Positive Periodic Solutions in Shifts $\delta_\pm$ for a Neutral Delay Logistic Equation on Time Scales

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Abstract: Let $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_\pm$ with period $P \in (t_0, \infty)_\mathbb{T}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed. By using a fixed point theorem of strict-set-contraction, some criteria are established for the existence of positive periodic solutions in shifts $\delta_\pm$ for a neutral delay logistic equation on time scales of the form

$$x^\Delta(t) = x(t)\left[r(t) - a(t)x(t) - \sum_{j=1}^n b_j(t)x(\delta_-(\tau_j, t)) - \sum_{j=1}^n c_j(t)x^\Delta(\delta_-(\xi_j, t))\right], t \in \mathbb{T}.$$

Finally, two numerical examples are presented to illustrate the feasibility and effectiveness of the results.

Key–Words: Positive periodic solution; Neutral delay logistic equation; Strict-set-contraction; Shift operator; Time scale.

1 Introduction

In 1993, Kuang [1] proposed an open problem (open problem 9.2) to obtain sufficient conditions for the existence of positive periodic solutions to

$$\begin{aligned} x'(t) &= x(t)\big[a(t) - \beta(t)x(t) - b(t)x(t - \tau(t)) \\ &\quad -c(t)x'(t - \tau(t))\big], \end{aligned}$$

where a, β, b, c, τ are nonnegative continuous periodic functions. Since then, different classes of neutral functional differential and difference equations have been extensively studied; see, for example, [2-5].

However, in the natural world, there are many species whose developing processes are both continuous and discrete. Hence, using the only differential equation or difference equation can't accurately describe the law of their developments, see, for example, [6,7]. Therefore, there is a need to establish correspondent dynamic models on new time scales.

The theory of calculus on time scales (see [8] and references cited therein) was initiated by Stefan Hilger [9] in order to unify continuous and discrete analysis. Therefore, the study of dynamic equations on time scales, which unifies differential, difference, h-difference, and q-differences equations and more, has received much attention; see [10-14].

The existence problem of periodic solutions is an important topic in qualitative analysis of functional dynamic equations. Up to now, there are only a few results concerning periodic solutions of neutral dynamic equations on time scales; see, for example, [15,16]. In these papers, authors considered the existence of periodic solutions for dynamic equations on time scales satisfying the condition "there exists a $\omega > 0$ such that $t \pm \omega \in \mathbb{T}, \forall t \in \mathbb{T}$." Under this condition all periodic time scales are unbounded above and below. However, there are many time scales such as $\overline{q^\mathbb{Z}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$ which do not satisfy the condition. Adıvar and Raffoul introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm \omega$ for a fixed $\omega > 0$. They defined a new periodicity concept with the aid of shift operators $\delta_\pm$ which are first defined in [17] and then generalized in [18].

Recently, by using the cone theory techniques, many researchers studied the existence of positive periodic solutions in shifts $\delta_\pm$ for some nonlinear first-order functional dynamic equations on time scales; see [19-22]. However, to the best of our knowledge, there are few papers published on the existence of positive periodic solutions in shifts $\delta_\pm$ for a neutral functional differential equation on time scales.

The main purpose of this paper is by using a fixed point theorem of strict-set-contraction to establish some criteria to guarantee the existence of pos-

itive periodic solutions in shifts $\delta_\pm$ of the following neutral delay logistic equation on time scales

$$
x^\Delta(t) = x(t)\Bigg[r(t) - a(t)x(t)
$$
$$
- \sum_{j=1}^n b_j(t)x(\delta_-(\tau_j, t))
$$
$$
- \sum_{j=1}^n c_j(t)x^\Delta(\delta_-(\xi_j, t))\Bigg], \quad (1)
$$

where $t \in \mathbb{T}$, $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_\pm$ with period $P \in [t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is non-negative and fixed; $r, a, b_j, c_j \in C(\mathbb{T}, (0, \infty))(j = 1, 2, \ldots, n)$ are Δ-periodic in shifts $\delta_\pm$ with period ω; $\tau_j, \xi_j(j = 1, 2, \ldots, n)$ are fixed if $\mathbb{T} = \mathbb{R}$ and $\tau_j, \xi_j \in [P, \infty)_{\mathbb{T}}$ if $\mathbb{T}$ is periodic in shifts $\delta_\pm$ with period P.

For convenience, we introduce the notation

$$
\Theta := e_r(t_0, \delta_+^\omega(t_0)),
$$
$$
\Gamma := \int_{t_0}^{\delta_+^\omega(t_0)} \Bigg[\Theta a(s) + \sum_{j=1}^n \Theta b_j(s) - \sum_{j=1}^n c_j(s)\Bigg]\Delta s,
$$
$$
\Pi := \int_{t_0}^{\delta_+^\omega(t_0)} \Bigg[a(s) + \sum_{j=1}^n b_j(s) + \sum_{j=1}^n c_j(s)\Bigg]\Delta s,
$$

and

$$
f^M = \sup_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} f(t), \quad f^m = \inf_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} f(t),
$$

where f is a continuous ω-periodic function in shifts $\delta_\pm$.

Throughout this paper, we assume that

(H_1) $\Theta := e_r(t_0, \delta_+^\omega(t_0)) < 1$;

(H_2) $\Theta a(t) + \sum_{j=1}^n \Theta b_j(t) - \sum_{j=1}^n c_j(t) \geq 0$;

(H_3) $(1+r^m)\frac{\Theta^2\Gamma}{1-\Theta} \geq \sup_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} \Bigg\{a(t) + \sum_{j=1}^n b_j(t) + \sum_{j=1}^n c_j(t)\Bigg\}$;

(H_4) $\frac{\Pi(r^M-1)}{\Theta(1-\Theta)} \leq \inf_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} \Bigg\{\Theta a(t) + \sum_{j=1}^n \Theta b_j(t) - \sum_{j=1}^n c_j(t)\Bigg\}$;

(H_5) $\frac{1-\Theta}{\Theta^2\Gamma}\sum_{j=1}^n c_j^M < 1$.

2 Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$
\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},
$$
$$
\rho(t) = \sup\{s \in \mathbb{T} : s < t\}
$$
$$
\mu(t) = \sigma(t) - t.
$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C(\mathbb{T}) = C(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is continuous is denoted by $C^1(\mathbb{T}) = C^1(\mathbb{T}, \mathbb{R})$.

For the basic theories of calculus on time scales, see [8].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. If r is a regressive function, then the generalized exponential function e_r is defined by

$$
e_r(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\}
$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$
\xi_h(z) = \begin{cases} \frac{\text{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}
$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$
p \oplus q = p + q + \mu pq, \quad \ominus p = -\frac{p}{1+\mu p}, \quad p \ominus q = p \oplus (\ominus q).
$$

Lemma 1. [8] *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*
(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $(e_{\ominus p}(t, s))^\Delta = (\ominus p)(t)e_{\ominus p}(t, s)$.

The following definitions, lemmas about the shift operators and the new periodicity concept for time scales which can be found in [20,23].

Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}^*$ be a fixed number, define operators $\delta_\pm : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$. The operators δ_+ and δ_- associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be forward and backward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, \infty)_{\mathbb{T}}$ in $\delta_\pm(s, t)$ is called the shift size. The value $\delta_+(s, t)$ and $\delta_-(s, t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets

$$\mathbb{D}_\pm := \{(s, t) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_\mp(s, t) \in \mathbb{T}^*\}$$

are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Definition 2. [23] *(Periodicity in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$ if there exists $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if*

$$P := \inf\{p \in (t_0, \infty)_{\mathbb{T}^*} : (p, t) \in \delta_\pm, \forall t \in \mathbb{T}^*\} \neq t_0,$$

then P is called the period of the time scale $\mathbb{T}$.

Definition 3. [23] *(Periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ and $f(\delta_\pm^\omega(t)) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.*

Definition 4. [23] *(Δ-periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$, the shifts $\delta_\pm^\omega$ are Δ-differentiable with rd-continuous derivatives and $f(\delta_\pm^\omega(t))\delta_\pm^{\Delta\omega}(t) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.*

Lemma 5. [23] $\delta_+^\omega(\sigma(t)) = \sigma(\delta_+^\omega(t))$ *and* $\delta_-^\omega(\sigma(t)) = \sigma(\delta_-^\omega(t))$ *for all* $t \in \mathbb{T}^*$.

Lemma 6. [20] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^\omega$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $\omega \in [P, \infty)_{\mathbb{T}^*}$ and $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period ω. Then*

(i) $e_p(\delta_\pm^\omega(t), \delta_\pm^\omega(t_0)) = e_p(t, t_0)$ *for* $t, t_0 \in \mathbb{T}^*$;

(ii) $e_p(\delta_\pm^\omega(t), \sigma(\delta_\pm^\omega(s))) = e_p(t, \sigma(s)) = \frac{e_p(t,s)}{1+\mu(t)p(t)}$ *for* $t, s \in \mathbb{T}^*$.

Lemma 7. [23] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P, and let f be a Δ-periodic function in shifts $\delta_\pm$ with the period $\omega \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^t f(s)\Delta s = \int_{\delta_\pm^\omega(t_0)}^{\delta_\pm^\omega(t)} f(s)\Delta s.$$

Lemma 8. [8] *Suppose that r is regressive and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous. Let $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$, then the unique solution of the initial value problem*

$$y^\Delta = r(t)y + f(t), \ y(t_0) = y_0$$

is given by

$$y(t) = e_r(t, t_0)y_0 + \int_{t_0}^t e_r(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Set

$$C_\omega^0 = \{x : x \in C(\mathbb{T}, \mathbb{R}), x(\delta_+^\omega(t)) = x(t)\}$$

with the norm defined by $|x|_0 = \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |x(t)|$, and

$$C_\omega^1 = \{x : x \in C^1(\mathbb{T}, \mathbb{R}), x(\delta_+^\omega(t)) = x(t)\}$$

with the norm defined by $|x|_1 = \max\{|x|_0, |x^\Delta|_0\}$. Then C_ω^0 and C_ω^1 are all Banach spaces.

By using Lemmas 1, 5 and 8, we can obtain the following lemma.

Lemma 9. $x(t) \in C_\omega^1$ *is an ω-periodic solution in shifts $\delta_\pm$ of system (1) if and only if $x(t)$ is an ω-periodic solution in shifts $\delta_\pm$ of*

$$x(t) = \int_t^{\delta_+^\omega(t)} G(t, s)x(s)\Bigg[a(s)x(s)$$
$$+ \sum_{j=1}^n b_j(s)x(\delta_-(\tau_j, s))$$
$$+ \sum_{j=1}^n c_j(s)x^\Delta(\delta_-(\xi_j, s))\Bigg]\Delta s, \quad (2)$$

where

$$G(t, s) = \frac{e_r(t, \sigma(s))}{1 - e_r(t_0, \delta_+^\omega(t_0))}.$$

It is easy to verify that the Green's function $G(t,s)$ satisfies the property

$$\frac{\Theta}{1-\Theta} \leq G(t,s) \leq \frac{1}{1-\Theta}, \forall s \in [t, \delta_+^\omega(t)]_\mathbb{T}, \quad (3)$$

where $\Theta := e_r(t_0, \delta_+^\omega(t_0))$. By Lemma 6, we have

$$G(\delta_+^\omega(t), \delta_+^\omega(s)) = G(t,s), \quad (4)$$

where $t \in \mathbb{T}^*, s \in [t, \delta_+^\omega(t)]_\mathbb{T}$.

In order to obtain the existence of periodic solutions in shifts $\delta_\pm$ of system (1), we first make the following preparations:

Let E be a Banach space and K be a cone in E. The semi-order induced by the cone K is denoted by "$\leq$", that is, $x \leq y$ if and only if $y - x \in K$. In addition, for a bounded subset $A \subset E$, let $\alpha_E(A)$ denote the (Kuratowski) measure of non-compactness defined by

$$\alpha_E(A) = \inf \Big\{ d > 0 : \text{there is a finite number of}$$

$$\text{subsets } A_i \subset A, \text{ such that } A = \bigcup_i A_i$$

$$\text{and } \operatorname{diam}(A_i) \leq d \Big\},$$

where $\operatorname{diam}(A_i)$ denotes the diameter of the set A_i.

Let E, F be two Banach spaces and $D \subset E$, a continuous and bounded map $\Phi : \overline{\Omega} \to F$ is called k-set contractive if for any bounded set $S \subset D$ we have

$$\alpha_F(\Phi(S)) \leq k\alpha_E(S).$$

Φ is called strict-set-contractive if it is k-set-contractive for some $0 \leq k < 1$.

Lemma 10. [24, 25] *Let K be a cone of the real Banach space X and $K_{r,R} = \{x \in K | r \leq x \leq R\}$ with $R > r > 0$. Suppose that $\Phi : K_{r,R} \to K$ is strict-set-contractive such that one of the following two conditions is satisfied:*

(i) $\Phi x \nleq x, \forall x \in K, ||x|| = r$ *and* $\Phi x \ngeq x, \forall x \in K, ||x|| = R$.

(ii) $\Phi x \ngeq x, \forall x \in K, ||x|| = r$ *and* $\Phi x \nleq x, \forall x \in K, ||x|| = R$.

Then Φ has at least one fixed point in $K_{r,R}$.

Define K, a cone in C_ω^1, by

$$K = \{x \in C_\omega^1 : x(t) \geq \Theta |x|_1, t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}\}, \quad (5)$$

and an operator $\Phi : K \to C_\omega^1$ by

$$(\Phi x)(t) = \int_t^{\delta_+^\omega(t)} G(t,s)x(s) \Big[a(s)x(s)$$

$$+ \sum_{j=1}^n b_j(s)x(\delta_-(\tau_j, s))$$

$$+ \sum_{j=1}^n c_j(s)x^\Delta(\delta_-(\xi_j, s)) \Big] \Delta s. \quad (6)$$

In the following, we will give some lemmas concerning K and Φ defined by (5) and (6), respectively.

Lemma 11. *Assume that $(H_1) - (H_3)$ hold.*

(i) *If $r^M \leq 1$, then $\Phi : K \to K$ is well defined.*

(ii) *If (H_4) holds and $r^M > 1$, then $\Phi : K \to K$ is well defined.*

Proof. For any $x \in K$, it is clear that $\Phi x \in C^1(\mathbb{T}, \mathbb{R})$. In view of (6), for $t \in \mathbb{T}$, we obtain

$$(\Phi x)(\delta_+^\omega(t))$$

$$= \int_{\delta_+^\omega(t)}^{\delta_+^\omega(\delta_+^\omega(t))} G(\delta_+^\omega(t), s)x(s) \Big[a(s)x(s)$$

$$+ \sum_{j=1}^n b_j(s)x(\delta_-(\tau_j, s))$$

$$+ \sum_{j=1}^n c_j(s)x^\Delta(\delta_-(\xi_j, s)) \Big] \Delta s$$

$$= \int_t^{\delta_+^\omega(t)} G(\delta_+^\omega(t), \delta_+^\omega(u))x(\delta_+^\omega(u))$$

$$\times \Big[a(\delta_+^\omega(u))\delta_+^{\Delta\omega}(u)x(\delta_+^\omega(u))$$

$$+ \sum_{j=1}^n b_j(\delta_+^\omega(u))\delta_+^{\Delta\omega}(u)x(\delta_-(\tau_j, \delta_+^\omega(u)))$$

$$+ \sum_{j=1}^n c_j(\delta_+^\omega(u))\delta_+^{\Delta\omega}(u)x^\Delta(\delta_-(\xi_j, \delta_+^\omega(u))) \Big] \Delta u$$

$$= \int_t^{\delta_+^\omega(t)} G(t,u)x(u) \Big[a(u)x(u)$$

$$+ \sum_{j=1}^n b_j(u)x(\delta_-(\tau_j, u))$$

$$+ \sum_{j=1}^n c_j(u)x^\Delta(\delta_-(\xi_j, u)) \Big] \Delta u$$

$$= (\Phi x)(t),$$

that is, $(\Phi x)(\delta_+^\omega(t)) = (\Phi x)(t), t \in \mathbb{T}$. So $\Phi x \in C_\omega^1$. In view of (H_2), for $x \in K, t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}$, we have

$$a(t)x(t) + \sum_{j=1}^n b_j(t)x(\delta_-(\tau_j, t))$$

$$+ \sum_{j=1}^n c_j(t)x^\Delta(\delta_-(\xi_j, t))$$

$$\geq \; a(t)x(t) + \sum_{j=1}^{n} b_j(t)x(\delta_-(\tau_j, t))$$

$$- \sum_{j=1}^{n} c_j(t)|x^\Delta(\delta_-(\xi_j, t))|$$

$$\geq \; \Theta a(t)|x^\Delta|_1 + \sum_{j=1}^{n} \Theta b_j(t)|x^\Delta|_1$$

$$- \sum_{j=1}^{n} c_j(t)|x^\Delta|_1$$

$$= \; \left[\Theta a(t) + \sum_{j=1}^{n} \Theta b_j(t) - \sum_{j=1}^{n} c_j(t) \right]|x^\Delta|_1$$

$$\geq \; 0. \tag{7}$$

Therefore, for $x \in K, t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}$, we find

$$\begin{aligned} |\Phi x|_0 \; \leq \; & \frac{1}{1-\Theta} \int_{t_0}^{\delta_+^\omega(t_0)} x(s)\Big[a(s)x(s) \\ & + \sum_{j=1}^{n} b_j(s)x(\delta_-(\tau_j, s)) \\ & + \sum_{j=1}^{n} c_j(s)x^\Delta(\delta_-(\xi_j, s))\Big]\Delta s \end{aligned}$$

and

$$\begin{aligned} (\Phi x)(t) \; \geq \; & \frac{\Theta}{1-\Theta} \int_{t}^{\delta_+^\omega(t)} x(s)\Big[a(s)x(s) \\ & + \sum_{j=1}^{n} b_j(s)x(\delta_-(\tau_j, s)) \\ & + \sum_{j=1}^{n} c_j(s)x^\Delta(\delta_-(\xi_j, s))\Big]\Delta s \\ = \; & \frac{\Theta}{1-\Theta} \int_{t_0}^{\delta_+^\omega(t_0)} x(s)\Big[a(s)x(s) \\ & + \sum_{j=1}^{n} b_j(s)x(\delta_-(\tau_j, s)) \\ & + \sum_{j=1}^{n} c_j(s)x^\Delta(\delta_-(\xi_j, s))\Big]\Delta s \\ \geq \; & \Theta|\Phi x|_0. \tag{8} \end{aligned}$$

Now, we show that $(\Phi x)^\Delta(t) \geq \Theta|\Phi x|_1, t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}$. From (6), we have

$$(\Phi x)^\Delta(t)$$
$$= G(t, \delta_+^\omega(t))x(\delta_+^\omega(t))\Big[a(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t)x(\delta_+^\omega(t))$$

$$+ \sum_{j=1}^{n} b_j(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t)x(\delta_-(\tau_j, \delta_+^\omega(t)))$$

$$+ \sum_{j=1}^{n} c_j(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t)x^\Delta(\delta_-(\xi_j, \delta_+^\omega(t)))\Big]$$

$$- G(t, t)x(t)\Big[a(t)x(t) + \sum_{j=1}^{n} b_j(t)x(\delta_-(\tau_j, t))$$

$$+ \sum_{j=1}^{n} c_j(t)x^\Delta(\delta_-(\xi_j, t))\Big] + r(t)(\Phi x)(t)$$

$$= r(t)(\Phi x)(t) - x(t)\Big[a(t)x(t)$$

$$+ \sum_{j=1}^{n} b_j(t)x(\delta_-(\tau_j, t))$$

$$+ \sum_{j=1}^{n} c_j(t)x^\Delta(\delta_-(\xi_j, t))\Big]. \tag{9}$$

It follows from (7) and (9) that if $(\Phi x)^\Delta(t) \geq 0$, then

$$(\Phi x)^\Delta(t) \leq r(t)(\Phi x)(t) \leq r^M(\Phi x)(t) \leq (\Phi x)(t). \tag{10}$$

On the other hand, from (8), (9) and (H_3), if $(\Phi x)^\Delta(t) < 0$, then

$$-(\Phi x)^\Delta(t)$$

$$= x(t)\Big[a(t)x(t) + \sum_{j=1}^{n} b_j(t)x(\delta_-(\tau_j, t))$$

$$+ \sum_{j=1}^{n} c_j(t)x^\Delta(\delta_-(\xi_j, t))\Big] - r(t)(\Phi x)(t)$$

$$\leq |x|_1^2\Big[a(t) + \sum_{j=1}^{n} b_j(t) + \sum_{j=1}^{n} c_j(t)\Big] - r^m(\Phi x)(t)$$

$$\leq (1+r^m)\frac{\Theta^2}{1-\Theta}|x|_1^2 \int_{t_0}^{\delta_+^\omega(t_0)}\Big[\Theta a(s)$$

$$+ \sum_{j=1}^{n} \Theta b_j(s) - \sum_{j=1}^{n} c_j(s)\Big]\Delta s - r^m(\Phi x)(t)$$

$$= (1+r^m)\int_{t_0}^{\delta_+^\omega(t_0)} \frac{\Theta}{1-\Theta}\Theta|x|_1\Big[\Theta|x|_1 a(s)$$

$$+ \sum_{j=1}^{n} \Theta|x|_1 b_j(s) - \sum_{j=1}^{n} |x|_1 c_j(s)\Big]\Delta s$$

$$- r^m(\Phi x)(t)$$

$$\leq (1+r^m)\int_{t}^{\delta_+^\omega(t)} G(t, s)x(s)\Big[a(s)x(s)$$

$$+ \sum_{j=1}^{n} b_j(s) x(\delta_-(\tau_j, s))$$

$$- \sum_{j=1}^{n} c_j(s) |x^\Delta(\delta_-(\xi_j, s))| \Bigg] \Delta s - r^m (\Phi x)(t)$$

$$\leq (1 + r^m) \int_t^{\delta_+^\omega(t)} G(t, s) x(s) \Bigg[a(s) x(s)$$

$$+ \sum_{j=1}^{n} b_j(s) x(\delta_-(\tau_j, s))$$

$$+ \sum_{j=1}^{n} c_j(s) x^\Delta(\delta_-(\xi_j, s)) \Bigg] \Delta s - r^m (\Phi x)(t)$$

$$= (1 + r^m)(\Phi x)(t) - r^m (\Phi x)(t)$$
$$= (\Phi x)(t). \tag{11}$$

It follows from (10) and (11) that $|(\Phi x)^\Delta|_0 \leq |\Phi x|_0$. So $|\Phi x|_1 = |\Phi x|_0$. By (8), we have $(\Phi x)(t) \geq \Theta |\Phi x|_1$. Hence, $\Phi x \in K$. The proof of (i) is complete.

(ii) In view of the proof of (i), we only need to prove that $(\Phi x)^\Delta(t) \geq 0$ implies

$$(\Phi x)^\Delta(t) \leq (\Phi x)(t).$$

From (7), (9), (H_2) and (H_4), we obtain

$$(\Phi x)^\Delta(t)$$

$$\leq r(t)(\Phi x)(t) - \Theta |x|_1 \Bigg[a(t) x(t)$$

$$+ \sum_{j=1}^{n} b_j(t) x(\delta_-(\tau_j, t))$$

$$- \sum_{j=1}^{n} c_j(t) |x^\Delta(\delta_-(\xi_j, t))| \Bigg]$$

$$\leq r(t)(\Phi x)(t)$$

$$- \Theta |x|_1^2 \Bigg[a(t) + \sum_{j=1}^{n} b_j(t) - \sum_{j=1}^{n} c_j(t) \Bigg]$$

$$\leq r^M (\Phi x)(t) - \Theta |x|_1^2 \frac{r^M - 1}{\Theta(1 - \Theta)} \int_{t_0}^{\delta_+^\omega(t_0)} \Bigg[a(s)$$

$$+ \sum_{j=1}^{n} b_j(s) + \sum_{j=1}^{n} c_j(s) \Bigg] \Delta s$$

$$\leq r^M (\Phi x)(t)$$

$$- (r^M - 1) \int_t^{\delta_+^\omega(t)} \frac{1}{1 - \Theta} |x|_1 \Bigg[a(s) |x|_1$$

$$+ \sum_{j=1}^{n} b_j(s) |x|_1 + \sum_{j=1}^{n} c_j(s) |x|_1 \Bigg] \Delta s$$

$$\leq r^M (\Phi x)(t)$$

$$- (r^M - 1) \int_t^{\delta_+^\omega(t)} G(t, s) x(s) \Bigg[a(s) x(s)$$

$$+ \sum_{j=1}^{n} b_j(s) x(\delta_-(\tau_j, s))$$

$$+ \sum_{j=1}^{n} c_j(s) |x^\Delta(\delta_-(\xi_j, s))| \Bigg] \Delta s$$

$$\leq r^M (\Phi x)(t)$$

$$- (r^M - 1) \int_t^{\delta_+^\omega(t)} G(t, s) x(s) \Bigg[a(s) x(s)$$

$$+ \sum_{j=1}^{n} b_j(s) x(\delta_-(\tau_j, s))$$

$$+ \sum_{j=1}^{n} c_j(s) x^\Delta(\delta_-(\xi_j, s)) \Bigg] \Delta s$$

$$= r^M (\Phi x)(t) - (r^M - 1)(\Phi x)(t)$$
$$= (\Phi x)(t).$$

The proof of (ii) is complete. $\square$

Lemma 12. *Assume that* $(H_1) - (H_3)$ *hold and* $R \sum_{j=1}^{n} c_j^M < 1$.

(i) *If* $r^M \leq 1$, *then* $\Phi : K \bigcap \bar{\Omega}_R \to K$ *is strict-set-contractive,*

(ii) *If* (H_4) *holds and* $r^M > 1$, *then* $\Phi : K \bigcap \bar{\Omega}_R \to K$ *is strict-set-contractive,*

where $\Omega_R = \{x \in C_\omega^1 : |x|_1 < R\}$.

Proof. We only need to prove (i), since the proof of (ii) is similar. It is easy to see that Φ is continuous and bounded. Now we prove that $\alpha_{C_\omega^1}(\Phi(S)) \leq \left(R \sum_{j=1}^{n} c_j^M \right) \alpha_{C_\omega^1}(S)$ for any bounded set $S \subset \bar{\Omega}_R$. Let $\eta = \alpha_{C_\omega^1}(S)$. Then, for any positive number $\varepsilon < \left(R \sum_{j=1}^{n} c_j^M \right) \eta$, there is a finite family of subsets $\{S_i\}$ satisfying $S = \bigcup_i S_i$ with $\text{diam}(S_i) \leq \eta + \varepsilon$. Therefore

$$|x - y|_1 \leq \eta + \varepsilon \quad \text{for any } x, y \in S_i. \tag{12}$$

As S and S_i are precompact in C_ω^0, it follows that there is a finite family of subsets $\{S_{ij}\}$ of S_i such that $S_i = \bigcup_j S_{ij}$ and

$$|x - y|_0 \leq \varepsilon \quad \text{for any } x, y \in S_{ij}. \tag{13}$$

In addition, for any $x \in S$ and $t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}$, we have

$$
|(\Phi x)(t)| = \int_t^{\delta_+^\omega(t)} G(t,s)x(s)\Bigg[a(s)x(s)
$$
$$
+ \sum_{j=1}^n b_j(s)x(\delta_-(\tau_j,s))
$$
$$
+ \sum_{j=1}^n c_j(s)x^\Delta(\delta_-(\xi_j,s))\Bigg]\Delta s
$$
$$
\leq \frac{R^2}{1-\Theta} \int_{t_0}^{\delta_+^\omega(t_0)} \Bigg[a(s) + \sum_{j=1}^n b_j(s)
$$
$$
+ \sum_{j=1}^n c_j(s)\Bigg]\Delta s := H
$$

and

$$
|(\Phi x)^\Delta(t)| = \Bigg| r(t)(\Phi x)(t) - x(t)\Bigg[a(t)x(t)
$$
$$
+ \sum_{j=1}^n b_j(t)x(\delta_-(\tau_j,t))
$$
$$
+ \sum_{j=1}^n c_j(t)x^\Delta(\delta_-(\xi_j,t))\Bigg]\Bigg|
$$
$$
\leq r^M H + R^2 a^M + R^2 \sum_{j=1}^n (b_j^M + c_j^M).
$$

Applying the Arzela-Ascoli Theorem, we know that $\Phi(S)$ is precompact in C_ω^0. Then, there is a finite family of subsets $\{S_{ijk}\}$ of S_{ij} such that $S_{ij} = \bigcup_k S_{ijk}$ and

$$
|\Phi x - \Phi y|_0 \leq \varepsilon \quad \text{for any } x,y \in S_{ijk}. \tag{14}
$$

From (7), (9) and (12)-(14) and (H_2), for any $x,y \in S_{ijk}$, we obtain

$$
|(\Phi x)^\Delta - (\Phi y)^\Delta|_0
$$
$$
= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \Bigg\{\Bigg| r(t)(\Phi x)(t) - r(t)(\Phi y)(t)
$$
$$
- x(t)\Bigg[a(t)x(t) + \sum_{j=1}^n b_j(t)x(\delta_-(\tau_j,t))
$$
$$
+ \sum_{j=1}^n c_j(t)x^\Delta(\delta_-(\xi_j,t))\Bigg]
$$
$$
+ y(t)\Bigg[a(t)y(t) + \sum_{j=1}^n b_j(t)y(\delta_-(\tau_j,t))
$$
$$
+ \sum_{j=1}^n c_j(t)y^\Delta(\delta_-(\xi_j,t))\Bigg]\Bigg|\Bigg\}
$$

$$
\leq \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \{|r(t)[(\Phi x)(t) - (\Phi y)(t)]|\}
$$
$$
+ \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \Bigg\{\Bigg| x(t)\Bigg[a(t)x(t)
$$
$$
+ \sum_{j=1}^n b_j(t)x(\delta_-(\tau_j,t))
$$
$$
+ \sum_{j=1}^n c_j(t)x^\Delta(\delta_-(\xi_j,t))\Bigg]
$$
$$
- y(t)\Bigg[a(t)y(t) + \sum_{j=1}^n b_j(t)y(\delta_-(\tau_j,t))
$$
$$
+ \sum_{j=1}^n c_j(t)y^\Delta(\delta_-(\xi_j,t))\Bigg]\Bigg|\Bigg\}
$$
$$
\leq r^M |(\Phi x) - (\Phi y)|_0
$$
$$
+ \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \Bigg\{\Bigg| x(t)\Bigg[\Bigg(a(t)x(t)
$$
$$
+ \sum_{j=1}^n b_j(t)x(\delta_-(\tau_j,t))
$$
$$
+ \sum_{j=1}^n c_j(t)x^\Delta(\delta_-(\xi_j,t))\Bigg)
$$
$$
- \Bigg(a(t)y(t) + \sum_{j=1}^n b_j(t)y(\delta_-(\tau_j,t))
$$
$$
+ \sum_{j=1}^n c_j(t)y^\Delta(\delta_-(\xi_j,t))\Bigg)\Bigg]\Bigg|\Bigg\}
$$
$$
+ \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \Bigg\{\Bigg| y(t)\Bigg[a(t)y(t)
$$
$$
+ \sum_{j=1}^n b_j(t)y(\delta_-(\tau_j,t))
$$
$$
+ \sum_{j=1}^n c_j(t)y^\Delta(\delta_-(\xi_j,t))\Bigg][x(t)-y(t)]\Bigg|\Bigg\}
$$
$$
\leq r^M \varepsilon + R \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \Bigg\{a(t)|x(t)-y(t)|
$$
$$
+ \sum_{j=1}^n b_j(t)|x(\delta_-(\tau_j,t)) - y(\delta_-(\tau_j,t))|
$$
$$
+ \sum_{j=1}^n c_j(t)|x^\Delta(\delta_-(\xi_j,t)) - y^\Delta(\delta_-(\xi_j,t))|\Bigg\}
$$
$$
+\varepsilon \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \Bigg\{a(t)y(t) + \sum_{j=1}^n b_j(t)y(\delta_-(\tau_j,t))
$$

$$+ \sum_{j=1}^{n} c_j(t)|y^\Delta(\delta_-(\xi_j, t))| \Big\}$$

$$\leq r^M \varepsilon + R\varepsilon \left(a^M + \sum_{j=1}^{n} b_j^M \right)$$

$$+ R(\eta + \varepsilon) \left(\sum_{j=1}^{n} c_j^M \right)$$

$$+ R\varepsilon \left(a^M + \sum_{j=1}^{n} b_j^M + \sum_{j=1}^{n} c_j^M \right)$$

$$= \left(R\eta \sum_{j=1}^{n} c_j^M \right) + \hat{H}\varepsilon, \qquad (15)$$

where $\hat{H} = r^M + 2Ra^M + 2R \sum_{j=1}^{n} b_j^M + 2R \sum_{j=1}^{n} c_j^M$.

From (14) and (15) we have

$$|\Phi x - \Phi y|_1 \leq \left(R \sum_{j=1}^{n} c_j^M \right)\eta + \hat{H}\varepsilon,$$

for any $x, y \in S_{ijk}$. As ε is arbitrary small, it follows that

$$\alpha_{C_\omega^1}(\Phi(S)) \leq \left(R \sum_{j=1}^{n} c_j^M \right)\alpha_{C_\omega^1}(S).$$

Therefore, Φ is strict-set-contractive. The proof is complete. $\square$

3 Main Result

Our main result of this paper is as follows:

Theorem 13. *Assume that* $(H_1) - (H_3)$, (H_5) *hold.*

(i) *If* $r^M \leq 1$, *then system (1) has at least one positive ω-periodic solution in shifts $\delta_\pm$.*

(ii) *If* (H_4) *holds and* $r^M > 1$, *then system (1) has at least one positive ω-periodic solution in shifts $\delta_\pm$.*

Proof. We only need to prove (i), since the proof of (ii) is similar. Let $R = \frac{1-\Theta}{\Theta^2\Gamma}$ and $0 < r < \frac{\Theta(1-\Theta)}{\Pi}$, then we have $0 < r < R$. From Lemmas 11 and 12, we know that Φ is strict-set-contractive on $K_{r,R}$. In view of (9), we see that if there exists $x^* \in K$ such that $\Phi x^* = x^*$, then x^* is one positive ω-periodic solution in shifts $\delta_\pm$ of system (1). Now, we shall prove that condition (ii) of Lemma 10 holds.

First, we prove that $\Phi x \not\geq x$, $\forall x \in K$, $|x|_1 = r$. Otherwise, there exists $x \in K$, $|x|_1 = r$ such that

$\Phi x \geq x$. So $|x| > 0$ and $\Phi x - x \in K$, which implies that

$$(\Phi x)(t) - x(t) \geq \Theta|\Phi x - x|_1 \geq 0, \qquad (16)$$

for any $t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}$.

Moreover, for $t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}$, we have

$$\begin{aligned}(\Phi x)(t) &= \int_t^{\delta_+^\omega(t)} G(t, s)x(s)\Big[a(s)x(s) \\ &\quad + \sum_{j=1}^{n} b_j(s)x(\delta_-(\tau_j, s)) \\ &\quad + \sum_{j=1}^{n} c_j(s)x^\Delta(\delta_-(\xi_j, s))\Big]\Delta s \\ &\leq \frac{1}{1-\Theta}r|x|_0 \int_{t_0}^{\delta_+^\omega(t_0)} \Big[a(s) + \sum_{j=1}^{n} b_j(s) \\ &\quad + \sum_{j=1}^{n} c_j(s)\Big]\Delta s \\ &= \frac{r}{1-\Theta}\Pi|x|_0 \\ &< \Theta|x|_0. \qquad (17)\end{aligned}$$

In view of (16) and (17), we have

$$|x|_0 \leq |\Phi x| < \Theta|x|_0 < |x|_0,$$

which is a contradiction. Finally, we prove that $\Phi x \not\leq x$, $\forall x \in K$, $|x|_1 = R$ also holds. For this case, we only need to prove that

$$\Phi x \not\leq x \quad x \in K, |x|_1 = R.$$

Suppose, for the sake of contradiction, that there exists $x \in K$ and $|x|_1 = R$ such that $\Phi x < x$. Thus $x - \Phi x \in K \setminus \{0\}$. Furthermore, for any $t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}$, we have

$$x(t) - (\Phi x)(t) \geq \Theta|x - \Phi x|_1 > 0. \qquad (18)$$

In addition, for any $t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}$, we find

$$\begin{aligned}(\Phi x)(t) &= \int_t^{\delta_+^\omega(t)} G(t, s)x(s)\Big[a(s)x(s) \\ &\quad + \sum_{j=1}^{n} b_j(s)x(\delta_-(\tau_j, s)) \\ &\quad + \sum_{j=1}^{n} c_j(s)x^\Delta(\delta_-(\xi_j, s))\Big]\Delta s \\ &\geq \frac{\Theta}{1-\Theta}\Theta|x|_1^2 \int_{t_0}^{\delta_+^\omega(t_0)} \Big[\Theta a(s) \\ &\quad + \sum_{j=1}^{n} \Theta b_j(s) - \sum_{j=1}^{n} c_j(s)\Big]\Delta s\end{aligned}$$

$$= \frac{\Theta^2}{1-\Theta}\Gamma R^2$$
$$= R. \tag{19}$$

From (18) and (19), we obtain

$$|x| > |\Phi x|_0 \geq R,$$

which is a contradiction. Therefore, conditions (i) and (ii) hold. By Lemma 10, we see that Φ has at least one nonzero fixed point in K. Therefore, system (1) has at least one positive ω-periodic solution in shifts $\delta_\pm$. The proof is complete. □

4   Numerical examples

Example 1. Consider the following system on time scales

$$x^\Delta(t) = x(t)\left[\frac{2+\cos\pi t}{8} - (5-2\sin\pi t)x(t)\right.$$
$$-(2+\sin\pi t)x(\delta_-(2,t))$$
$$\left. -\frac{1-\cos\pi t}{20}x^\Delta(\delta_-(2,t))\right]. \tag{20}$$

Obviously,

$$r = \frac{2+\cos\pi t}{8}, \, a = 5-2\sin\pi t,$$

$$b_1 = 2+\sin\pi t, \, c_1 = \frac{1-\cos\pi t}{20}.$$

Let $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, then $\delta_+^\omega(t) = t+2$. It is easy to verify $r(t), a(t), b_1(t), c_1(t)$ are Δ-periodic in shifts $\delta_\pm$ with period $\omega = 2$. By a direct calculation, we can get

$$\Theta = e^{-\frac{1}{2}}, \, \Gamma = 8.3914,$$

$$\inf_{t\in[0,2]_\mathbb{T}}\{\Theta a(t) + \Theta b_1(t) - c_1(t)\} = 2.3261,$$

$$\sup_{t\in[0,2]_\mathbb{T}}\{a(t)+b_1(t)+c_1(t)\} = 8.0512,$$

$$(1+r^m)\frac{\Theta^2}{1-\Theta}\Gamma = 8.8264,$$

and

$$\frac{1-\Theta}{\Theta^2\Gamma}c_1^M = 0.0128.$$

Hence, $(H_1)-(H_3)$, (H_5) hold and $r^M \leq 1$. According to Theorem 13, when $\mathbb{T} = \mathbb{R}$, system (20) has at least one positive 2-periodic solution in shifts $\delta_\pm$.

Example 2. Consider the following system on time scales

$$x^\Delta(t) = x(t)\left[\frac{1}{5t} - \frac{2}{t}x(t) - \frac{3}{t}x(\delta_-(4,t))\right.$$
$$\left. -\frac{1}{20t}x^\Delta(\delta_-(4,t))\right]. \tag{21}$$

Obviously,

$$r = \frac{1}{5t}, \, a = \frac{2}{t}, b_1 = \frac{3}{t}, \, c_1 = \frac{1}{20t}.$$

Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $t_0 = 1$, then $\delta_+^\omega(t) = 4t$. It is easy to verify $r(t), a(t), b_1(t), c_1(t)$ are Δ-periodic in shifts $\delta_\pm$ with period $\omega = 4$. By a direct calculation, we can get

$$\Theta = 0.6818, \, \Gamma = 4.6566,$$

$$\inf_{t\in[1,4]_\mathbb{T}}\{\Theta a(t) + \Theta b_1(t) - c_1(t)\} = 0.8022,$$

$$\sup_{t\in[1,4]_\mathbb{T}}\{a(t)+b_1(t)+c_1(t)\} = 5.0500,$$

$$(1+r^m)\frac{\Theta^2}{1-\Theta}\Gamma = 7.1429,$$

and

$$\frac{1-\Theta}{\Theta^2\Gamma}c_1^M = 0.0073.$$

Hence, $(H_1)-(H_3)$, (H_5) hold and $r^M \leq 1$. According to Theorem 13, when $\mathbb{T} = 2^{\mathbb{N}_0}$, system (21) has at least one positive 4-periodic solution in shifts $\delta_\pm$.

5   Conclusion

This paper is concerned with the existence of positive periodic solutions in shifts $\delta_\pm$ for a neutral delay logistic equation on time scales. Based on the theory of calculus on time scales, by using a fixed point theorem of strict-set-contraction, some existence results are established for the system.

The method used in this paper is the same in [3], but the results obtained in this paper extend and unify periodic differential, difference, h-difference and q-difference equations and more by a new periodicity concept on time scales.

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