi+\beta t)(\eta-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)}, \\ \quad \xi\le s\le\eta, s\le t, \\ \frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Lambda\Gamma(\alpha-1)} \\ +\frac{\gamma(1-\beta\xi+\beta t)(\eta-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)}, \\ \quad \xi\le s\le\eta, t\le s, \\ -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Lambda\Gamma(\alpha-1)}, \\ \quad \eta\le s, s\le t, \\ \frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Lambda\Gamma(\alpha-1)}, \quad \eta\le s, t\le s. \end{cases} \tag{6}$$

Proof: Integrating both sides of the equation (3), we can get

$$\phi_p(D_{0+}^\alpha u(t))=-\int_0^t h(s)ds,$$

that is

$$D_{0+}^\alpha u(t)=-\phi_q\Big(\int_0^t h(s)ds\Big).$$

Taking Lemma 4 into account,

$$\begin{aligned}u(t)=&-\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t(t-s)^{\alpha-1}\phi_q\Big(\int_0^s h(\tau)d\tau\Big)ds\\ &+A+Bt+Ct^2,\end{aligned}$$

the condition $u''(0)=0$ implies that $C=0$. So

$$\begin{aligned}u'(t)=&\ -\tfrac{1}{\Gamma(\alpha-1)}\textstyle\int_0^t(t-s)^{\alpha-2}ds\\ &\phi_q\Big(\textstyle\int_0^s h(\tau)d\tau\Big)+B,\end{aligned}$$

in view of condition (4), we have

$$\begin{aligned}&B+\tfrac{\beta}{\Gamma(\alpha)}\textstyle\int_0^\xi(\xi-s)^{\alpha-1}\phi_q\Big(\int_0^s h(\tau)d\tau\Big)ds\\ &\quad -A\beta-B\beta\xi=0,\end{aligned} \tag{7}$$

$$\begin{aligned}&-\tfrac{1}{\Gamma(\alpha-1)}\textstyle\int_0^1(1-s)^{\alpha-2}\phi_q\Big(\int_0^s h(\tau)d\tau\Big)ds\\ &+B-\tfrac{\gamma}{\Gamma(\alpha)}\textstyle\int_0^\eta(\eta-s)^{\alpha-1}\phi_q\Big(\int_0^s h(\tau)d\tau\Big)ds\\ &+A\gamma+B\eta\gamma=0.\end{aligned} \tag{8}$$

Solving (7), (8), we get

$$\begin{aligned}u(t)=&-\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t(t-s)^{\alpha-1}\phi_q\Big(\int_0^s h(\tau)d\tau\Big)ds\\ &+\tfrac{1-\beta\xi+\beta t}{\Lambda\Gamma(\alpha-1)}\textstyle\int_0^1(1-s)^{\alpha-2}\phi_q\Big(\int_0^s h(\tau)d\tau\Big)ds\\ &+\tfrac{(1-\beta\xi)\gamma+\beta\gamma t}{\Lambda\Gamma(\alpha)}\textstyle\int_0^\eta(\eta-s)^{\alpha-1}\phi_q\Big(\int_0^s h(\tau)d\tau\Big)ds\\ &+\tfrac{\beta(1+\eta\gamma)-\beta\gamma t}{\Lambda\Gamma(\alpha)}\textstyle\int_0^\xi(\xi-s)^{\alpha-1}\phi_q\Big(\int_0^s h(\tau)d\tau\Big)ds.\end{aligned}$$

This completes the proof.

Denote

$$N=\frac{\beta(1+\gamma\eta)+(\alpha-1)(1-\beta\xi+\beta)+\gamma(1-\beta\xi+\beta)}{\Lambda\Gamma(\alpha)},$$

$$G=\frac{(\alpha-1)(1-\beta\xi+\frac{1}{4}\beta)-\Lambda}{\beta(1+\gamma\eta)+(\alpha-1)(1-\beta\xi+\beta)+\gamma(1-\beta\xi+\beta)}.$$

Lemma 6 *The function $G(t,s)$ defined by (6) has the following properties.*

1. $G\in C([0,1]\times[0,1]),\ 0\le G(t,s)\le N(1-s)^{\alpha-2},\ \frac{\partial G(t,s)}{\partial t}\le\frac{(\beta+\beta\gamma)(1-s)^{\alpha-2}}{\Lambda\Gamma(\alpha-1)},\ t,s\in(0,1);$
2. *There exists a number $G>0$ such that*

$$\min_{\frac14\le t\le\frac34}G(t,s)\ge GN(1-s)^{\alpha-2},\ \ 0<s<1. \tag{9}$$

Definition 7 *[18] The map β is said to be a nonnegative continuous convex functional on a cone K of a real Banach space E provided that $\beta:\ K\to[0,\infty)$ is continuous and*

$$\beta(tx+(1-t)y)\le t\beta(x)+(1-t)\beta(y)$$

for all $x,y\in K$ and $0\le t\le 1$.

Definition 8 *[18] The map α is said to be a nonnegative continuous concave functional on a cone K of a real Banach space E provided that $\alpha:\ K\to[0,\infty)$ is continuous and*

$$\alpha(tx+(1-t)y)\ge t\alpha(x)+(1-t)\alpha(y)$$

for all $x,y\in K$ and $0\le t\le 1$.

Let γ, θ be nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K. For positive real numbers a, b, c, and d, we define the following convex sets:

$$\begin{aligned}&K(\gamma,\ d) = \{x \in K \mid \gamma(x) < d\},\\ &K(\gamma,\ \alpha,\ b,\ d) = \{x \in K \mid b \le \alpha(x),\ \gamma(x) \le d\},\\ &K(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d) = \{x \in K \mid b \le \alpha(x),\\ &\qquad\qquad\qquad \theta(x) \le c, \gamma(x) \le d\},\\ &R(\gamma,\ \psi,\ a,\ d) = \{x \in K \mid\ a \le \psi(x),\ \gamma(x) \le d\}.\end{aligned}$$

The following fixed point theorem due to Avery and Peterson is fundamental in the proofs of our main results in section 4.

Lemma 9 *[18] Let K be a cone in a Banach space E. Let γ and θ be nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K satisfying*

$$\psi(\lambda x) \le \lambda\psi(x)\ \text{for}\ 0 \le \lambda \le 1,$$

such that for some positive numbers M and d,

$$\alpha(x) \le \psi(x), \quad \|x\| \le M\gamma(x)\ \text{for}\ x \in \overline{K(\gamma,\ d)}, \tag{10}$$

where $\overline{K(\gamma,\ d)}$ is the closure of the set $K(\gamma,\ d)$. Suppose

$$T:\ \overline{K(\gamma,\ d)} \to \overline{K(\gamma,\ d)}$$

is completely continuous and there exist positive numbers $a,\ b$, and c with $a < b$ such that

(S_1) $x \in K(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)\ |\alpha(x) > b\} \ne \emptyset$ and $\alpha(Tx) > b$ for $x \in K(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)$;

(S_2) $\alpha(Tx) > b$ for $x \in K(\gamma,\ \alpha,\ b,\ d)$ with $\theta(Tx) > c$;

(S_3) $0 \notin R(\gamma,\ \psi,\ a,\ d)$ and $\psi(Tx) < a$ for $x \in R(\gamma,\ \psi,\ a,\ d)$ with $\psi(x) = a$.

Then T has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{K(\gamma,\ d)}$, such that

$$\gamma(x_i) \le d, \quad i = 1, 2, 3;\ \ b < \alpha(x_1);$$

$$a < \psi(x_2),\ \ \alpha(x_2) < b,\ \ \psi(x_3) < a.$$

3 Existence and iteration of positive solutions for the problem (1) and (2)

Consider the Banach space $E = C^1[0, 1]$ equipped with the norm

$$\|u\| = \max\left\{\max_{0\le t\le 1} |u(t)|,\ \max_{0\le t\le 1} |u'(t)|\right\}.$$

Define the cone $P \subset E$ by

$$P = \{u \in E \mid u(t) \ge 0,\ \ 0 \le t \le 1\}. \tag{11}$$

For $u \in P$, we define the operator T by

$$(Tu)(t) = \int_0^1 G(t,s)ds\ \phi_q\left(\int_0^s a(\tau)f(\tau, u(\tau), u'(\tau))d\tau\right) \tag{12}$$

where $G(t, s)$ is given by (6).

Obviously, $u(t)$ is a solution of boundary value problem (1), (2) if and only if $u(t)$ satisfies the equation $u = Tu$.

Lemma 10 *The operator $T:\ P \to P$ is completely continuous.*

The statement of the main result needs to introduce the notations.

$$M = \max\left\{\frac{N\phi_q\left(\int_0^1 a(s)ds\right)}{\alpha - 1}, \frac{(\beta + \beta\gamma)\phi_q\left(\int_0^1 a(s)ds\right)}{(\alpha - 1)\Lambda\Gamma(\alpha - 1)}\right\}.$$

Theorem 11 *Suppose that $(H_1), (H_2)$ hold. Furthermore, there exists a number $m > 0$ such that*

(A_1) $f(t, x_1, y_1) \le f(t, x_2, y_2)$, for $0 \le t \le 1,\ 0 \le x_1 \le x_2 \le m,\ 0 \le |y_1| \le |y_2| \le m$;

(A_2) $\max\limits_{0\le t\le 1} f(t, m, m) \le \phi_p(\frac{m}{M})$;

(A_3) $f(t, 0, 0) \not\equiv 0$ on $0 \le t \le 1$.

Then the fractional boundary value problem (1), (2) has at least one positive solution $\omega^* \in P$ with $0 < \omega^* \le m,\ 0 < |(\omega^*)'| \le m$ and $\lim\limits_{n\to\infty} \omega_n = \lim\limits_{n\to\infty} T^n\omega_0 = \omega^*$, where

$$\omega_0(t) = \frac{m}{M}\phi_q\left(\int_0^1 a(s)ds\right)\int_0^1 G(t,s)ds,\ 0 \le t \le 1.$$

Proof: Set $P_m = \{u \in P:\ \|u\| \le m\}$. Next, we first prove $TP_m \subset P_m$.

Let $u \in P_m$, then

$$0 \le u(t) \le \max_{0\le t\le 1} |u(t)| \le \|u\| \le m,$$

$$|u'(t)| \le \max_{0\le t\le 1} |u'(t)| \le \|u\| \le m,$$

taking (A_1) and (A_2) into account,

$$\begin{aligned}&0 \le f(t, u(t), u'(t)) \le f(t, m, m)\\ &\le \max_{0\le t\le 1} f(t, m, m) \le \phi_p(\tfrac{m}{M}),\ 0 \le t \le 1,\end{aligned}$$

(12) implies that $(Tu)(t) \geq 0$. Moreover,

$$\begin{aligned} |(Tu)(t)| &\leq \int_0^1 \max_{0\leq t\leq 1} G(t,s) \\ &\quad \phi_q\bigg(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\bigg)ds \\ &\leq N\tfrac{m}{M}\phi_q\bigg(\int_0^1 a(s)ds\bigg)\int_0^1(1-s)^{\alpha-2}ds \\ &= \tfrac{Nm}{M(\alpha-1)}\phi_q\bigg(\int_0^1 a(s)ds\bigg) \leq m, \end{aligned}$$

and

$$\begin{aligned} |(Tu)'(t)| &\leq \int_0^1 \max_{0\leq t\leq 1} \tfrac{\partial G(t,s)}{\partial t} \\ &\quad \phi_q\bigg(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\bigg)ds \\ &\leq \tfrac{m}{M}\phi_q\bigg(\int_0^1 a(s)ds\bigg)\tfrac{\beta+\beta\gamma}{\Lambda\Gamma(\alpha-1)} \\ &\quad \int_0^1(1-s)^{\alpha-2}ds \\ &\leq \tfrac{m}{M(\alpha-1)}\tfrac{\beta+\beta\gamma}{\Lambda\Gamma(\alpha-1)}\phi_q\bigg(\int_0^1 a(s)ds\bigg) \\ &\leq m. \end{aligned}$$

Thus, we have $\|Tu\| \leq m$, which implies $TP_m \subset P_m$.

Let

$$\omega_0(t) = \frac{m}{M}\phi_q\bigg(\int_0^1 a(s)ds\bigg)\int_0^1 G(t,s)ds,$$

then

$$\begin{aligned} \max_{0\leq t\leq 1}\omega_0(t) &= \tfrac{m}{M}\phi_q\bigg(\int_0^1 a(s)ds\bigg)\int_0^1 \max_{0\leq t\leq 1} G(t,s)ds \\ &\leq \tfrac{m}{M}\phi_q\bigg(\int_0^1 a(s)ds\bigg)N\int_0^1(1-s)^{\alpha-2}ds \\ &= \tfrac{Nm}{M(\alpha-1)}\phi_q\bigg(\int_0^1 a(s)ds\bigg) \leq m, \end{aligned}$$

and

$$\begin{aligned} \max_{0\leq t\leq 1}|\omega_0'(t)| &\leq \tfrac{m}{M}\phi_q\bigg(\int_0^1 a(s)ds\bigg)\int_0^1 \max_{0\leq t\leq 1}\tfrac{\partial G(t,s)}{\partial t}ds \\ &\leq \tfrac{m}{M}\phi_q\bigg(\int_0^1 a(s)ds\bigg)\tfrac{\beta+\beta\gamma}{\Lambda\Gamma(\alpha-1)} \\ &\quad \int_0^1(1-s)^{\alpha-2}ds \\ &= \tfrac{m}{M(\alpha-1)}\tfrac{\beta+\beta\gamma}{\Lambda\Gamma(\alpha-1)}\phi_q\bigg(\int_0^1 a(s)ds\bigg) \\ &\leq \tfrac{mM}{M} = m. \end{aligned}$$

So we have $\omega_0(t) \in P_m$. Let $\omega_1 = T\omega_0$, then $\omega_1 \in P_m$. Denote

$$\omega_{n+1} = T\omega_n = T^{n+1}\omega_0, \quad (n = 0,1,2,\cdots),$$

since $TP_m \subset P_m$, we have $\omega_n \in P_m$, $(n = 0,1,2,\cdots)$, $\{\omega_n\}_{n=1}^{\infty}$ is a sequentially compact set since T is a completely continuous operator.

$$\begin{aligned} &\omega_1(t) = T\omega_0(t) \\ &= \int_0^1 G(t,s)\phi_q\bigg(\int_0^s a(\tau)f(\tau,\omega_0(\tau),\omega_0'(\tau))d\tau\bigg)ds \\ &\leq \tfrac{m}{M}\phi_q\bigg(\int_0^1 a(s)ds\bigg)\int_0^1 G(t,s)ds \\ &= \omega_0(t), \quad 0\leq t\leq 1, \end{aligned}$$

and

$$\begin{aligned} &|\omega_1'(t)| = |(T\omega_0)'(t)| \\ &= \int_0^1 |\tfrac{\partial G(t,s)}{\partial t}|\phi_q\bigg(\int_0^s a(\tau)f(\tau,\omega_0(\tau),\omega_0'(\tau))d\tau\bigg)ds \\ &\leq \tfrac{m}{M}\phi_q\bigg(\int_0^1 a(s)ds\bigg)\int_0^1\left|\tfrac{\partial G(t,s)}{\partial t}\right|ds \\ &= |\omega_0'(t)|, \quad 0\leq t\leq 1, \end{aligned}$$

which implies

$$\omega_1(t) \leq \omega_0(t), \quad |\omega_1'(t)| \leq |\omega_0'(t)|, \quad 0\leq t\leq 1,$$

so

$$\omega_2(t) = T\omega_1(t) \leq T\omega_0(t) = \omega_1(t),$$
$$|\omega_2'(t)| = |(T\omega_1)'(t)| \leq |(T\omega_0)'(t)| = |\omega_1'(t)|.$$

Thus we have

$$\omega_{n+1} \leq \omega_n, \quad |\omega_{n+1}'(t)| \leq |\omega_n'(t),$$
$$0\leq t\leq 1, \;\; n = 0,1,2,\cdots.$$

So we get T has a fixed point $\omega^* \in P_m$, and $\omega^* = \lim_{n\to\infty}\omega_n$, moreover, $\|\omega^*\| > 0$, since the zero function is not a solution of boundary value problem (1), (2). According to the properties of the $G(t,s)$, we have $\omega^*(t) \geq G\|\omega^*\| > 0$, $\frac{1}{4} \leq t \leq \frac{3}{4}$, that is ω^* is a positive solution of (1) and (2). □

Remark If the equation (1) is the integer order p-Laplacian equation, we consider the following boundary value problem

$$(\phi_p(u'(t)))' + q(t)f(t,u(t),u'(t)) = 0, \quad t \in (0,1),$$
$$u'(0) - \alpha u(\xi) = 0, \qquad u'(1) + \beta u(\eta) = 0,$$

where $\phi_p(s) = s|s|^{p-2}$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\xi,\eta \in (0,1)$ and $\xi < \eta$, $\alpha \in (0,\frac{1}{\xi})$,$\beta \in (0,\frac{1}{1-\eta})$.

Let

$$A = \max\left\{\begin{array}{l} \phi_q(\int_0^1 q(\tau)d\tau)(1+\frac{1}{\alpha}), \\ \phi_q(\int_0^1 q(\tau)d\tau)(1+\frac{1}{\beta}) \end{array}\right\}.$$

(H_1) $f \in C([0,1]\times[0,\infty)\times(-\infty,\infty),(0,\infty))$;

(H_2) $q(t)$ is a nonnegative measurable function defined on $(0,1)$, $q(t) \not\equiv 0$ on any subinterval of $(0,1)$. In addition, $\int_0^1 q(t)dt < +\infty$;

(H_3) $\xi, \eta \in (0,1)$ and $\xi < \eta$, $\alpha \in (0, \frac{1}{\xi})$, $\beta \in (0, \frac{1}{1-\eta})$. We have the following Theorem.

Theorem: [40] Assume that (H_1), (H_2) and (H_3) hold, and there exists $a > 0$, such that

(C_1) $f(t,\ x_1,\ y_1) \le f(t,\ x_2,\ y_2)$, for any $0 \le t \le 1$,
$0 \le x_1 \le x_2 \le a,\ 0 \le |y_1| \le |y_2| \le a$;

(C_2) $\max\limits_{0\le t\le 1} f(t,a,a) \le \phi_p\left(\frac{a}{A}\right)$;

(C_3) $f(t,\ 0,\ 0) \not\equiv 0\ for\ 0 \le t \le 1$.

Then the above boundary value problem has one positive solution $\omega^* \in K$ such that $0 < \omega^* \le a$, $0 < |(\omega^*)'| \le a$ and $\lim\limits_{n\to\infty} T^n\omega_0 = \omega^*$, $\lim\limits_{n\to\infty}(T^n\omega_0)' = (\omega^*)'$ where

$$\omega_0(t) = a\frac{\max\left\{(\frac{1}{\alpha}+t),\ (\frac{1}{\beta}+(1-t))\right\}}{\max\left\{(\frac{1}{\alpha}+1),\ (\frac{1}{\beta}+1)\right\}},\ 0 \le t \le 1.$$

Corollary: [40] Assume $(\mathrm{H}_1)-(\mathrm{H}_3)$, (C_1), (C_3) hold, and there exist $0 < a_1 < a_2 < \cdots < a_n$, such that

(C_2') $\max\limits_{0\le t\le 1} f(t,a_k,a_k) \le \phi_p(\frac{a_k}{A})$, $k = 1,2,\cdots,n$, particularly, $\liminf\limits_{r\to+\infty}\max\limits_{0\le t\le 1} f(t,r,a_k) = 0$, $k = 1,2,\cdots,n)$.

Then the above boundary value problem has n positive solutions $\omega_k^* \in K$ such that $0 < \omega_k^* \le a_k$, $0 < |(\omega_k^*)'| \le a_k$ and $\lim\limits_{n\to\infty} T^n\omega_{k_0} = \omega_k^*$, $\lim\limits_{n\to\infty}(T^n\omega_{k_0})' = (\omega_k^*)'$ where

$$\omega_{k_0}(t) = a_k\frac{\max\left\{(\frac{1}{\alpha}+t),\ (\frac{1}{\beta}+(1-t))\right\}}{\max\left\{(\frac{1}{\alpha}+1),\ (\frac{1}{\beta}+1)\right\}},\ 0 \le t \le 1.$$

4 Three positive solutions of the problem (1) and (2)

Lemma 12 *Given $h(t) \in C[0,1]$, $2 < \alpha \le 3$. Assume that $u(t)$ is a solution of problem (3), (4),then*

$$\begin{aligned} u'(t) &= -\tfrac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &+\tfrac{\beta}{\Lambda\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &+\tfrac{\beta\gamma}{\Lambda\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &-\tfrac{\beta\gamma}{\Lambda\Gamma(\alpha)}\int_0^\xi(\xi-s)^{\alpha-1}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds. \end{aligned} \tag{13}$$

Proof: From Lemma 5, we get

$$\begin{aligned} u(t) &= -\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &+\tfrac{1-\beta\xi+\beta t}{\Lambda\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &+\tfrac{(1-\beta\xi)\gamma+\beta\gamma t}{\Lambda\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &+\tfrac{\beta(1+\eta\gamma)-\beta\gamma t}{\Lambda\Gamma(\alpha)}\int_0^\xi(\xi-s)^{\alpha-1}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds. \end{aligned}$$

Then equation (13) is established.

Lemma 13 *Assume that $h(t) > 0$ and $u(t)$ is a solution of problem (3), (4). Then there exists a positive constant G_1 such that*

$$\max_{0\le t\le 1}|u(t)| \le G_1\max_{0\le t\le 1}|u'(t)|, \tag{14}$$

where $G_1 = \frac{N\Lambda\Gamma(\alpha-1)}{\beta} > 0$.

Proof: From Lemmas 5 and 6, we obtain

$$\begin{aligned} \max_{0\le t\le 1}|u(t)| &= \max_{0\le t\le 1}\left|\int_0^1 G(t,s)\phi_q\left(\int_0^s h(\tau)d\tau\right)ds\right| \\ &\le \int_0^1 N(1-s)^{\alpha-2}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds, \end{aligned} \tag{15}$$

$$\begin{aligned} \max_{0\le t\le 1}|u'(t)| &\ge |u'(0)| \\ &= \tfrac{\beta}{\Lambda\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &+\tfrac{\beta\gamma}{\Lambda\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &-\tfrac{\beta\gamma}{\Lambda\Gamma(\alpha)}\int_0^\xi(\xi-s)^{\alpha-1}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &> \tfrac{\beta}{\Lambda\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds. \end{aligned} \tag{16}$$

Thus,

$$\max_{0\le t\le 1}|u(t)| = \max_{0\le t\le 1}\left|\int_0^1 G(t,s)\phi_q\left(\int_0^s h(\tau)d\tau\right)ds\right|$$
$$\le \int_0^1 N(1-s)^{\alpha-2}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds,$$
$$\le \int_0^1 G_1\frac{\beta}{\Lambda\Gamma(\alpha-1)}(1-s)^{\alpha-2}\phi_q\left(\int_0^s h(\tau)d\tau\right)ds$$
$$\le G_1\max_{0\le t\le 1}|u'(t)|. \tag{17}$$

Let the space $E=C^1[0,1]$ endowed with the norm,

$$\|u\| = \max\left\{\max_{0\le t\le 1}|u(t)|, \max_{0\le t\le 1}|u'(t)|\right\}. \tag{18}$$

It is well known that E is a Banach space. Define the cone K by

$$K=\{u\in E|\ u(t)\ge 0, \min_{\frac14\le t\le\frac34}u(t)\ge G\max_{0\le t\le 1}u(t),$$
$$\max_{0\le t\le 1}|u(t)|\le G_1\max_{0\le t\le 1}|u'(t)|\}.$$

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functional θ, γ and the nonnegative continuous functional ψ be defined on the cone K by

$$\gamma(u)=\max_{0\le t\le 1}|u'(t)|,\ \psi(u)=\theta(u)=\max_{0\le t\le 1}|u(t)|,$$
$$\alpha(u)=\min_{\frac14\le t\le\frac34}|u(t)|,\ \text{for } u\in K.$$

With lemmas 6 and 13, for all $u\in K$, the functionals defined above satisfy that

$$G\theta(u)\le\alpha(u)\le\theta(u)=\psi(u),\ \|u\|\le G_2\gamma(u),$$

where $G_2=\max\{G_1,1\}$. Therefore, the condition (10) of Lemma 9 is satisfied.

Define an operator $T: K\to E$ by

$$(Tu)(t) = \int_0^1 G(t,s)ds\ \phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right) \tag{19}$$

From Lemma 10, we have $T: K\to K$ is completely continuous. □

Theorem 14 *Assume that* (H_1), (H_2) *hold. Let* $0<a<b\le Gd$, *with*

$$b[\alpha\Lambda+\beta\gamma(\xi^\alpha+\eta^\alpha)+\alpha\beta]\phi_q(\int_0^1 a(s)ds)$$
$$<\Lambda\Gamma(\alpha+1)dGN\int_{\frac14}^{\frac34}(1-s)^{\alpha-2}\phi_q(\int_{\frac14}^s a(\tau)d\tau)ds,$$

and suppose that f satisfies the following conditions:

(A_1) $f(t,h,k)\le\phi_p\left(\frac{\Lambda\Gamma(\alpha+1)d}{[\alpha\Lambda+\beta\gamma(\xi^\alpha+\eta^\alpha)+\alpha\beta]\phi_q(\int_0^1 a(s)ds)}\right)$, *for* $(t,h,k)\in[0,1]\times[0,G_2d]\times[-d,d]$;

(A_2) $f(t,h,k)>\phi_p\left(\frac{b}{GN\int_{\frac14}^{\frac34}(1-s)^{\alpha-2}\phi_q(\int_{\frac14}^s a(\tau)d\tau)ds}\right)$, *for* $(t,h,k)\in[\frac14,\frac34]\times[b,\frac{b}{G}]\times[-d,d]$;

(A_3) $f(t,h,k)<\phi_p\left(\frac{a(\alpha-1)}{N\phi_q(\int_0^1 a(s)ds)}\right)$, *for* $(t,h,k)\in[0,1]\times[0,a]\times[-d,d]$.

Then the boundary value problem (1),(2) has at least three positive solutions u_1, u_2, u_3 *such that* $\max_{0\le t\le 1}|u_i'(t)|\le d$, $i=1,2,3$; $b<\min_{\frac14\le t\le\frac34}|u_1(t)|$; $a<\max_{0\le t\le 1}|u_2(t)|$, $\min_{\frac14\le t\le\frac34}|u_2(t)|<b$, $\max_{0\le t\le 1}|u_3(t)|<a$.

Proof: Boundary value problem (1),(2) has a solution $u=u(t)$ if and only if u solves the operator equation $u=Tu$. Thus we set out to prove that T satisfies the Avery-Peterson fixed point theorem which will prove the existence of three fixed points of T which satisfy the conclusion of the theorem.

For $u\in\overline{K(\gamma,d)}$, there is $\gamma(u)=\max_{0\le t\le 1}|u'(t)|\le d$. With lemma 13, there is $\max_{0\le t\le 1}|u(t)|\le G_1d\le G_2d$, then condition (A_1) implies

$$f(t,u(t),u'(t))$$
$$\le\phi_p\left(\frac{\Lambda\Gamma(\alpha+1)d}{[\alpha\Lambda+\beta\gamma(\xi^\alpha+\eta^\alpha)+\alpha\beta]\phi_q(\int_0^1 a(s)ds)}\right).$$

Then

$$\gamma(Tu)=\max_{0\le t\le 1}|(Tu)'(t)|$$
$$=\max_{0\le t\le 1}\left|-\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\right.$$
$$\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)ds$$
$$+\frac{\beta}{\Lambda\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}$$
$$\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)ds$$
$$+\frac{\beta\gamma}{\Lambda\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}$$
$$\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)ds$$
$$-\frac{\beta\gamma}{\Lambda\Gamma(\alpha)}\int_0^\xi(\xi-s)^{\alpha-1}$$

$$\phi_q\bigg(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\bigg)ds\bigg|$$
$$\le \left[\frac{1}{(\alpha-1)\Gamma(\alpha-1)}+\frac{\beta\gamma(\xi^\alpha+\eta^\alpha)+\alpha\beta}{\Lambda\Gamma(\alpha+1)}\right]$$
$$\frac{\Lambda\Gamma(\alpha+1)d}{[\alpha\Lambda+\beta\gamma(\xi^\alpha+\eta^\alpha)+\alpha\beta]\phi_q(\int_0^1 a(s)ds)}$$
$$\phi_q(\int_0^1 a(s)ds)$$
$$= d.$$

Therefore, $T:\ \overline{K(\gamma,d)}\to\overline{K(\gamma,d)}$.

We choose $u(t)=\frac{b}{G}$. It is easy to see that $u(t)\in K(\gamma,\ \theta,\ \alpha,\ b,\ \frac{b}{G},\ d)$ and $\alpha(u)=\alpha(\frac{b}{G})=\frac{b}{G}>b$, and so $\{u\in K(\gamma,\ \theta,\ \alpha,\ b,\ \frac{b}{G},\ d)|\alpha(u)>b\}\neq\emptyset$. So for $u\in K(\gamma,\ \theta,\ \alpha,\ b,\ \frac{b}{G},\ d)$, we have $b\le u(t)\le\frac{b}{G},\ \ |u'(t)|\le d$ for $\frac14\le t\le\frac34$.
From (A_2), we get

$$f(t,u(t),u'(t))$$
$$>\phi_p\left(\frac{b}{GN\int_{\frac14}^{\frac34}(1-s)^{\alpha-2}\phi_q(\int_{\frac14}^s a(\tau)d\tau)ds}\right)$$

for $\frac14\le t\le\frac34$, thus

$$\alpha(Tu)=\min_{\frac14\le t\le\frac34}|(Tu)(t)|$$
$$=\min_{\frac14\le t\le\frac34}\bigg|\int_0^1 G(t,s)\phi_q\bigg(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\bigg)ds\bigg|$$
$$>\int_0^1 GN(1-s)^{\alpha-2}\phi_q\bigg(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau$$
$$>GN\int_{\frac14}^{\frac34}(1-s)^{\alpha-2}\phi_q\bigg(\int_{\frac14}^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\bigg)ds\bigg|$$
$$>\frac{b}{GN\int_{\frac14}^{\frac34}(1-s)^{\alpha-2}\phi_q(\int_{\frac14}^s a(\tau)d\tau)ds}$$
$$GN\int_{\frac14}^{\frac34}(1-s)^{\alpha-2}\phi_q(\int_{\frac14}^s a(\tau)d\tau)ds=b,$$

which implies $\alpha(Tu)>b$ for all $u\in K(\gamma,\ \theta,\ \alpha,\ b,\ \frac{b}{G},\ d\}$. These ensure that the condition (S_1) of Lemma 9 is satisfied.

Secondly, for all $u\in K(\gamma,\ \alpha,\ b,d)$ with $\theta(Tu)>\frac{b}{G}$,

$$\alpha(Tu)\ge G\theta(Tu)>G\frac{b}{G}=b.$$

This shows that (S_2) of Lemma 9 is satisfied.

Finally, we test that condition (S_3) of Lemma 9 also holds. Clearly, $\psi(0)=0<a$, this shows $0\notin R(\gamma,\ \psi,\ a,\ d)$. If $u\in R(\gamma,\ \psi,\ a,\ d)$ with $\psi(u)=a$. From the condition (A_3), we have

$$\psi(Tu)=\max_{0\le t\le1}|(Tu)(t)|$$
$$=\max_{0\le t\le1}\bigg|\int_0^1 G(t,s)\phi_q\bigg(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\bigg)ds\bigg|$$
$$\le\int_0^1 N(1-s)^{\alpha-2}\frac{a(\alpha-1)}{N\phi_q(\int_0^1 a(s)ds)}\phi_q(\int_0^1 a(s)ds)=a.$$

So all the conditions of Lemma 9 are satisfied. Therefore, boundary value problem (1),(2) has at least three positive solutions $u_1,\ u_2,\ u_3$ satisfing $\max_{0\le t\le1}|u_i'(t)|\le d,\ i=1,2,3;\ b<\min_{\frac14\le t\le\frac34}|u_1(t)|;\ a<\max_{0\le t\le1}|u_2(t)|$, $\min_{\frac14\le t\le\frac34}|u_2(t)|<b,\ \max_{0\le t\le1}|u_3(t)|<a$. The proof is complete. □

5 Example

Example: We consider the following four-point fractional boundary value problem

$$(D_{0^+}^{\frac52}u(t))'+f(t,u(t),u'(t))=0,\ \ 0<t<1,\ \ (20)$$

$$u'(0)-\frac12u(\frac14)=0,\ \ u''(0)=0,\ \ u'(1)+\frac12u(\frac12)=0,\ \ (21)$$

where

$$f(t,x,y)=-t^2+t+\frac{1}{400}x+\frac{1}{1600}y^2.$$

Then $(20),(21)$ has at least one positive solution ω^* such that

$$0<\omega^*\le4,\quad 0<|(\omega^*)'|\le4$$

and

$$\lim_{n\to\infty}\omega_n=\lim_{n\to\infty}T^n\omega_0=\omega^*.$$

Proof: We can easily see that $p=2,\alpha=\frac52,\beta=\gamma=\eta=\frac12,\xi=\frac14,a(s)=1$, by computating, $\Lambda=\frac{17}{16},N=\frac{72}{17\sqrt{\pi}},M=\frac{48}{17\sqrt{\pi}}$, choose $m=4$, we claim all the assumptions in Lemma 9 hold.

(A_1) $f(t,x_1,y_1)\le f(t,x_2,y_2)$ for any $0\le t\le1,0\le x_1\le x_2\le4,0\le|y_1|\le|y_2|\le4$;

(A_2) $\max_{0\le t\le1}f(t,4,4)=\max_{0\le t\le1}(-t^2+t+\frac{1}{400}\times4+\frac{1}{1600}\times16)=\frac14+\frac{1}{200}<\phi_p(\frac{m}{M})=\frac{m}{M}=\frac{17\sqrt{\pi}}{12}$,

(A_3) $f(t,0,0)\not\equiv0$, for $0\le t\le1$.

Therefore, by Lemma 9, the fractional boundary value problem $(20),(21)$ has at least one positive solution ω^*. □

6 Conclusion

This paper is motivated from some recent papers treating the boundary value problems for four-point fractional differential equations with p-Laplacian operator. In section 2, we first give some notations, recall some concepts and preparation results. In section 3, we firstly use the monotone iterative technique to investigate the existence and monotone positive solutions and establish a corresponding iterative scheme for the mixed-order four-point boundary value problem with p-Laplacian (1), (2). In section 4, we use Avery and Peterson fixed point theorem to study the existence of three positive solutions for the fractional boundary value problem (1), (2). In section 5, an example is given to demonstrate the use of the main results in section 3. To the best of our knowledge, no work has been done to get existence and monotone positive solutions and three positive solutions of the four-point fractional boundary value problem for $2 < \alpha \leq 3$. The aim of this paper is to fill the gap in the relevant literatures. Such investigations will provide an important platform for gaining a deeper understanding of our environment.

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