Optimal Reinsurance and Investment Problem with Stochastic Interest Rate and Stochastic Volatility in the Mean-variance Framework

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Abstract: This paper studied an optimal reinsurance and investment problem for insurers under the mean-variance criterion in the stochastic interest rate and stochastic volatility environment, where the financial market consists of two assets: one is the risk-free asset (i.e bond) and the other is the risky-asset (i.e stock) whose volatility satisfying the Heston model. Assume that the interest rate is driven by Vasicek interest rate model and the surplus process is approximated by diffusion approximation model. In order to hedge the risk of the insurance, proportional reinsurance is considered. And the insurer wishes to look for the optimal reinsurance and investment strategies to minimize the variance of the terminal wealth for a given expected terminal wealth. By employing dynamic programming principle and Lagrange duality theorem, the optimal reinsurance and investment strategies and the efficient frontier are explicitly obtained. Finally, some special cases and sensitivity analysis are provided to illustrate our results.

Key–Words: Optimal reinsurance and investment strategy; Vasicek interest rate; Stochastic volatility; mean-variance framework; the efficient frontier;

1 Introduction

The optimal reinsurance and investment problems have drawn great attention due to the fact that the reinsurance is an effective way to spread risk and that the investment is an increasingly important element in the insurance business in recent years. Many literatures have studied optimization problems under different financial markets and various objectives in insurance risk management, such as maximizing the expected utility function of terminal wealth, minimizing the probability of ruin for the insurers, and mean-variance criterion. Mean-variance criterion is pioneered in portfolio selection problem by Markowitz [1] which has been regarded as the milestone of modern portfolio theory. Since Markowitz, Li and Ng [2] first extend a single-period investment model into multi-period model and the explicit solution is derived by a linear quadratic (LQ) control that is developed by them. Zhou and Li [3] focus on continuous-time mean-variance portfolio selection problems, and the efficient investment strategy and the efficient frontier are obtained by applying the LQ approach. Fu et al.[4] obtain the closed-form solution to the optimal strategy and the efficient frontier by the dynamic programming principle and Lagrange duality theorem. In addition, some scholars point out that the mean-variance criterion could be of interest in insurance applications. For example, Bauerle [5] considers the proportional reinsurance and obtains the optimal strategy under the mean-variance criterion where the surplus process is modeled by the classical Cramer-Landberg (CL) model. Bai and Zhang [6] investigate the reinsurance and investment problem under the assumption that the surplus of the insurer is modeled by a CL model and a diffusion approximation (DA) model respectively, and obtain the optimal reinsurance and investment policies under the mean-variance criterion by LQ method. Zeng and Li [7] also consider investing multiple risky assets whose price processes follow geometric Brownian motions (GBM), and the optimal time-consistent policies of investment-reinsurance problem are obtained under the mean-variance criterion. The concern is that the above mentioned risky asset models of which the volatilities are deterministic and the interest rates are fixed.

It is all well-known that deterministic volatility is unrealistic in the real-world environment where inflation, wars, and disaster will come, and that the interest rate is not always fixed. Ever since the oil crisis in the last century, the interest rate has appeared more volatile in many western countries. So, it is very reasonable to incorporate stochastic interest rate and stochastic volatility into the reinsurance and investment problem. Previous works on reinsurance-investment problem with stochastic volatility are as follows. Bauerle and Blatter [8] derive the optimal investment and reinsurance policies under the assumption that both the surplus process and the stock index are driven by Levy process. Gu et al.[9], Lin and Li [10] and Liang et al.[11] investigate the optimal investment-proportional reinsurance strategies under the constant elasticity of variance (CEV) model. Li et. al.[12] get the time-consistent investment and reinsurance strategies for the mean-variance problem under Heston's stochastic volatility model.

However, it does not appeared that someone works on reinsurance-investment problem with stochastic interest rate. But most of optimal investment problems with stochastic interest have been studied, such as Bajeux-Besnainou et al.[13], Korn and Kraft [14], Chang et al.[15], Chang and Lu [16] and so on. Moreover, some scholars have investigated optimal investment problems with stochastic interest rate and stochastic volatility. For example, Liu [17] obtained the optimal investment-consumption strategy in the stochastic framework, in which the interest rate and volatility are stochastic parameters that are described as the functions of stochastic factors. Li and Wu [18] obtain the optimal investment strategy under the assumption that the interest rate is govern by the Cox-Ingersoll-Ross (CIR) model and the risky asset is modeled by the Heston model. Liu et al.[19] study the dynamic portfolios with the CIR interest rate under the Heston model whose expression is different to Li and Wu [18], where the manager invests his wealth to a zero-coupon bond, a riskless asset and a stock, and get the explicit solution of optimal portfolio strategy. Chang and Rong [20] investigate an investment and consumption problem on the basis of Li and Wu [18], and get the closed-form expression of the optimal investment and consumption strategy in the stochastic interest rate and stochastic volatility. Guan and Liang [21] study an optimal investment problem of DC pension plan, and obtain the optimal investment strategy under the stochastic framework, in which the interest rate is described by the affine model and the risky-asset is assumed to be driven by the Heston model.

As far as we know, little of literatures for insurers considers the optimal reinsurance and investment problem with both stochastic interest rate and stochastic volatility under mean-variance criterion. And it is clear that the optimal strategy under stochastic interest rate and stochastic volatility will be more practical. So, in this paper, we will focus on studying the optimal reinsurance and investment problem for an insurer in the dual stochastic environment, where the interest rate is assumed to follow the Vasicek interest rate model and the price of the risky asset is driven by the Heston model. The insurer will purchase proportional reinsurance and allocate one risk-free asset and one risky asset into the financial market. And, the objective of the insurer is to minimize the variance of the terminal wealth for a given expected terminal wealth. In this paper, by employing the stochastic dynamic programming method, we derive the Hamilton-Jacobi-Bellman (HJB) equation for the value function. After using a variable change technique and separate variable approach, we obtain the closed-form solution of the optimal reinsurance and investment strategy. And by applying Lagrange duality theorem, we get the explicit expression of the efficient frontier. As a result, sensitive analysis is given to illustrate the results obtained, and the effects of market parameters on the optimal reinsurance and investment strategies are analyzed. This paper has three main contributions:

(i) Based on Li and Wu [18], the reinsurance is introduced and the optimal reinsurance and investment problem under stochastic interest rate and stochastic volatility is studied, in which interest rate is described by the Vasicek interest rate model.

(ii) The mean-variance model with Vasicek interest rate and stochastic volatility is studied.

(iii) We adopt dynamic programming principle and Lagrange duality theorem to successfully obtain the optimal reinsurance and investment policy and the efficient frontier.

The reminder of this paper is organized as follows. In Section 2, the problem framework is described. In Section 3, Using Lagrange multiplier technique and dynamic programming principle, we get the closed-form solutions of the optimal reinsurance and investment policies for the insurers. In section 4, the explicit expression of the efficient frontier is obtained by applying Lagrange duality theorem. In Section 5, some special cases are given. In Section 6, a sensitivity analysis is provided to demonstrate our results. In Section 7, some important conclusions are drawn.

2 The Model

In this section, we provide the problem framework, which is composed of four parts: the surplus process, the financial market, the wealth process and the optimization criterion. Suppose that the financial market consists of two assets: one risk-free asset and one risky asset. In addition, the insurers can buy proportional reinsurance or new business.

Throughout this paper $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0\leq t\leq T})$ is a given complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual condition, where T is a finite constant and P is the real world probability; $\mathcal{F}_t$ stands for the information available until time t. All stochastic processes in this paper are supposed to be well defined and adapted processes in this probability space.

2.1 Surplus Process

We consider the surplus process of an insurer is given by the DA model:

$$dR_0(t) = \mu_0 dt + \sigma_0 dW_1(t), \tag{1}$$

where $\mu_0 > 0$ stands for the premium return rate of the insurer ; $\sigma_0 > 0$ can be identified as the volatility of the insurers' surplus; $\{W_1(t)\}$ is a one-dimensional standard Brownian motion.

Suppose that insurers can purchase proportional reinsurance and acquire new business to control the insurance risk. For each $t \in [0, T]$, the value of risk exposure is denoted as $m(t) \in [0, +\infty]$ representing the retention level of reinsurance or new business. When $m(t) \in [0, 1]$, it corresponds to a proportional reinsurance cover; in this case, the insurer should divert the premium to the reinsurer at the rate of $(1 - m(t))\mu_1$, where μ_1 is the premium return rate of the reinsurer satisfying $\mu_1 > \mu_0$. Otherwise, arbitrage will exist; meanwhile, for each claim, the insurer will pay $100m(t)\%$ of the claim, and the reinsurer pays the rest $100(1 - m(t))\%$. And, $m(t) \in (1, +\infty)$ corresponds to acquiring new business. Incorporating purchasing proportional reinsurance or acquiring new business, the DA dynamics for the surplus process associated with such a reinsurance strategy $\{m(t) : t \in [0, T]\}$ becomes:

$$dR(t) = [\mu_0 - (1 - m(t))\mu_1]dt + \sigma_0 m(t)dW_1(t). \tag{2}$$

2.2 Financial Market

Assume that the financial market consists two assets: one risk-free asset (e.g a bond or a bank account) and one risky asset (e.g a stock); the price process of the risk-free asset $B(t)$ is modeled by:

$$dB(t) = r(t)B(t)dt, \tag{3}$$

where $r(t)$ is the interest rate and it is described by a Vasicek interest rate model:

$$dr(t) = [\theta - cr(t)]dt + k_1 dW_1(t), \tag{4}$$

with initial value $r(0) > 0$, where θ, c, and k_1 are constants. Clearly, $r(t) > 0$ for all $t \geq 0$.

The price process of the risky asset $S(t)$ is assumed to be driven by the following Heston model ,i.e

$$dS(t) = S(t)[(r(t) + k\eta(t))dt + \sigma_1\sqrt{\eta(t)}dW_2(t)], \tag{5}$$

$$d\eta(t) = [b - a\eta(t)]dt + \sigma_1\sqrt{\eta(t)}dW_2(t), \tag{6}$$

with initial value $\eta(0) > 0$, where k, b, a, and σ_1 are positive constants satisfying $2b > \sigma_1^2$, we also have $\eta(t) > 0$ for all $t \geq 0$. $\{W_2(t)\}$ is a standard Brownian motion independent of $\{W_1(t)\}$.

2.3 Wealth Process

$n(t)$ is denoted by the proportion of the total wealth invested in the risky asset, and $X(t)$ is the wealth of the investor at time t with initial value $X(0) = x_0$. Then the amount invested in the risky asset at time t is $n(t) * X(t)$. In this paper, the insurer is allowed to buy reinsurance/acquire new business and invest in the financial market during the time $[0, T]$ to reduce the risk. So, a trading strategy is a pair of stochastic processes, which should be denoted by $\pi(t) = \{(m(t), n(t))\}_{t\in[0,T]}$, where $m(t)$ is the value of risk exposure at time t. Then $X^\pi(t)$ satisfies the following stochastic differential equation (SDE):

$$\begin{aligned} dX^\pi(t) = {} & dR(t) + n(t)X^\pi(t)\frac{dS(t)}{S(t)} \\ & + [1 - n(t)]X^\pi(t)\frac{dB(t)}{B(t)}. \end{aligned} \tag{7}$$

Substituting (2),(3) and (5) back into (7) shows that

$$\begin{aligned} dX^\pi(t) = {} & [\mu_0 - (1 - m(t))\mu_1 + r(t)X^\pi(t) \\ & + k\eta(t)n(t)X^\pi(t)]dt + \sigma_0 m(t)dW_1(t) \\ & + \sigma_1 n(t)X^\pi(t)\sqrt{\eta(t)}dW_2(t), \\ X^\pi(0) = {} & x_0, \end{aligned} \tag{8}$$

where x_0 is the wealth at time 0.

2.4 The Optimization Criterion

Definition 1 *(Admissible strategy). An investment-reinsurance strategy $\pi(t) = \big(m(t), n(t)\big)$ is said to be admissible, if the following conditions are satisfied.*

(i) $\big(m(t), n(t)\big)$ is $\mathcal{F}_t$ -progressively measurable;

(ii) $0 \leq n(t) \leq 1$, $m(t) \geq 0$;

(iii)$E\bigg(\int_0^T \big(\sigma_1^2 n(t)^2 X(t)^2\eta(t) + \sigma_0^2 m(t)^2\big)dt\bigg) < \infty$;

(iv) For all initial conditions $(t_0, r_0, \eta_0, x_0) \in [0, T] \times (0, \infty)^3$, the equation (8) has a unique strong solution.

Suppose that the set of all admissible strategies is denoted by $\Pi = \{(m(t), n(t))\}_{t\in[0,T]}$. The insurer's objective is to find an optimal reinsurance and investment strategy $(m(t), n(t))$ such that the expected terminal wealth satisfies $E(X(T)) = C$, for some $C \in R$, while the risk measured by the variance of the terminal wealth

$$\begin{aligned} VarX(T) &= E(X(T) - E(X(T)))^2 \\ &= E(X(T) - C)^2, \end{aligned} \tag{9}$$

is minimized. The problem of finding the optimal strategy $(m(t), n(t))$ is regarded as the mean-variance problem. Therefore, the mean-variance problem can be written as a linearly constrained stochastic optimization problem:

$$\begin{aligned} &\underset{\pi(t)\in\Pi}{Minimize} VarX(T) = E(X(T) - C)^2 \\ &\qquad s.tE(X(T)) = C. \end{aligned} \tag{10}$$

Finally, an optimal reinsurance-investment policy of the above problem is called an efficient strategy corresponding to some constant C, and the corresponding $(C, VarX(T))$ is called an efficient point. Whereas, the set of the efficient points is called an efficient frontier.

3 The Optimal Reinsurance and Investment Policy

In order to find the optimal reinsurance and investment strategy, a Lagrange multiplier $2\lambda \in R$ is introduced and the objective function can be written as:

$$\begin{aligned} &\hat{L}(m(t), n(t), \lambda) \\ =&E((X(T) - C)^2 + 2\lambda(X(T) - C)) \\ =&E(X(T) - (C - \lambda))^2 - \lambda^2. \end{aligned} \tag{11}$$

Letting $l = C - \lambda$, we get the following stochastic control problem:

$$\begin{aligned} &Minimize \bar{L}(m(t), n(t), l) \\ &\qquad =E(X(T) - l)^2 - (C - l)^2. \end{aligned} \tag{12}$$

The link between the problem (10) and the problem (12) is provided by Lagrange duality theorem(see Fu et al.[4]). Then, we have

$$\begin{aligned} MinVarX(T) &= \underset{\lambda\in R}{Max}\ \underset{\pi(t)\in\Pi}{Min}\ \hat{L}(m(t), n(t), \lambda) \\ &= \underset{l\in R}{Max}\ \underset{\pi(t)\in\Pi}{Min}\ \bar{L}(m(t), n(t), l). \end{aligned} \tag{13}$$

For a fixed constant C and l, the objective (12) is equivalent to

$$\underset{\pi(t)\in\Pi}{Minimize}\, E(X(T) - l)^2. \tag{14}$$

Then, the value function is defined as

$$\begin{aligned} V(t, x, r, \eta) = &\underset{\pi\in\Pi}{Min}\, E[(X(T) - l)^2 \\ &|r(t) = r, \eta(t) = \eta, X(t) = x]. \end{aligned} \tag{15}$$

And, for any $V(t, x, r, \eta) \in [0, T] \times R \times R \times R$, we define a variational operator :

$$\begin{aligned} &\mathcal{A}^\pi V(t, x, \eta, r) = V_t + [\mu_0 - (1 - m(t))\mu_1 \\ &+rx + k\eta n(t)x]V_x + (\theta - cr)V_r \\ &+(b - a\eta)V_\eta + [\frac{1}{2}\sigma_1^2 n(t)^2 x^2 \eta \\ &+\frac{1}{2}\sigma_0^2 m(t)^2]V_{xx} + \frac{1}{2}\sigma_1^2\eta V_{\eta\eta} \\ &+\frac{1}{2}k_1^2 V_{rr} + \sigma_0 m(t)k_1 V_{xr} + \sigma_1^2\eta x n(t)V_{x\eta}, \end{aligned} \tag{16}$$

where, $\mathcal{A}$ is a variational operator, V_t, V_x, V_r, V_η, V_{xx}, V_{rr}, $V_{\eta\eta}$, V_{xr} and $V_{x\eta}$ represent first-order and second-order partial derivatives with respect to the variables t, x, r, and η.

Based on the reference Fleming and Soner [22], $V(t, x, r, \eta)$ satisfies the following HJB equation:

$$\begin{cases} \underset{\pi(t)\in\Pi}{Min}\{\mathcal{A}^\pi V(t, x, r, \eta)\} = 0 \\ V(T, x, r, \eta) = (x - l)^2. \end{cases} \tag{17}$$

In order to solve the nonlinear partial differential equation(17) and find the optimal strategy $\pi^*(t) = (m^*(t), n^*(t))$, we assume that the HJB equation(17) has a classical solution V satisfying $V_x > 0$ and $V_{xx} > 0$. Differentiating with respect to $m(t)$ and $n(t)$ in the equation(17), the optimizer can be derived as follows:

$$m^*(t) = -\frac{k_1 V_{xr}}{\sigma_0 V_{xx}} - \frac{\mu_1}{\sigma_0^2} \cdot \frac{V_x}{V_{xx}}, \tag{18}$$

$$n^*(t) = -\frac{V_{x\eta}}{xV_{xx}} - \frac{k}{\sigma_1^2} \cdot \frac{V_x}{xV_{xx}}. \tag{19}$$

Substituting (18) and (19) back into (17) , by means of some simplification ,the HJB equation (13) is equivalent to the following partial differential equation:

$$\begin{aligned} &V_t + (\mu_0 - \mu_1 + rx)V_x + (\theta - cr)V_r \\ &+ (b - a\eta)V_\eta - \frac{1}{2}\Big(\frac{\mu_1^2}{\sigma_0^2} + \frac{k^2\eta}{\sigma_1^2}\Big)\frac{V_x^2}{V_{xx}} - \frac{1}{2}\sigma_1^2\eta\frac{V_{x\eta}^2}{V_{xx}} \\ &- k\eta\frac{V_x V_{x\eta}}{V_{xx}} + \frac{1}{2}\sigma_1^2\eta V_{\eta\eta} + \frac{1}{2}k_1^2 V_{rr} \\ &- \frac{1}{2}k_1^2\frac{V_{xr}^2}{V_{xx}} - \frac{k_1\mu_1}{\sigma_0} \cdot \frac{V_x V_{xr}}{Vxx} = 0, \end{aligned} \tag{20}$$

with terminal condition: $V(T,x,r,\eta)=(x-l)^2$.

We try to conjecture the value function V with the following form:

$$V(t,r,x,\eta)=f(t,r,\eta)\big[x-g(t,r)\big]^2, \tag{21}$$

with $f(T,r,\eta)=1$ and $g(T,r)=0$ for all η and r. The partial derivatives with respect to $V(t,x,r,\eta)$ are following:

$$\begin{aligned}
&V_t=f_t(x-g)^2-2f(x-g)g_t,\\
&V_x=2f(x-g),\quad V_\eta=f_\eta(x-g)^2,\\
&V_r=f_r(x-g)^2-2f(x-g)g_r,\\
&V_{rr}=f_{rr}(x-g)^2-4f_rg_r(x-g)+2fg_r^2\\
&-2fg_{rr}(x-g),\\
&V_{\eta\eta}=f_{\eta\eta}(x-g)^2,\quad V_{xr}=2[f_r(x-g)-fg_r],\\
&V_{xx}=2f,\quad V_{x\eta}=2f_\eta(x-g).
\end{aligned} \tag{22}$$

Putting the above partial derivatives(22) back into (20) leads to an equation for $f(t,r,\eta)$ and $g(t,r)$:

$$\begin{aligned}
&(x-g)^2\bigg[f_t+2rf+(\theta-cr)f_r+(b-a\eta)f_\eta\\
&-\left(\frac{\mu_1^2}{\sigma_0^2}+\frac{k^2\eta}{\sigma_1^2}\right)f-\sigma_1^2\eta\frac{f_\eta^2}{f}-2k\eta f_\eta+\frac{1}{2}\sigma_1^2\eta f_{\eta\eta}\\
&-k_1^2\frac{f_r^2}{f}-\frac{2k_1\mu_1}{\sigma_0}f_r+\frac{1}{2}k_1^2f_{rr}\bigg]\\
&-2(x-g)f\bigg[g_t-rg+(\theta-\frac{k_1\mu_1}{\sigma_0}-cr)g_r\\
&+\frac{1}{2}k_1^2g_{rr}+(\mu_1-\mu_0)\bigg]=0
\end{aligned} \tag{23}$$

To solve the equation(23), we decomposed it into the following two equations:

$$\begin{aligned}
&f_t+2rf+(\theta-cr)f_r+(b-a\eta)f_\eta\\
&-\left(\frac{\mu_1^2}{\sigma_0^2}+\frac{k^2\eta}{\sigma_1^2}\right)f-\sigma_1^2\eta\frac{f_\eta^2}{f}-2k\eta f_\eta\\
&+\frac{1}{2}\sigma_1^2\eta f_{\eta\eta}-k_1^2\frac{f_r^2}{f}-\frac{2k_1\mu_1}{\sigma_0}f_r+\frac{1}{2}k_1^2f_{rr}=0;
\end{aligned} \tag{24}$$

$$\begin{aligned}
&g_t-rg+(\theta-cr-\frac{k_1\mu_1}{\sigma_0})g_r+\frac{1}{2}k_1^2g_{rr}\\
&+(\mu_1-\mu_0)=0,\\
&g(T,r)=l.
\end{aligned} \tag{25}$$

Lemma 2 *Assume that a solution of the equation(24) is conjectured as*

$$f(t,r,\eta)=\Psi(t)\exp\big\{\phi(t)\eta+h(t)r\big\}, \tag{26}$$

with boundary conditions given by $\Psi(T)=1$ and $\phi(T)=h(T)=0$, then $\Psi(t),\phi(t)$ and $h(t)$ are given by (36),(33), and (35) respectively.

Proof: Introducing (26) into (24), after some simplifications, we obtain

$$\begin{aligned}
&\Psi(t)\eta\bigg(\phi'(t)-a\phi(t)-\frac{k^2}{\sigma_1^2}-\sigma_1^2\phi^2(t)-2k\phi(t)\\
&+\frac{1}{2}\sigma_1^2\phi^2(t)\bigg)+\Psi(t)r\Big(2+h'(t)-ch(t)\Big)\\
&+\Psi'(t)+\Psi(t)\bigg(\Big(\theta-\frac{2k_1\mu_1}{\sigma_0}\Big)h(t)+b\phi(t)\\
&-\frac{\mu_1^2}{\sigma_0^2}-\frac{1}{2}k_1^2h^2(t)\bigg)=0,
\end{aligned} \tag{27}$$

here, $\Psi'(t)$, $\phi'(t)$ and $h'(t)$ denote the derivatives with respect to t. Comparing the coefficients on the both sides of (27), we have

$$\begin{aligned}
&\phi'(t)-(2k+a)\phi(t)-\frac{1}{2}\sigma_1^2\phi^2(t)-\frac{k^2}{\sigma_1^2}=0,\\
&\phi(T)=0
\end{aligned} \tag{28}$$

$$h'(t)-ch(t)+2=0,\quad h(T)=0; \tag{29}$$

$$\begin{aligned}
&\Psi'(t)+\Psi(t)\bigg(\Big(\theta-\frac{2k_1\mu_1}{\sigma_0}\Big)h(t)+b\phi(t)\\
&-\frac{\mu_1^2}{\sigma_0^2}-\frac{1}{2}k_1^2h^2(t)\bigg)=0,\quad \Psi(T)=1.
\end{aligned} \tag{30}$$

Suppose that Δ_ϕ denotes the discriminant of the quadratic function

$$\frac{1}{2}\sigma_1^2\phi^2(t)+(2k+a)\phi(t)+\frac{k^2}{\sigma_1^2}=0. \tag{31}$$

Obviously, $\Delta_\phi=(2k+a)^2-2k^2>0$, the equation (31) has two different roots, and the two roots are given by

$$\phi_{1,2}=\frac{-(2k+a)\pm\sqrt{(2k+a)^2-2k^2}}{\sigma_1^2}.$$

Further, the equation (28) can be rewritten in the form

$$\phi'(t)=\frac{1}{2}\sigma_1^2(\phi(t)-\phi_1)(\phi(t)-\phi_2). \tag{32}$$

After some calculation, integrating from t to T, we obtain

$$\begin{aligned}
&\frac{1}{\phi_1-\phi_2}\int_t^T\left(\frac{1}{(\phi-\phi_1)(\phi-\phi_2)}\right)d\phi(t)\\
=&\frac{1}{2}\sigma_1^2(T-t).
\end{aligned}$$

Solving the above integral, by some easy calculations, we have

$$\phi(t) = \frac{\phi_1\phi_2(1-\exp\{\frac{1}{2}\sigma_1^2(T-t)(\phi_1-\phi_2)\})}{\phi_1-\phi_2\exp\{\frac{1}{2}\sigma_1^2(T-t)(\phi_1-\phi_2)\}}. \tag{33}$$

Since (29) is a first-order linear differential equation, we can rewrite it as following form

$$\frac{dh(t)}{ch(t)-2} = dt. \tag{34}$$

Through the integration of the both sides of equation (34), we obtain the solution of (29)

$$h(t) = \frac{2}{c}(1-e^{-c(T-t)}). \tag{35}$$

Substituting (33) and (35) back into (30), we can get the solution of equation (30)

$$\Psi(t) = \exp\Bigg(\Big(\theta - \frac{2k_1\mu_1}{\sigma_0}\Big)\int_t^T h(s)ds + b\int_t^T \phi(s)ds - \frac{1}{2}k_1^2\int h^2(s)ds - \frac{\mu_1^2}{\sigma_0^2}(T-t)\Bigg), \tag{36}$$

where,

$$\int_t^T h(s)ds = \frac{2}{c}(T-t) - \frac{2}{c^2}(1-e^{-c(T-t)}),$$

$$\int_t^T \phi(s)ds = \phi_2(T-t) + \frac{2k^2}{\sigma_1^4}\ln\frac{\phi_1-\phi_2}{\phi_1-\phi_2\exp\{\frac{1}{2}\sigma_1^2(\phi_1-\phi_2)(T-t)\}},$$

$$\int_t^T h^2(s)ds = \frac{4}{c^2}(T-t) - \frac{8}{c^3}(1-e^{-c(T-t)}) + \frac{2}{c^3}(1-e^{-2c(T-t)}).$$

The proof is completed. ■

Lemma 3 *Suppose that a solution of (25) is of the structure $g(t,r) = (\mu_1-\mu_0)\int_t^T \hat{g}(s,r)ds + l\hat{g}(t,r)$, then $\hat{g}(t,r)$ satisfies the following partial differential equation :*

$$\hat{g}_t - r\hat{g} + \Big(\theta - \frac{k_1\mu_1}{\sigma_0} - cr\Big)\hat{g}_r + \frac{1}{2}k_1^2\hat{g}_{rr} = 0, \tag{37}$$

with the boundary condition given by $\hat{g}(T,t) = 1$.

Proof: For convenience, we denote the following variational operator on any function $g(t,r)$:

$$\nabla g(t,r) = -rg + \Big(\theta - \frac{k_1\mu_1}{\sigma_0} - cr\Big)g_r + \frac{1}{2}k_1^2 g_{rr}. \tag{38}$$

(21) can be written as

$$\begin{cases} \frac{\partial g(t,r)}{\partial t} + \nabla g(t,r) + (\mu_1-\mu_0) = 0 \\ g(T,r) = l. \end{cases} \tag{39}$$

Considering

$$g(t,r) = (\mu_1-\mu_0)\int_t^T \hat{g}(s,r)ds + l\hat{g}(t,r),$$

we derive

$$\begin{aligned} \frac{\partial g(t,r)}{\partial t} &= -(\mu_1-\mu_0)\hat{g}(t,r) + l\frac{\partial \hat{g}(t,r)}{\partial t} \\ &= (\mu_1-\mu_0)\Big(\int_t^T \frac{\partial \hat{g}(s,r)}{\partial s}ds - \hat{g}(T,r)\Big) \\ &\quad + l\frac{\partial \hat{g}(t,r)}{\partial t} \end{aligned} \tag{40}$$

$$\nabla g(t,r) = (\mu_1-\mu_0)\int_t^T \nabla\hat{g}(s,r)ds + l\nabla\hat{g}(t,r). \tag{41}$$

Putting (40) and (41) into (39), and we get

$$(\mu_1-\mu_0)\Bigg(\int_t^T \Big(\frac{\partial \hat{g}(s,r)}{\partial s} + \nabla\hat{g}(s,r)\Big)ds - \hat{g}(T,r) + 1\Bigg) + l\Big(\frac{\partial \hat{g}(t,r)}{\partial t} + \nabla\hat{g}(t,r)\Big) = 0. \tag{42}$$

Therefore, we obtain

$$\begin{cases} \frac{\partial \hat{g}(s,r)}{\partial s} + \nabla\hat{g}(s,r) = 0 \\ \hat{g}(T,r) = 1. \end{cases} \tag{43}$$

In a result, we complete the proof. ■

Lemma 4 *Assume that $\hat{g}(t,r) = \exp\{A(t)+B(t)r\}$ is a solution of the equation (37), with boundary condition $A(T)=0$ and $B(T)=0$, then $A(t)$ and $B(t)$ are given by (49) and (48), respectively.*

Proof: The partial derivatives of $\hat{g}(t,r)$ with respect to t and r are given as follows:

$$\begin{aligned} \hat{g}_t &= \exp\{A(t)+B(t)r\}\big[A'(t)+B'(t)r\big], \\ \hat{g}_r &= \exp\{A(t)+B(t)r\}B(t), \\ \hat{g}_{rr} &= \exp\{A(t)+B(t)r\}B^2(t). \end{aligned} \tag{44}$$

Plugging (44) into (37), after some simple calculations, we derive

$$\exp\{A(t)+B(t)r\}\Big[A'(t)+\Big(\theta-\frac{k_1\mu_1}{\sigma_0}\Big)B(t) +\frac{1}{2}k_1^2B^2(t)+r\big(B'(t)-1-cB(t)\big)\Big]=0. \tag{45}$$

The equation (45) can be decomposed into the following two equations:

$$B'(t)-1-cB(t)=0, \quad B(T)=0; \tag{46}$$

$$A'(t)+\Big(\theta-\frac{k_1\mu_1}{\sigma_0}\Big)B(t)+\frac{1}{2}k_1^2B^2(t)=0, \quad A(T)=0; \tag{47}$$

Using the same approach as (29), the solution of (46) is given by

$$B(t)=\frac{1}{c}(e^{-c(T-t)}-1). \tag{48}$$

Further, for (47), after some integration, we obtain

$$\begin{aligned}A(t)=&\Big(\theta-\frac{k_1\mu_1}{\sigma_0}\Big)\int_t^T B(s)ds\\&+\frac{1}{2}k_1^2\int_t^T B^2(s)ds\\=&\Big(\frac{\theta}{c^2}-\frac{k_1\mu_1}{c^2\sigma_0}-\frac{k_1^2}{c^3}\Big)\cdot\Big(1-e^{-c(T-t)}\Big)\\&+\Big(\frac{k_1^2}{2c^2}-\frac{\theta}{c}+\frac{k_1\mu_1}{c\sigma_0}\Big)\Big(T-t\Big)\\&+\frac{k_1^2}{4c^3}(1-e^{-2c(T-t)}).\end{aligned} \tag{49}$$

In a result, the proof of lemma4 is completed. ■

Further, under equation (22), we obtain

$$\frac{V_x}{V_{xx}}=X(t)-g, \tag{50}$$

$$\frac{V_{x\eta}}{V_{xx}}=\phi(t)\big(X(t)-g\big), \tag{51}$$

$$\frac{V_{xr}}{V_{xx}}=h(t)\big(X(t)-g\big)-g_r. \tag{52}$$

To sum up, we get the following conclusion.

Theorem 5 *For a given l and C , if a solution of HJB equation (17) is given by $V(t,x,r,\eta)$, then the optimal investment and reinsurance strategies for the problem (12) and (14) are given by*

$$n^*(t)=-\frac{(X(t)-g(t,r))}{X(t)}\Big(\phi(t)+\frac{k}{\sigma_1^2}\Big),$$

$$m^*(t)=-\Big(\frac{k_1}{\sigma_0}h(t)+\frac{\mu_1}{\sigma_0^2}\Big)(X(t)-g(t,r))+\frac{k_1}{\sigma_0}g_r,$$

where, $g(t,r)=(\mu_1-\mu_0)\int_t^T\hat{g}(s,r)ds+l\hat{g}(t,r)$ is given by lemma (3) and lemma (4). Equation (33) is the solution of $\phi(t)$.

Remark 6 *In the investment-reinsurance strategies, we find that the optimal reinsurance strategy $m^*(t)$ is affected by both the parameters of reinsurance market and the parameters of interest rate model in financial market. However, the optimal reinsurance strategy is not affected by the parameters of the risky-asset, which is the same to the conclusion that the parameters of the risky asset have no influence on the optimal reinsurance strategy in Li et al.(2012). But, it has some different points to them. For example, since they don't consider the stochastic interest rate model, they obtain that the reinsurance strategy is only affected by the parameters of the reinsurance market. And, the parameters of insurance market have no effect on the optimal investment strategy. In our results, it is obviously that the optimal investment $n^*(t)$ is not only related to the parameters of financial market, it is also relevant to the parameters of insurance market.*

4 The Efficient Frontier

In this section, we will derive the explicit expression of the efficient frontier by the Lagrange duality theorem. Considering the problem (17), we get the minimized value of (14) as follows.

$$f(0,r_0,\eta_0)\big(x_0-g(0,r_0)\big)^2.$$

So, the minimized value of (12) is given by

$$\begin{aligned}&\bar{L}_{\min}(m^*(t),n^*(t),l)\\=&f(0,r_0,\eta_0)\big(x_0-g(0,r_0)\big)^2-(C-l)^2\\=&f(0,r_0,\eta_0)\Big(x_0-(\mu_1-\mu_0)\times\\&\int_0^T\hat{g}(s,r)ds-l\hat{g}(0,r_0)\Big)^2-(C-l)^2\\=&f(0,r_0,\eta_0)\big(\psi^2-2l\psi\hat{g}(0,r_0)+l^2\hat{g}^2(0,r_0)\big)\\&-(C^2-2Cl+l^2)\\=&\big(f(0,r_0,\eta_0)\hat{g}^2(0,r_0)-1\big)l^2\\&-2\big(\psi f(0,r_0,\eta_0)\hat{g}(0,r_0)-C\big)l\\&+\psi^2f(0,r_0,\eta_0)-C^2,\end{aligned} \tag{53}$$

where,

$$\psi=x_0-(\mu_1-\mu_0)\int_0^T\hat{g}(s,r)ds. \tag{54}$$

Noting that

$$\begin{aligned}&f(0,r_0,\eta_0)\hat{g}^2(0,r_0)-1\\=&\Psi(0)\exp\{2A(0)+\phi(0)\eta+(h(0)+2B(0))r\}\\&-1,\end{aligned} \tag{55}$$

we have the following conclusion.

Lemma 7 *Under the condition:* $\frac{k_1\mu_1}{\sigma_0}+\frac{k_1^2}{2c}<\theta<\frac{2k_1\mu_1}{\sigma_0}$, $\phi(t)$, $B(t)$, $A(t)$ *and* $\Psi(t)$ *increase with respect to the variable t. But,* $h(t)$ *decreases with respect to t. Moreover, we have:* $\phi(t)<0$, $B(t)<0$, $A(t)<0$, $h(t)>0$ *and* $\Psi(t)<1$, *for* $\forall t\in[0,T)$. *But, we have* $h(t)+2B(t)=0$.

Proof: Differentiating $\phi(t)$, $B(t)$, $A(t)$, $\Phi(t)$, and $h(t)$ with respect to t, we have

$$\begin{aligned}\phi'(t)=&\frac{\frac{k^2}{\sigma_1^2}(\phi_1-\phi_2)^2\exp\{\frac{1}{2}\sigma_1^2(\phi_1-\phi_2)(T-t)\}}{(\phi_1-\phi_2\exp\{\frac{1}{2}\sigma_1^2(\phi_1-\phi_2)(T-t)\})^2},\\B'(t)=&e^{-c(T-t)},\\A'(t)=&-(\theta-\frac{k_1\mu_1}{\sigma_0})B(t)-\frac{1}{2}k_1^2B^2(t)\\=&-B(t)\left(\theta-\frac{k_1\mu_1}{\sigma_0}+\frac{1}{2}k_1^2B(t)\right)\\=&-B(t)\left(\theta-\frac{k_1\mu_1}{\sigma_0}+\frac{k_1^2}{2c}(e^{-c(T-t)}-1)\right)\\=&-B(t)\left(\frac{2\theta\sigma_0c-2ck_1\mu_1-k_1^2\sigma_0}{2\sigma_0c}\right.\\&\left.+\frac{k_1^2}{2c}e^{-c(T-t)}\right),\\h'(t)=&-2e^{-c(T-t)},\\\Psi'(t)=&-\Psi(t)\left(\left(\theta-\frac{2k_1\mu_1}{\sigma_0}\right)h(t)+b\phi(t)\right.\\&\left.-\frac{\mu_1^2}{\sigma_0^2}-\frac{1}{2}k_1^2h^2(t)\right).\end{aligned}$$

Clearly, we find that $\phi'(t)>0$, $B'(t)>0$, and obtain $\phi(t)<\phi(T)=0$, $B(t)<B(T)=0$. In addition, noticing that under the condition: $\theta>\frac{k_1\mu_1}{\sigma_0}+\frac{k_1^2}{2c}$, we have $A'(t)>0$ and $A(t)<A(T)=0$. But, it is obviously that $h'(t)<0$, then we get $h(t)>h(T)=0$, for $\forall t\in[0,T)$. From (36), we find that $\Psi(t)>0$. Thus, under the condition $\theta<\frac{2k_1\mu_1}{\sigma_0}$, $\Psi'(t)>0$, then we obtain $\Psi(t)<\Psi(T)=1$. From (35) and (48), it is clear that $h(t)+2B(t)=0$.

Therefore, Lemma 7 holds. ■

According to Lemma 7 , under the condition $\frac{k_1\mu_1}{\sigma_0}+\frac{k_1^2}{2c}<\theta<\frac{2k_1\mu_1}{\sigma_0}$, we obtain

$$\begin{aligned}&f(0,r_0,\eta_0)\hat{g}^2(0,r_0)-1\\=&\Psi(0)\exp\{2A(0)+\phi(0)\eta+(h(0)+2B(0))r\}\\&-1<0.\end{aligned} \tag{56}$$

Therefore, $\bar{L}_{\min}(m^*(t),n^*(t),l)$ can be maximized when l is given by

$$l^*=\frac{\psi f(0,r_0,\eta_0)\hat{g}(0,r_0)-C}{f(0,r_0,\eta_0)\hat{g}^2(0,r_0)-1}. \tag{57}$$

In addition, the maximized value of

$$\bar{L}_{\min}(m^*(t),n^*(t),l)$$

is obtained

$$\begin{aligned}&\bar{L}_{\max\min}(m^*,n^*,l^*)\\=&\frac{f(0,r_0,\eta_0)\hat{g}^2(0,r_0)}{1-f(0,r_0,\eta_0)\hat{g}^2(0,r_0)}(C-\psi\hat{g}^{-1}(0,r_0))^2.\end{aligned} \tag{58}$$

As a result, we summarize the above results in the following proposition.

Theorem 8 *For a given constant C, under the conditions* $\frac{k_1\mu_1}{\sigma_0}+\frac{k_1^2}{2c}<\theta<\frac{2k_1\mu_1}{\sigma_0}$, *the optimal reinsurance and investment strategies for the mean-variance problem (9) corresponding to* $E(X(T))=C$ *are given by*

$$n^*(t)=-\frac{(X(t)-g(t,r))}{X(t)}(\phi(t)+\frac{k}{\sigma_1^2}),$$

$$m^*(t)=-\left(\frac{k_1}{\sigma_0}h(t)+\frac{\mu_1}{\sigma_0^2}\right)(X(t)-g(t,r))+\frac{k_1}{\sigma_0}g_r,$$

with the efficient frontier given by

$$\begin{aligned}Var(X(T))=&\frac{1}{\Psi^{-1}(0)e^{-2A(0)-\phi(0)\eta_0}-1}\\&\cdot\left(E(X(T))-\psi e^{-A(0)-B(0)r_0}\right)^2,\end{aligned}$$

where,

$$\begin{aligned}\psi=&x_0-(\mu_1-\mu_0)\int_0^T\hat{g}(s,r)ds,\\l^*=&\frac{\psi\Psi(0)e^{A(0)+\phi(0)\eta_0}-C}{\Psi(0)e^{2A(0)+\phi(0)\eta_0}-1},\\g(t,r)=&(\mu_1-\mu_0)\int_t^Te^{A(s)+B(s)r(s)}ds\\&+l^*e^{A(t)+B(t)r(t)}.\\g_r=&(\mu_1-\mu_0)\int_t^TB(s)e^{A(s)+B(s)r(s)}ds\\&+l^*B(t)e^{A(t)+B(t)r(t)}.\end{aligned}$$

Here, $\Psi(t)$, $\phi(t)$, $h(t)$, $A(t)$, and $B(t)$ are given by Lemma 2 and Lemma 3, respectively.

Remark 9 *From Theorem 8, we can draw some conclusions. (i) the optimal investment strategy $n^*(t)$ is correlated with the parameters of the insurance market μ_0, μ_1, σ_0, the parameters of interest rate model k_1, c, θ, and the parameters of risky-asset model a, k, σ_1. The reason of this result is that the Brownian motion describing the dynamic of surplus process is the same as that describing the dynamics of interest rate. But the expectation of volatility b has no influence on the investment strategy, it surprised us. (ii) the reinsurance strategy $m^*(t)$ is affected by the parameters of reinsurance market μ_0, μ_1,σ_0 and the parameters of interest rate k_1, c, θ. (iii) The efficient frontier has correlation with all the parameters. That is, the risk for an insurer depends on the volatilities of interest rate, risky asset, and the insurers' surplus.*

Remark 10 *The optimal polices and efficient frontier are not deterministic functions, but are dynamic functions. They depend on the stochastic interest rate $r(t)$, but does not depend on the stochastic volatility process $\eta(t)$. However, the optimal investment strategy and the efficient frontier are related to the parameters of the dynamics of $\eta(t)$, but the optimal reinsurance strategy is not affected by them.*

Remark 11 *The efficient frontier is a straight line in the mean-variance deviation diagram, no matter at which state interest rate is. Let $\sigma[X(T)]$ be the standard deviation of the terminal wealth, then we have*

$$\begin{aligned} E(X(T)) =&\psi e^{-A(0)-B(0)r_0} + \sigma[X(T)] \\ &\times \sqrt{\Psi^{-1}(0)e^{-2A(0)-\phi(0)\eta_0} - 1}, \end{aligned}$$

which is called the capital market line.

When $\sigma[X(T)] = 0$, then we get

$$E(X(T)) = \psi e^{-A(0)-B(0)r_0}.$$

Therefore, when C runs over $[\psi e^{-A(0)-B(0)r_0}, +\infty]$, the efficient frontier consists of all the points.

5 Special Cases

In this section, we consider some special cases of our model in the previous sections, which is called the original model. The results of original model will be reduced to the following special cases.

Special case 1. Assumed that the volatility is constant, where $\eta(t) \equiv \eta$, $\forall t \in [0, T]$. In this case, the HJB equation (17) can be written as the following form:

$$\begin{aligned} &\underset{\pi\in\Pi}{Min}\Big(\big[\mu_0 - (1-m(t))\mu_1 + rx + k\eta n(t)x\big]V_x \\ &V_t + (\theta - cr)V_r + \big[\frac{1}{2}\sigma_1^2 n(t)^2x^2\eta + \frac{1}{2}\sigma_0^2 m(t)^2\big]V_{xx} \\ &+ \frac{1}{2}k_1^2 V_{rr} + \sigma_0 m(t)k_1 V_{xr}\Big) = 0, \\ &0 < t < T. \end{aligned} \tag{59}$$

Differentiating with respect to $m(t)$ and $n(t)$, the optimizer is given

$$m^*(t) = -\frac{k_1 V_{xr}}{\sigma_0 V_{xx}} - \frac{\mu_1}{\sigma_0^2}\cdot\frac{V_x}{V_{xx}}, \tag{60}$$

$$n^*(t) = -\frac{k}{\sigma_1^2}\cdot\frac{V_x}{xV_{xx}}. \tag{61}$$

Substituting (60) and (61) back into (59) , after some simplification ,the HJB equation (59) is changed into the following equation:

$$\begin{aligned} &V_t + (\mu_0 - \mu_1 + rx)V_x + (\theta - cr)V_r \\ &-\frac{1}{2}\Big(\frac{\mu_1^2}{\sigma_0^2} + \frac{k^2\eta}{\sigma_1^2}\Big)\frac{V_x^2}{V_{xx}} + \frac{1}{2}k_1^2 V_{rr} \\ &-\frac{1}{2}k_1^2\frac{V_{xr}^2}{V_{xx}} - \frac{k_1\mu_1}{\sigma_0}\cdot\frac{V_xV_{xr}}{V_{xx}} = 0, \end{aligned} \tag{62}$$

with terminal condition: $V(T, x, r, \eta) = (x - l)^2$.

Letting the value function V be the following form

$$V(t, r, x,) = \tilde{f}(t, r)\big[x - \tilde{g}(t, r)\big]^2, \tag{63}$$

with $\tilde{f}(T, r) = 1$ and $\tilde{g}(T, r) = l$. Putting (63) into (62) leads to

$$\begin{aligned} &(x-\tilde{g})^2\Big[\tilde{f}_t + 2r\tilde{f} + (\theta - cr)\tilde{f}_r - \Big(\frac{\mu_1^2}{\sigma_0^2} + \frac{k^2\eta}{\sigma_1^2}\Big)\tilde{f} \\ &- k_1^2\frac{\tilde{f}_r^2}{\tilde{f}} - \frac{2k_1\mu_1}{\sigma_0}\tilde{f}_r + \frac{1}{2}k_1^2\tilde{f}_{rr}\Big] \\ &- 2(x-\tilde{g})\tilde{f}\Big[\tilde{g}_t - r\tilde{g} + (\theta - \frac{k_1\mu_1}{\sigma_0} - cr)\tilde{g}_r \\ &+ \frac{1}{2}k_1^2\tilde{g}_{rr} + (\mu_1 - \mu_0)\Big] = 0. \end{aligned} \tag{64}$$

Then, we can get two equations

$$\begin{aligned} &\tilde{f}_t + 2r\tilde{f} + (\theta - cr)\tilde{f}_r - \Big(\frac{\mu_1^2}{\sigma_0^2} + \frac{k^2\eta}{\sigma_1^2}\Big)\tilde{f} \\ &- k_1^2\frac{\tilde{f}_r^2}{\tilde{f}} - \frac{2k_1\mu_1}{\sigma_0}\tilde{f}_r + \frac{1}{2}k_1^2\tilde{f}_{rr} = 0, \\ &\tilde{f}(T, r) = 1. \end{aligned} \tag{65}$$

$$\begin{aligned}&\tilde{g}_t - r\tilde{g} + (\theta - \frac{k_1\mu_1}{\sigma_0} - cr)\tilde{g}_r + \frac{1}{2}k_1^2\tilde{g}_{rr}\\&+ (\mu_1 - \mu_0) = 0, \quad \tilde{g}(T,r) = l.\end{aligned} \tag{66}$$

A solution of the equation(65) is conjectured as

$$\tilde{f}(t,r) = \tilde{\Psi}(t)\exp\{\tilde{h}(t)r\}. \tag{67}$$

After some calculations, we obtain

$$\begin{aligned}&\tilde{\Psi}(t)r\Big(2 + \tilde{h}'(t) - c\tilde{h}(t)\Big) + \tilde{\Psi}'(t)\\&+ \tilde{\Psi}(t)\Bigg(\Big(\theta - \frac{2k_1\mu_1}{\sigma_0}\Big)\tilde{h}(t) - \Big(\frac{\mu_1^2}{\sigma_0^2} + \frac{k^2\eta}{\sigma_1^2}\Big)\\&- \frac{1}{2}k_1^2\tilde{h}^2(t)\Bigg) = 0.\end{aligned} \tag{68}$$

Comparing the coefficients on the both sides of (68), we have

$$\tilde{h}'(t) - c\tilde{h}(t) + 2 = 0, \quad \tilde{h}(T) = 0; \tag{69}$$

$$\begin{aligned}&\tilde{\Psi}'(t) + \tilde{\Psi}(t)\Bigg(\Big(\theta - \frac{2k_1\mu_1}{\sigma_0}\Big)\tilde{h}(t) - \Big(\frac{\mu_1^2}{\sigma_0^2} + \frac{k^2\eta}{\sigma_1^2}\Big)\\&- \frac{1}{2}k_1^2\tilde{h}^2(t)\Bigg) = 0, \quad \tilde{\Psi}(T) = 1.\end{aligned} \tag{70}$$

And, comparing (69) to (29), we know $\tilde{h}(t) = h(t)$.

After some integrals, a solution of (70) is as follows

$$\begin{aligned}\tilde{\Psi}(t) = \exp\Bigg(&\Big(\theta - \frac{2k_1\mu_1}{\sigma_0}\Big)\int_t^T h(s)ds\\&- \frac{1}{2}k_1^2\int_t^T h^2(s)ds - \Big(\frac{\mu_1^2}{\sigma_0^2} + \frac{k^2\eta}{\sigma_1^2}\Big)(T-t)\Bigg),\end{aligned} \tag{71}$$

From (66) and (25), we find that $\tilde{g}(t,r) = g(t,r)$.

Then, we get the following corollary.

Corollary 12 *Assume that the volatility of risky-asset is constant, the optimal investment strategy $n^*(t)$ by the first expression in Theorem 8 is changed into*

$$n^*(t) = -\frac{k}{\sigma_1^2}\cdot\frac{(X(t) - g(t,r))}{X(t)},$$

and the optimal reinsurance strategy $m^*(t)$ given by the second expression in Theorem 8 does not change. The efficient frontier is given by

$$\begin{aligned}Var(X(T)) =& \frac{1}{\tilde{\Psi}^{-1}(0)e^{-2A(0)} - 1}\\&\cdot\Big(E(X(T)) - \psi e^{-A(0)-B(0)r_0}\Big)^2.\end{aligned}$$

Remark 13 *Comparing the corollary 12 to the theorem 8, $\phi(t)$ is changed into 0 as $\eta(t)$ is assumed to be a constant η, and $h(t)$, $g(t,r)$ don't change. So, the reinsurance strategy doesn't changed. It shows that the volatility of risky-asset have no influence on the optimal reinsurance strategy. And, we can conclude that $\phi(t)$ is an impact factor of the investment strategy produced by the parameters of volatility process $\eta(t)$. In addition, the efficient frontier is also affected.*

Special case 2. Suppose that the interest rate is constant, where $r(t) \equiv r$, $\forall t \in [0,T]$. Similar to the derivation of the original model, we solve the problem for this special case. Under this assumption, $\theta = 0$, $c = 0$, $k_1 = 0$, the equation (24) and (25) are changed into the following two equations:

$$\begin{aligned}&\bar{f}_t + 2r\bar{f} + (b - a\eta)\bar{f}_\eta - \Big(\frac{\mu_1^2}{\sigma_0^2} + \frac{k^2\eta}{\sigma_1^2}\Big)\bar{f}\\&- \sigma_1^2\eta\frac{\bar{f}_\eta^2}{\bar{f}} - 2k\eta\bar{f}_\eta + \frac{1}{2}\sigma_1^2\eta\bar{f}_{\eta\eta} = 0,\\&\bar{f}(T,\eta) = 1;\end{aligned} \tag{72}$$

$$\bar{g}_t - r\bar{g} + (\mu_1 - \mu_0) = 0, \quad \bar{g}(T) = l. \tag{73}$$

Assumed that the solution of (72) is the following form:

$$\bar{f}(t,\eta) = \bar{\Psi}(t)\exp\{\bar{\phi}(t)\eta\}, \tag{74}$$

with boundary conditions given by $\bar{\Psi}(T) = 1$ and $\bar{\phi}(T) = 0$. Substituting (74) into (72), we get

$$\begin{aligned}&\bar{\Psi}(t)\eta\Big(\bar{\phi}'(t) - a\bar{\phi}(t) - \frac{k^2}{\sigma_1^2} - 2k\bar{\phi}(t) - \frac{1}{2}\sigma_1^2\bar{\phi}^2(t)\Big)\\&+ \bar{\Psi}'(t) + \bar{\Psi}(t)\Big(b\bar{\phi}(t) + 2r - \frac{\mu_1^2}{\sigma_0^2}\Big) = 0,\end{aligned} \tag{75}$$

Then, the above equation can be decomposed into

$$\begin{aligned}&\bar{\phi}'(t) - a\bar{\phi}(t) - \frac{k^2}{\sigma_1^2} - 2k\bar{\phi}(t) - \frac{1}{2}\sigma_1^2\bar{\phi}^2(t) = 0,\\&\bar{\phi}(T) = 0;\end{aligned} \tag{76}$$

$$\begin{aligned}&\bar{\Psi}'(t) + \bar{\Psi}(t)\Big(b\bar{\phi}(t) + 2r - \frac{\mu_1^2}{\sigma_0^2}\Big) = 0,\\&\bar{\Psi}(T) = 1.\end{aligned} \tag{77}$$

Comparing (76) to (28), we find that $\bar{\phi}(t) = \phi(t)$. And after some integrals, the solution of (77) is

$$\bar{\Psi}(t) = \exp\Big(b\int_t^T \phi(s)ds + \Big(2r - \frac{\mu_1^2}{\sigma_0^2}\Big)(T-t)\Big). \tag{78}$$

For (73), it is a first-order linear differential equation, after some calculations, we obtain

$$\bar{g}(t) = \frac{\mu_0 - \mu_1}{r}(1 - e^{r(T-t)}) + le^{-r(T-t)}. \quad (79)$$

So, in this special case, the conclusion is as follows.

Corollary 14 *Supposed that the interest rate is constant, the optimal investment and reinsurance strategy are the following form*

$$n^*(t) = -\frac{(X(t) - \bar{g}(t))}{X(t)}\left(\phi(t) + \frac{k}{\sigma_1^2}\right),$$

$$m^*(t) = -\frac{\mu_1}{\sigma_0^2}(X(t) - \bar{g}(t)),$$

with the efficient frontier

$$Var(X(T)) = \frac{e^{\phi(0)\eta_0}}{\bar{\Psi}^{-1}(0) - e^{\phi(0)\eta_0 - 2rT}} \cdot \left(e^{-rT}E(X(T)) - \bar{\psi}\right)^2,$$

where, $\bar{\psi}(t) = x_0 - \frac{\mu_0 - \mu_1}{r}(1 - e^{-rT})$.

Remark 15 *From Lemma 3, we know that $g(t, r)$ is a dynamic function produced by the interest rate process $r(t)$. when $r(t)$ becomes to a constant r, $g(t, r)$ is changed into the function of t in corollary 9 and $h(t)$ becomes to 0. It suggests that $h(t)$ is an impact factor of the reinsurance strategy produced by the parameters of interest rate model. Further, the efficient frontier is affected. So, the optimal strategies and efficient frontier are all changed as the interest rate is fixed. It means that the stochastic interest rate has effect on optimal reinsurance and investment strategy.*

6 Numerical Analysis

In this section,a numerical example is provided to illustrate the impact of the parameters of interest rate, stochastic volatility and surplus process on the optimal reinsurance-investment strategy and the efficient frontier. Throughout this section, unless otherwise stated, the basic parameters are given by $\mu_0 = 0.8$, $\sigma_0 = 1$, $\mu_1 = 1.2$, $x_0 = 100$, $\theta = 0.1$, $c = 0.5$, $k_1 = 0.1$, $r(0) = 0.05$, $\eta_0 = 0.1$, $k = 1.5$, $a = 1.5$, $b = 0.06$, $\sigma_1 = 1.5$, $T = 1$, $t = 0$, $C = 107$.

6.1 Sensitivity Analysis on the Optimal Reinsurance Strategy

First, we study the effect of the parameters on the optimal reinsurance strategy. Some conclusions are as follows:

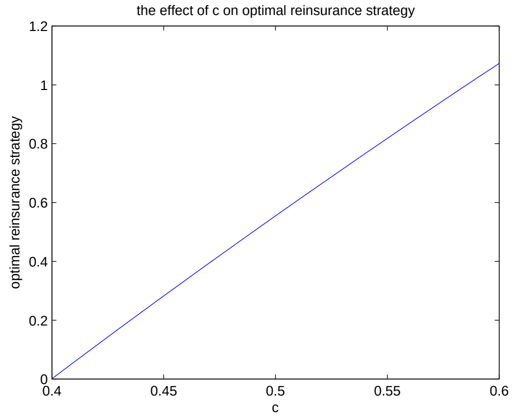

Figure 1: The effect of c on the optimal reinsurance strategy

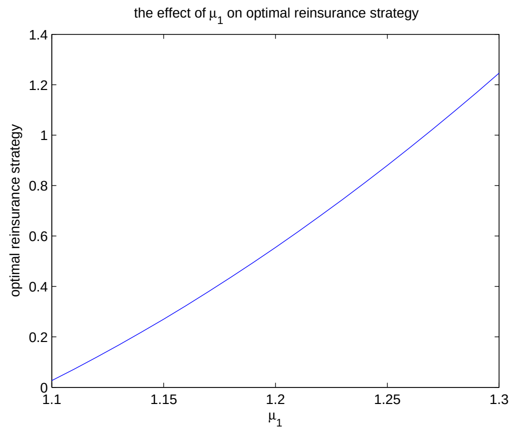

Figure 2: The effect of μ_1 on the optimal reinsurance strategy

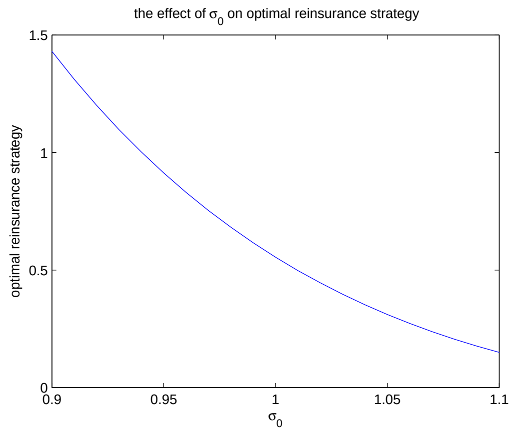

Figure 3: The effect of σ_0 on the optimal reinsurance strategy

(a1) Figure 1 shows that the parameter c of the interest rate model has an effect on the optimal reinsurance strategy $m^*(t)$, and $m^*(t)$ increases with respect to c. From the interest rate model, it is clear that as c becomes larger, the expectation of interest rate will be smaller. This means that when the expectation of interest rate goes down, the income of the insurance company will reduce. So, under the certain benefits, the insurer will cut down the amount of the reinsurance to meet the requirements of earning in the short term.

(a2) Figure2 and Figure 3 illustrate the parameters of surplus process how affect on the optimal reinsurance strategy. Figure 2 shows that $m^*(t)$ increases with respect to the parameter μ_1. It tells us that the larger μ_1 becomes, the less reinsurance or the more new business the insurer will purchase. Moreover, from Figure 3, we find that $m^*(t)$ decreases with respect to the parameter σ_0, which reflects the volatility of surplus process has impact on the optimal reinsurance policy. In other words, the larger σ_0 becomes, the more reinsurance is. It means that as the volatility becomes larger, the risk of surplus process increases. Then, the insurer will buy more reinsurance to spread the risk.

6.2 Sensitivity Analysis on the Optimal Investment Strategy

In this subsection, we give the impact of some parameters on the optimal investment $n^*(t)$, which is the optimal proportion of investing in the risky asset.

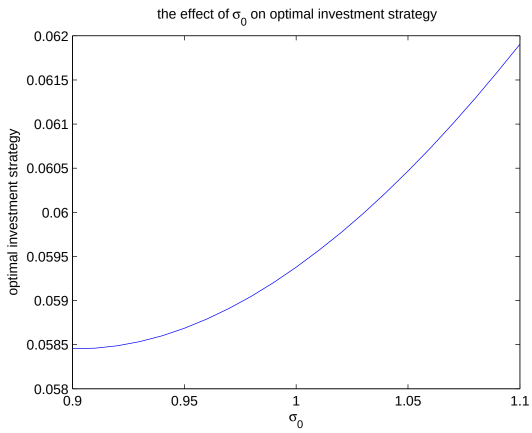

Figure 4: The effect of σ_0 on the optimal investment strategy

(b1) Figure 4 suggests that $n^*(t)$ increases with respect to σ_0. From (a2), we know that when σ_0 is larger, the insurer will buy more reinsurance to spread the risk. After having more reinsurance, the insurer's revenue will become lower. In order to achieve a fixed profit, the insurer will increase the proportion of investing the risky asset.

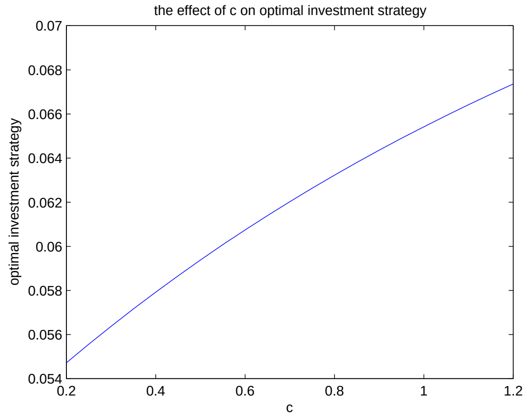

Figure 5: The effect of c on the optimal investment strategy

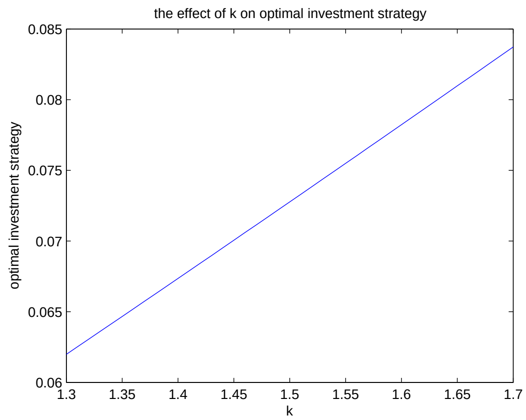

Figure 6: The effect of k on the optimal investment strategy

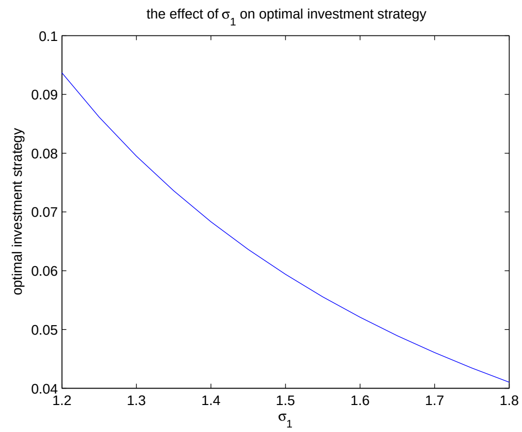

Figure 7: The effect of σ_1 on the optimal investment strategy

(b2) From Figure 5, we find that $n^*(t)$ increases as the parameter c becomes larger. In fact, the model of interest rate shows that the larger c, the smaller expectation of interest rate. Hence ,the insurer will buy more risky asset to get more profits. Figure 6 shows that $n^*(t)$ is all increasing when the parameter k is increasing. As the matter of the risky asset model, if k becomes larger, the more expectation of the risky asset. So, the insurer will buy more risky asset as k is increasing.

(b3) Figure 7 describes the parameter σ_1 which is interpreted as the volatility of the risky asset how effect on the optimal investment strategy. From it, as σ_1 is increasing, $n^*(t)$ is decreasing. It means that for the larger volatility of the risky asset , the insurer will purchase smaller risky asset to reduce the risk.

6.3 Sensitivity Analysis on the Efficient Frontier

Figure 8, Figure9, and Figure 10 illustrate the relationships between the standard deviation $\sigma[X(T)]$ and the expected return C. According to our observation on this Figures, we get the following some instructive conclusions:

(c1) $\sigma[X(T)]$ increases with respect to the parameters k_1 for a fixed expected return. From the points of economic implication of k_1, the larger value of k_1 will lead to a larger risk of interest rate, which leads to a larger risk level.

(c2) $\sigma[X(T)]$ also increases with respect to the parameters σ_0 for a fixed expected return. And from (b1), we know that a larger value of σ_0 will lead to a greater amount in the risky asset. It means that the risks from the risky asset and surplus process will greatly increase.

(c3) $\sigma[X(T)]$ is a decreasing function of the parameter σ_1 for the same expected return. From the conclusion (b3) in the previous subsection, we find that the larger value of σ_1, the smaller amount in the risky asset. It suggests that the risks resulted from the risky asset will descend, which leads to a smaller risk level.

7 Conclusions

Reinsurance is important for the insurer recently. This paper takes the proportional reinsurance into consideration and focuses on a dynamic mean-variance reinsurance-investment problem in the stochastic interest rate and stochastic volatility environment, in which the interest rate is modeled by the Vasicek model and the volatility of the risky-asset is driven by the Heston model. In order to hedge the risk of financial market, we purchase the risk-free asset and the risky-asset, and the proportional reinsurance

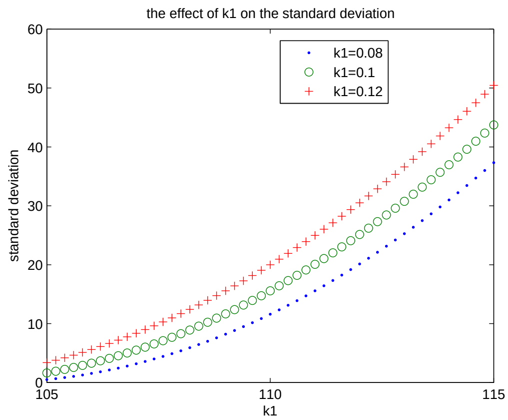

Figure 8: The effect of k_1 on the standard deviation

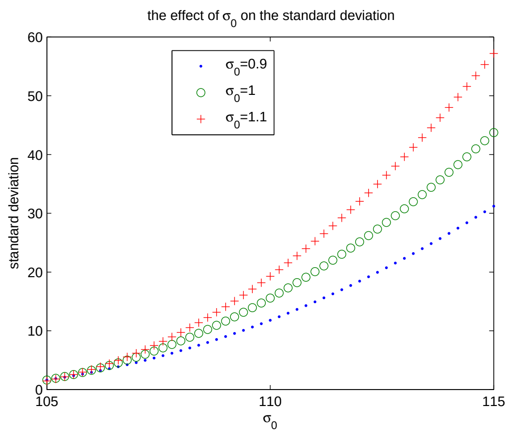

Figure 9: The effect of σ_0 on standard deviation

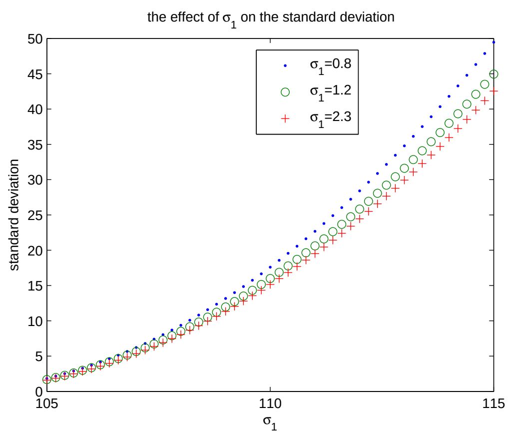

Figure 10: The effect of σ_1 on standard deviation

is considered to reduce the risk of the insurance. By employing dynamic programming principle and Lagrange duality theorem, we obtain the optimal reinsurance and investment strategies and the efficient frontier under the mean-variance criterion. Moreover, we consider some special cases and give the optimal results. Finally, we present a numerical example to illustrate the effect of model parameters on the optimal strategy and the efficient frontier at the end of the paper. Some interesting and main conclusions are as follows: (i) the optimal reinsurance strategy is not only affected by the parameters of reinsurance, it is also related to the parameters of interest rate model. But it is not influenced by the parameters of the risky asset model. And, $h(t)$ is an impact factor of reinsurance strategy produced by the parameters of interest rate model. (ii) the optimal investment strategy is not only affected by the parameters of the financial market, but also influenced by the parameters of insurance market. And, all the parameters have effect on the efficient frontier. (iii) the efficient reinsurance-investment strategies and the efficient frontier are all dynamic functions which depend on the interest rate process $r(t)$. (iv) The efficient frontier in the mean-standard deviation diagram is still a straight line, no matter at which state the interest rate is.

Acknowledgements: This research is supported by China Postdoctoral Science Foundation Funded Project (No.2014M560185), Tianjin Natural Science Foundation of China (No.15JCQNJC04000), and Humanities and Social Science Research Youth Foundation of Ministry of Education of China (No.11YJC790006).

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